40 Anosov systems

40.1 Anosov system

$f \in C^r(M, M)$ is Anosov, if $\Omega(f) = M$ and hyperbolic.

In more detailed words:[436]

An Anosov diffeomorphism $f : M \to M$ is a diffeomorphism which satisfies the following:
(a) There is a continuous splitting of the tangent bundle $TM = E^s + E^u$ which is preserved by the derivative Df.
(b) There exist constants $C > 0$, $C' > 0$ and $\lambda \in (0, 1)$ (i.e., hyperbolicity) and a Riemannian metric $\| \ \|$ on TM such that

$$\|Df^n(v)\| \leq C\lambda^n\|v\| \text{ for } v \in E^s, \tag{40.1}$$
$$\|Df^n(v)\| \geq C'\lambda^{-n}\|v\| \text{ for } v \in E^u. \tag{40.2}$$

40.2 Some properties of Anosov systems

Per(f) is countable and dense in M.

Anosov systems are structurally stable.
If the Lebesgue measure may be introduced on M, Anosov systems are ergodic with respect to it.

Thus, clearly being Anosov excludes being Morse-Smale **38.5**. Obviously, MS diffeomorphisms are not dense for $d \geq 2$.

40.3 Toral diffeomorphism

Integer matrices L with $|\det L| = 1$ on T^n (constructed from the unit cube with periodic boundary conditions) is called toral diffeomorphisms.

For $TM = E^s \oplus E^u$, if $\dim E^u$ or $\dim E^s = 1$, the toral diffeomorphisms is said to be codimension one.

Theorem [Franks] An Anosov diffeomorphism f is homomorphic to a toral diffeomorphism, if f is codimension 1.

[436]Taken from J Franks, 'Anosov diffeomorphisms on tori," Trans AMS 145, 117 (1969).

40.4 Group automorphism on T^2

Let A be a regular 2×2 integer matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{40.3}$$

This defines a homomorphism $T_A : T^2 \to T^2$:

$$T_A(x, y) = (ax + by, cx + dy) \bmod 1. \tag{40.4}$$

If $\det T_A = \pm 1$, this is an automorphism (homeomorphism onto itself).

Notice that A need not be normal (i.e., $AA^* = A^*A$), so even if eigenvalues of A are distinct, it need not be diagonalizable with an orthogonal transformation. Thus, the two eigendirections may not be orthogonal. These are well illustrated by Thom's diffeomorphism **40.5**.

40.5 Thom diffeomorphism

A toral diffeomorphism $T_A : T^2 \to T^2$ with

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \tag{40.5}$$

is called th Thom diffeomorphism (physicists often call its 'Arnold's cat map, since they read easy-reading books only).

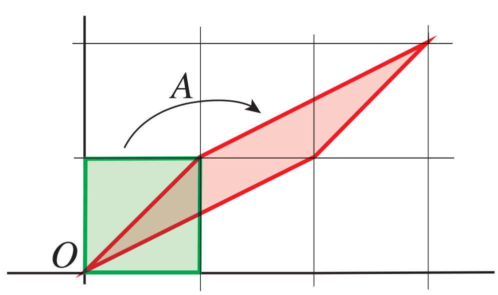

Figure 40.1: Thom's diffeomorphism is obtained from A by imposing a periodic boundary condition on the green square.

The origin corresponds to a hyperbolic fixed point p. The set of homoclinic points

$W^s(p) \cap W^u(p)$ is dense in T^2 (as you can guess from Fig. 40.4), so the periodic points are dense in T^2. This defines an Anosov system on T^2. It is called Arnold's cat map, because something like Fig. 40.2 was used by Arnold and Avez[437] what horrible things could happen for Anosov system.[438]

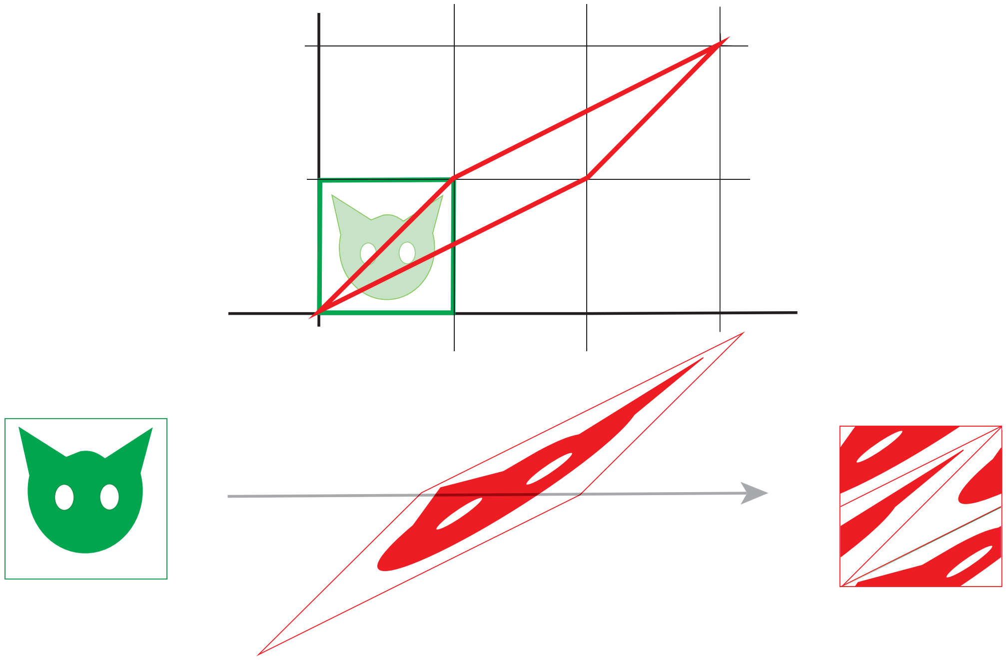

Figure 40.2: Thom's diffeomorphism

Note that $\det A = 1$. The eigenvalues and the corresponding eigenvectors of A are given by $(3 \pm \sqrt{5})/2$ (respectively) with $(1, (-1 \pm \sqrt{5})/2)^T$. We see that the Kolmogorov-Sinai entropy is $\log(3 + \sqrt{2})/2$.

[437] V I Arnold and A Avez, *Ergodic Problems of Classical Mechanics* (The Mathematical physics monograph series) (Benjamin 1968). This is a classic.

[438] A whole cat is kneaded here with T^2 illustrated: https://upload.wikimedia.org/wikipedia/commons/9/9e/Arnold%27s_cat_map.png.

40.6 Markov partition for Thom automorphism

Using the eigendirections in 40.5, we can make a Markov partition consisting of parallelograms, noting that stable and unstable manifolds must go through lattice points.

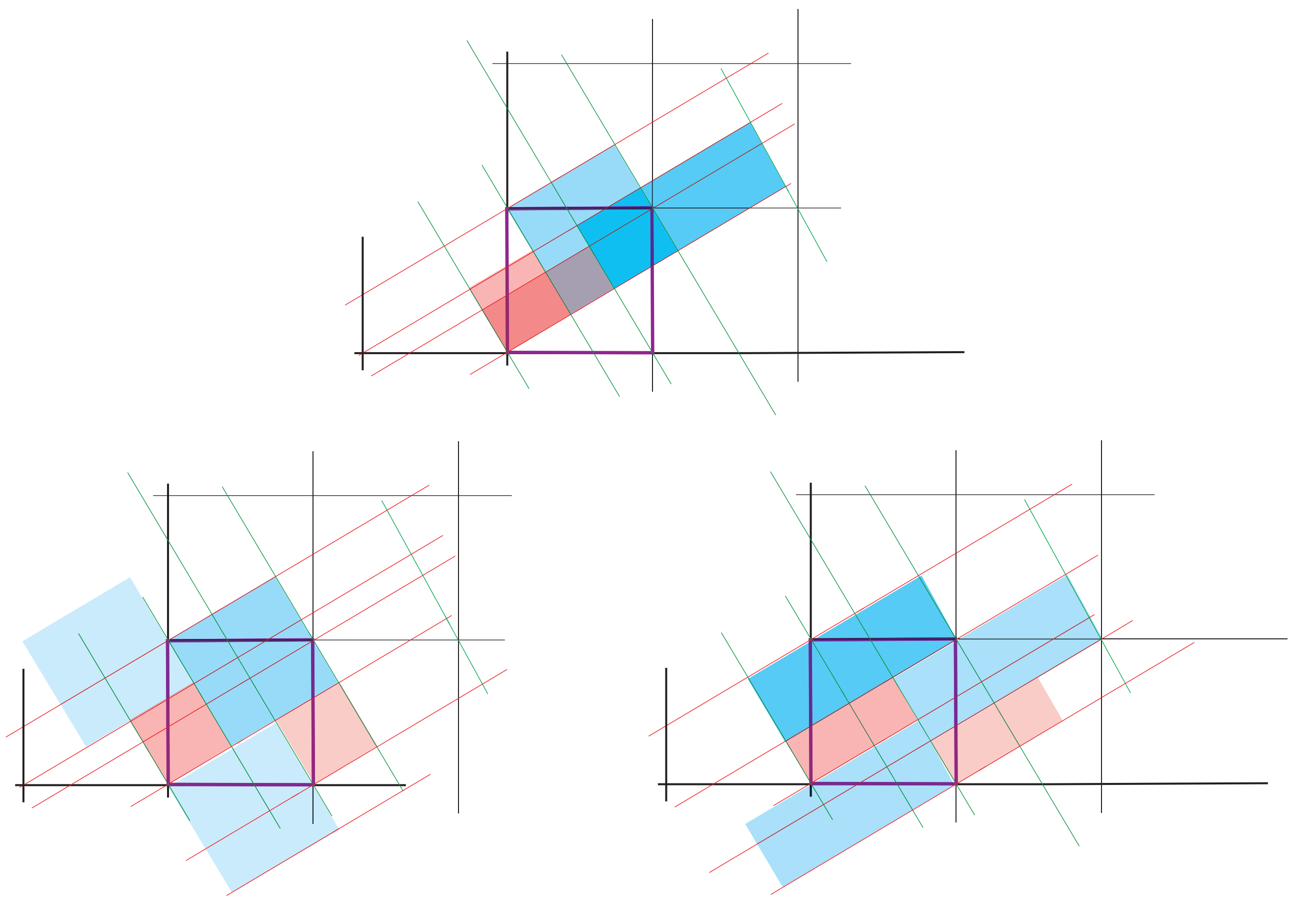

Figure 40.3: Markov partition; lower figures explain how to cover T^2 with the Markov partition above and its image. Redlines indicate W^u and the green W^s.

Needless to say, we can make many different Markov partitions, specifying the largest size of the piece.

Another example with nonnormal A is

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \tag{40.6}$$

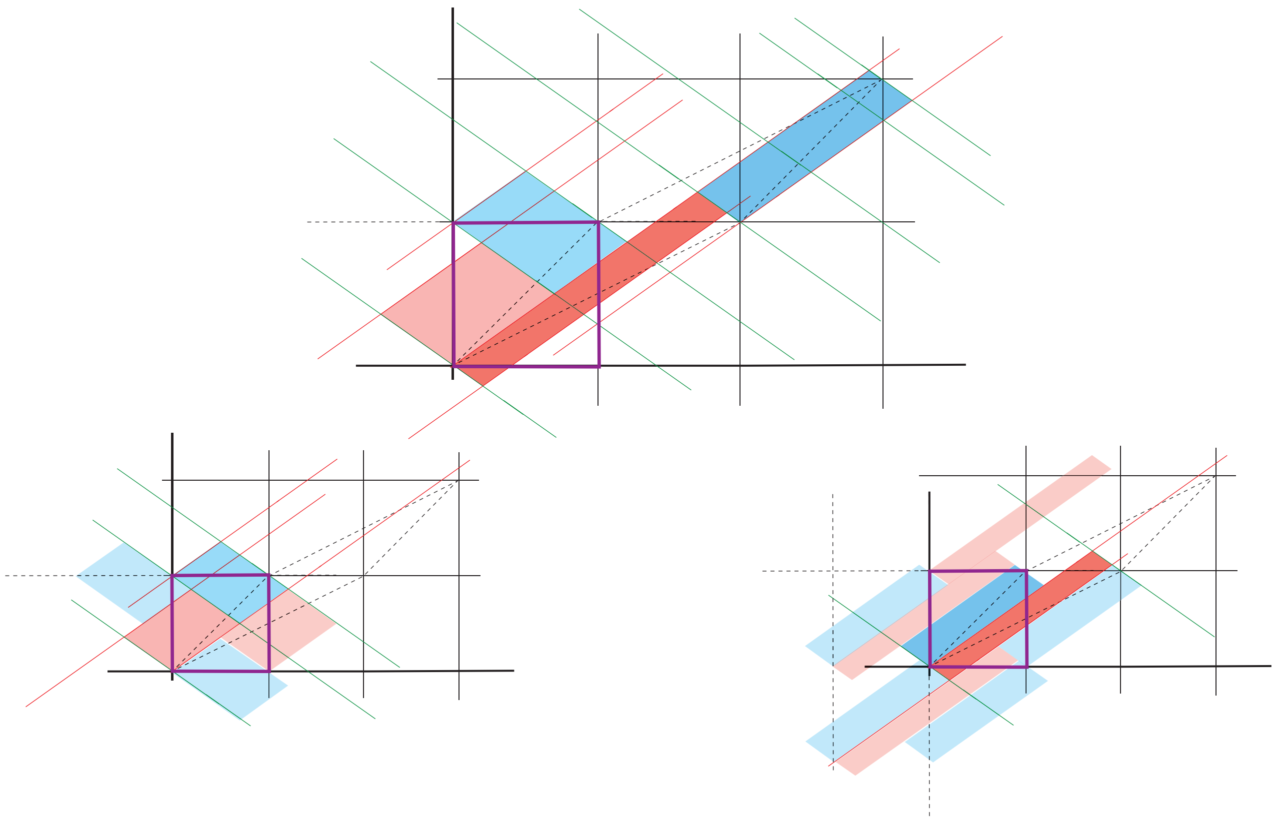

Figure 40.4: Markov partition; lower figures explain how to cover T^2 with the Markov partition above and its image. Redlines indicate W^u and the green W^s.

40.7 Pseudoorbit traceability

For a pseudoorbit $\{x_0, x_1, \cdots\}$, we can construct a true trajectory $T^k x$ always running close to it. This is the traceability of pseudoorbits.

For a system to have a traceability, necessary and sufficient condition (for C^1 systems) is that the system has a Markov partition (we have already seen this in Section 39). If a system has a Markov partition, the system is isomorphic to a symbolic dynamics called a Markovian subshift. Thus, Ornstein's theorem tells us that

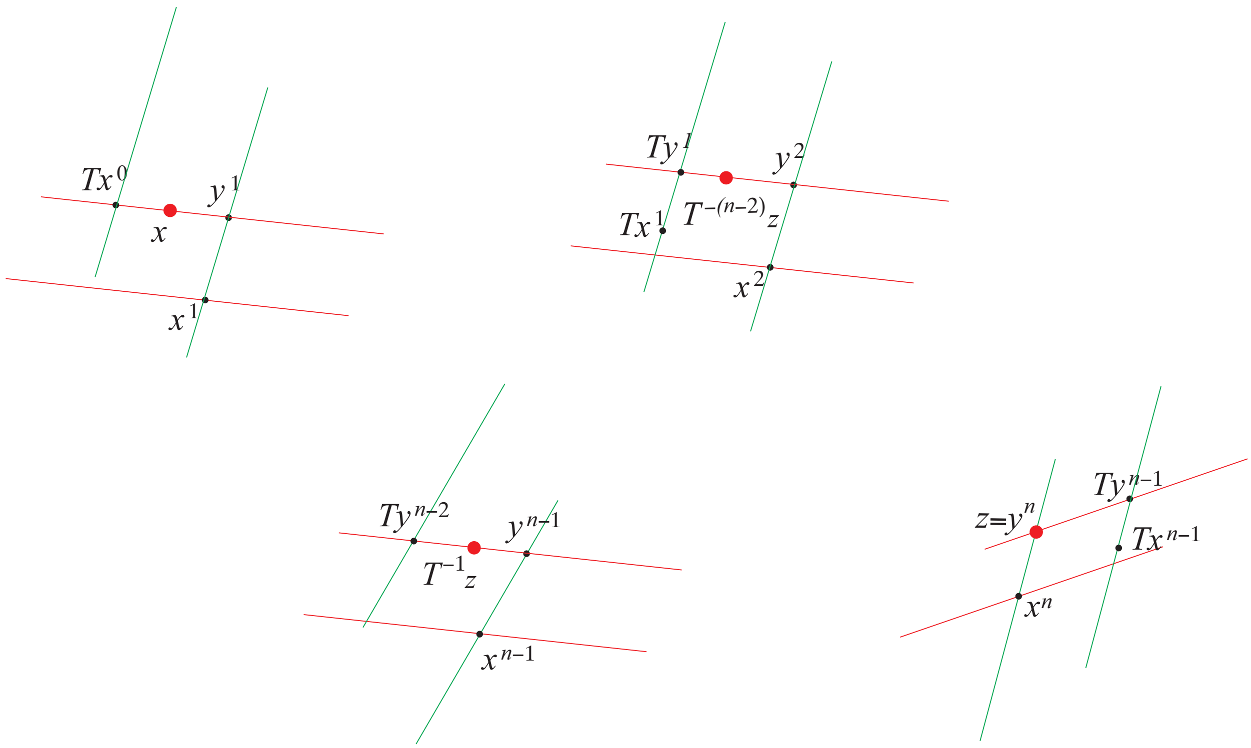

Figure 40.5: How to construct a shadowing orbit (red points)

the system is actually isomorphic to a Bernoulli system, a maximally chaotic system.

Thus ironically a numerically obtained trajectory can be a true trajectory only if the system is maximally chaotic.