39 Axiom A systems

39.1 Axiom A system

f is an axiom A system, iff $\Omega(f)$ is hyperbolic and

$$\Omega(f) = \overline{\{x : \text{ periodic point of } f\}} \tag{39.1}$$

Here, $\Omega(f)$ is te totality of the non-wandering set of f.

39.2 Hyperbolic set of f

Let X be a compact smooth manifold, $f : X \to X$ a diffeomorphism, and $Df : TM \to TM$ the differential of f. An f-invariant subset Λ of X (i.e., $f(\Lambda) = \Lambda$) is said to be hyperbolic, if the restriction to Λ of the tangent bundle of X (i.e., $T_\Lambda X$) admits a splitting into a Whitney sum of two Df-invariant subbundles, called the stable bundle and the unstable bundle (denoted E_s and E_u. With respect to some Riemannian metric on X, the restriction of Df to E_s must be a contraction and the restriction of Df to E_u must be an expansion.

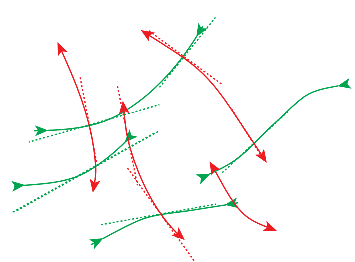

Figure 39.1: Hyperbolic set: red curves are unstable mfds and green curves are stable mfds. Dashed straight lines denote E^u and E^s.

39.3 How a small open set spreads

Let $f \in C^r(M, M)$, p its periodic point and $U \subset M$ is open.
(a)

$$U \cap W^s(p) \neq \emptyset \Rightarrow \overline{\cup_{m \in \mathbb{N}^+} f^m(U)} \supset W^u(p). \tag{39.2}$$

[If p is a sink $W^u(p) = \emptyset$, so the statement is trivially true.]

(b) Let $q \neq p$ be a periodic point, and x be a heteroclinic point of q and p: $x \in W^u(p) \cap W^s(q)$

$$U \cap W^s(p) \neq \emptyset, \ W^u(p) \overline{\pitchfork}_x W^s(q) \Rightarrow (\cup_{m \in \mathbb{N}^+} f^m(U)) \cap W^u(q) \neq \emptyset. \tag{39.3}$$

See the following Fig. 39.2.

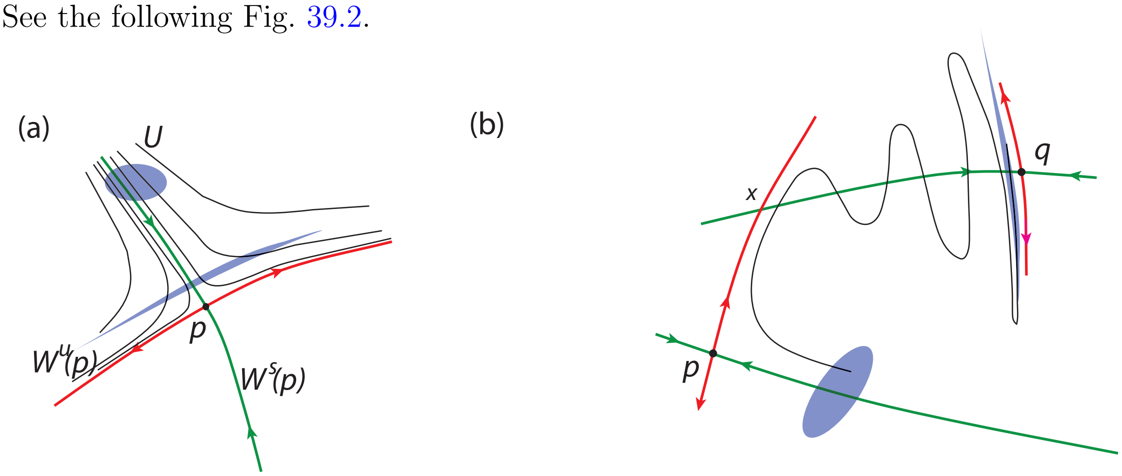

Figure 39.2: How small open sets on W^s spreads

39.4 Chain of periodic orbits

Let $f \in C^r(M, M)$, and p_i $(i = 0, \cdots, n)$ are hyperbolic periodic orbits $(p_0 = p_n)$ such that $x \in W^u(p_i) \cap W^s(p_{i+1})$ implies $W^u(p_i) \overline{\pitchfork}_x W^s(p_{i+1})$. Then, x is non-wandering (i.e., $x \in \Omega$).

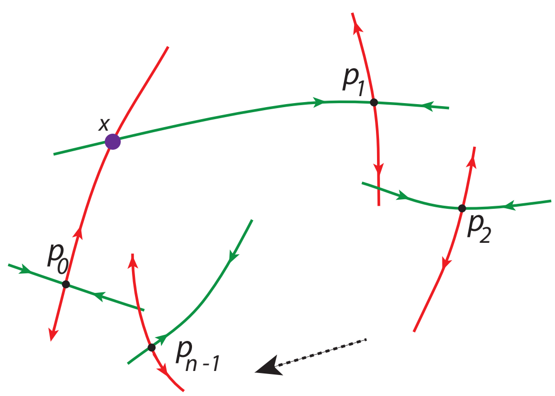

Figure 39.3: Chain of periodic orbits

39.5 Stability manifold theorem[432]

$f \in \mathrm{Diff}^r(X)$ and Λ is the hyperbolic set of f. Then, for small $\varepsilon > 0$

[432] Smale BAMS 73 747 (1967) Th(7.3) p781.

(a) Local stable and unstable mfd of x ($\in \Lambda$) is a C^r-disk in Λ and

$$T_x W^s_\varepsilon(x) = E^s_x,\ T_x W^u_\varepsilon(x) = E^u_x. \tag{39.4}$$

(b) On these local submanifolds f is just expanding or contracting as usual.
(c) $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$ depend on x continuously.

39.6 Existence of canonical coordinate[433]
Let f satisfy Axiom A. Then, for any small ε (> 0) there is $\delta > 0$ such that for $x, y \in \Omega(f)$ and for $d(x, y) \le \delta$,

$$W^s_\varepsilon(x) \cap W^u_\varepsilon(y) = \{[x, y]\} \subset \Omega(f), \tag{39.5}$$

and the crossing point [,] depends on x and y continuously.

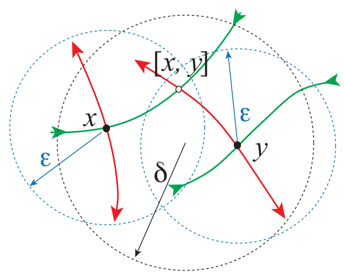

Figure 39.4: If we take $x, y \in \Lambda$ close enough (within δ).

The point is that for any ε we can find x and y in $\Omega(f)$ connected heteroclinically. Then, this is based on the following two lemmas (basically **39.3**):

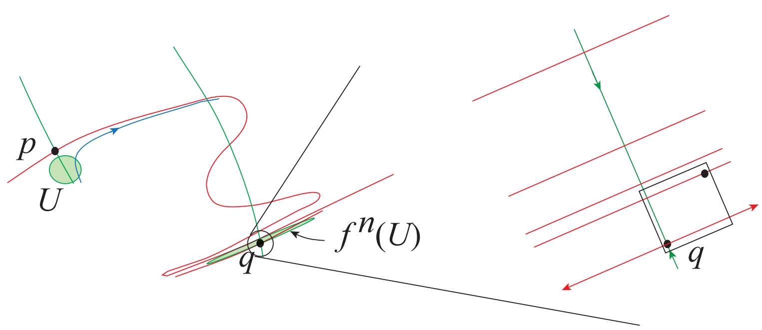

Figure 39.5: Lemma 1; Right: near q expanded, we can make a local cartesian coordinate system in the box.

[433]Bowen 3.3.

Lemma 1: Let p and q be periodic orbits and $W^u(p) \overline{\pitchfork} W^s(q)$. Take $U \cap W^s(p) \neq \emptyset$. Then, $\cup_{m>0} f^m(U) \cap W^s(q) \neq \emptyset$. See Fig. 39.5.
Lemma 2: Let $p_0, p_1, \cdots, p_n$ $(p_0 = p_n)$ be hyperbolic periodic points and $W^u(p_i) \overline{\pitchfork}_{x_i} W^s(p_{i+1})$. Then, $x_i \in \Omega(f)$.

This can be shown by applying Lemma 1 to the successive periodic points. See Fig. 39.6:

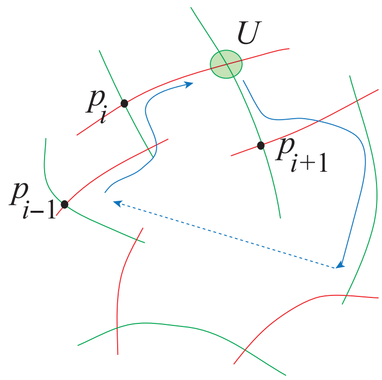

Figure 39.6: Lemma 2

The demonstration of the canonical coordinates goes as follows:
Periodic orbits are dense in $\Omega(f)$. If we take ε small enough, then $[x, y]$ becomes unique. If x and y are periodic, then Lemma 2 implies $[x, y] \in \Omega(f)$. Even if these are not periodic, they are dense in $\Omega(f)$, $[x, y]$ must be in $\Omega(f)$, because $\Omega(f)$ is a closed set.

39.7 Expansivity of hyperbolic set[434]

Let Λ be the hyperbolic set for f. Then there is $\varepsilon > 0$ such that $d(f^k(x), f^k(y)) > \varepsilon$ for some $k \in \mathbb{Z}$ for $x \in \Lambda$ and $y \in M$ $(x \neq y)$.

If there is no such ε, then $y \in W^s_\varepsilon(x) \cup W^u_\varepsilon(x)$, but this implies $x = y$, a contradiction.

39.8 Spectral decomposition theorem[435]

$\Omega(f)$ has the following structure:

[434] Bowen L3.4.
[435] Bowen Th 3.5

$$\Omega(f) = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_s \quad (\Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j), \tag{39.6}$$

where
(a) $f(\Omega_i) = \Omega_i$ and $f|_{\Omega_i}$ is topologically transitive.
(b) Each Ω_i has the following partition: $\Omega_i = X_{i,1} \vee X_{i,2} \vee \cdots \vee X_{i,n_i}$ with $f(X_{i,k}) = f_{i,k+1(\text{mod } n_i)}$ and $f^{n_i}|_{X_{i,j}}$ is topologically mixing.

39.9 Strategy to show spectral decomposition theorem

Let p be a periodic point. Make

$$X_p = \overline{W^u(p) \cap \Omega}. \tag{39.7}$$

$f(X_p) = X_{f(p)}$. If n is the period of p, $f^n(X_p) = X_p$ and this must be topologically mixing on X_p. We collect X_p and its images $f^k(X_p)$ to make Ω_1.

Repeat this until this procedure exhaust Ω to realize (39.6).

39.10 Shadowing = tracing

We say the point sequence $\boldsymbol{x} = \{x_i\}$ is β-shadowed by a trajectory starting from x (by x, for short), if

$$d(f^n x, x_n) \leq \beta \tag{39.8}$$

for $i \in [a,b]$ $(a, b \in \mathbb{Z})$.

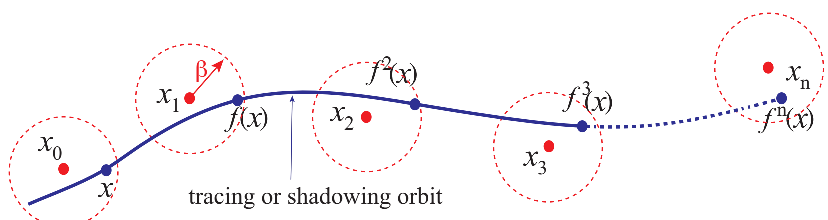

Figure 39.7: β-shadowing

39.11 α-pseudo orbit

$\boldsymbol{x}$ is called an α-pseudo orbit, if

$$d(fx_n, x_{n+1}) < \alpha. \tag{39.9}$$

39.12 Shadowing of pseudo orbits

For any $\beta > 0$, there is a $\alpha > 0$ such that any α-pseudo orbit $\boldsymbol{x}$ is β-shadowed by some point x.

If we choose α sufficiently small, then any α-pseudo orbit $\boldsymbol{y}$ satisfies

$$d(f^n y_0, y_n) < \delta/2, \tag{39.10}$$

where δ is the same δ as in **39.6**.

39.13 Strategy to show shadowability of pseudo orbits

(1) Note first that for a finite time span N an α-pseudo orbit $\boldsymbol{y} = \{y_i\}_{i=0}^{M}$, if α is sufficiently small, can satisfy, for all $i \in [0, M]$,

$$d(f^i y_0, y_i) < \delta/2. \tag{39.11}$$

(2) Let us make an α-pseudo orbit $\boldsymbol{x} = \{x_i\}_{i=0}^{rM}$, connecting such $\boldsymbol{y}$.
(3) The shadowing orbit $\boldsymbol{x}'$ is chosen as follows:
Starting from $x_0 = x_0'$, we recursively apply **39.6** as follows

$$x'_{(k+1)M} = [x_{(k+1)M}, f^M(x'_k)]. \tag{39.12}$$

See Fig. 39.8.

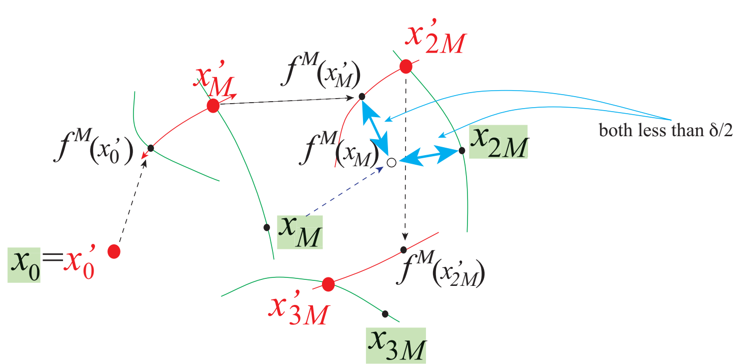

Figure 39.8: Constructing a orbit shadowing the α-pseudo orbit (pale green)

This way, we can find x_{rM}. The basic idea is to find the point we should aim at. That is x_{rM}.

Now, choose the initial condition such that $x = f^{-rM}(x_{rM})$, that is, from there we reach x_{rM}. Although we must carefully show that this orbit indeed δ-shadow th α-pseudo orbit, intuitively it should be clear by construction.

39.14 Closing α-pseudo periodic orbit

Let $x \in \Omega$ and $d(f^n(x), x) < \alpha$. For any $\beta > 0$ we can choose α so that there is a β-shadowing periodic orbit for this not-closed orbit such that $f^n(x') = x'$.

Concatenating $\{x, f(x), \cdots, f^{n-1}(x)\}$, we can make an infinite α-pseudo orbit which can be β-shadowed by $\boldsymbol{y}$ for any $\beta >$. Then,

$$\begin{aligned} d[f^i(y), f^i(f^n(y))] \;\le\; & d[f^i(y), f^i(x)] + d[f^i(x), f^i(f^n(y))] \\ & \le d[f^i(y), f^i(x)] + d[f^i(f^n(x)), f^i(f^n(y))] + d[f^i(x), f^i(f^n(x))]. \end{aligned} \tag{39.13}$$

The first two terms are less than β due to shadowing, and the last term is zero by the periodic concatenation. Thus, for any $\beta > 0$ $d[f^i(y), f^i(f^n(y))] < 2\beta$ for any $i \in \mathbb{Z}$. However, since $x, y \in \Omega$ and since Ω is expansive, this means $y = f^n(y)$ (due to the contraposition of **39.7**).

39.15 Rectangle, proper rectangle

$R \subset \Omega_s$ is a rectangle, if and only if for $x, y \in R$ $[x, y] \in R$. R is called a proper rectangle if $R = [R^\circ]$ (i.e., R is identical to the closure of its open kernel).

39.16 Rectangles are 'registered to' stable and unstable manifolds

Let $\partial^s R$ (resp., $\partial^u R$) be the boundary of rectangle R parallel to W^s (resp., W^u). Then,

$$\partial R = \partial^s R \cup \partial^u R. \tag{39.14}$$

More precisely,

$$\partial^s R \;=\; \{x \in R, x \notin \text{ int } W^u(x, R) \equiv W^u_\varepsilon(x) \cap R\}, \tag{39.15}$$

$$\partial^u R \;=\; \{x \in R, x \notin \text{ int } W^s(x, R) \equiv W^s_\varepsilon(x) \cap R\}. \tag{39.16}$$

39.17 Markov partition

A partition consisting of proper rectangles $\mathcal{R} = \{R_1, \cdots, R_m\}$ satisfying the following conditions is called a Markov partition:
(a) $\text{int } R_i \cap \text{ int } R_j = \emptyset$ for $i \neq j$,
(b) If $x \in \text{ int } R_i$ and $f(x) \in \text{ int } R_j$,

$$fW^u(x, R_i) \supset W^u(fx, R_j), \tag{39.17}$$
$$fW^s(x, R_i) \subset W^s(fx, R_j). \tag{39.18}$$

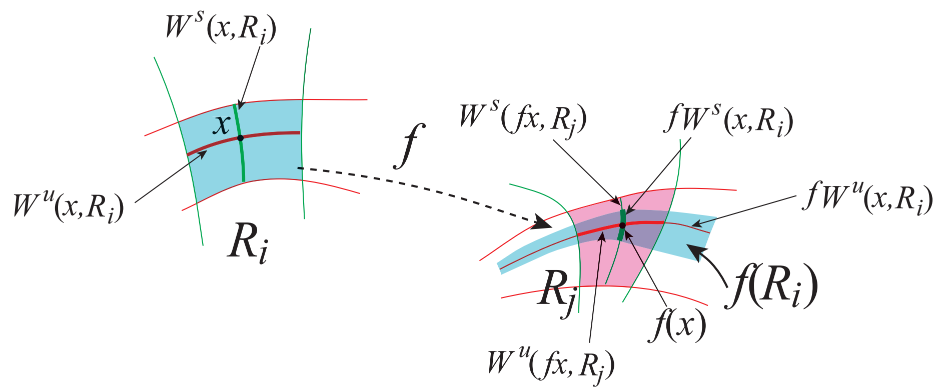

Figure 39.9: Markov partition

As seen later actually we may demand

$$f^{-1}(\partial^u R_i) \subset \partial^u R_k, \tag{39.19}$$
$$f(\partial^s R_i) \subset \partial^s R_l \tag{39.20}$$

for some k and l.

39.18 Axiom A system has Markov partition

Let Ω_s be a basic set for an Axiom A diffeomorphism f. Then, Ω_s has Markov partitions $\mathcal{R}$ of arbitrarily small diameter.

An outline of the logic to demonstrate this key theorem is as follows:
(1) Cover Ω_s with a net whose mesh size is no more than γ, such that for any $x, y \in \Omega_s$ if $d(x, y) < \gamma$, $d(fx, fy) < \alpha/2$, where α is the α appearing in the pseudo orbits. Let $P = \{p_1, \cdots, p_n\}$ be the totality of the net vertices just constructed.

(2) Make the set $\Sigma(P)$ consisting of all the α-pseudo orbits using the points in P. Take $p_s \in P$ and collect all the α-pseudo orbits $\boldsymbol{q}$ starting from p_s (do not forget that we go to the negative time direction as well; the pseudo orbits are defined for all $j \in \mathbb{Z}$). For each such pseudo orbit we can choose a unique β-shadowing orbit: for any $\boldsymbol{q}$ there is a unique point $\theta(\boldsymbol{q})$ that β-shadows it. This is intuitively clear due to the expansivity of f.
(3) Let T_s be the totality of $\theta(\boldsymbol{q})$ with $q_0 = p_s \in P$. This is a rectangle (**39.15**).
(4) T_i and T_j may overlap, so choosing smaller γ to remove the overlaps. Thus obtained T_ss make $\mathcal{R}$.
(5) Finally, we demonstrate (b).

Let us try to understand why T_s's can make a nice Markov partition.

39.19 How to determine T_s

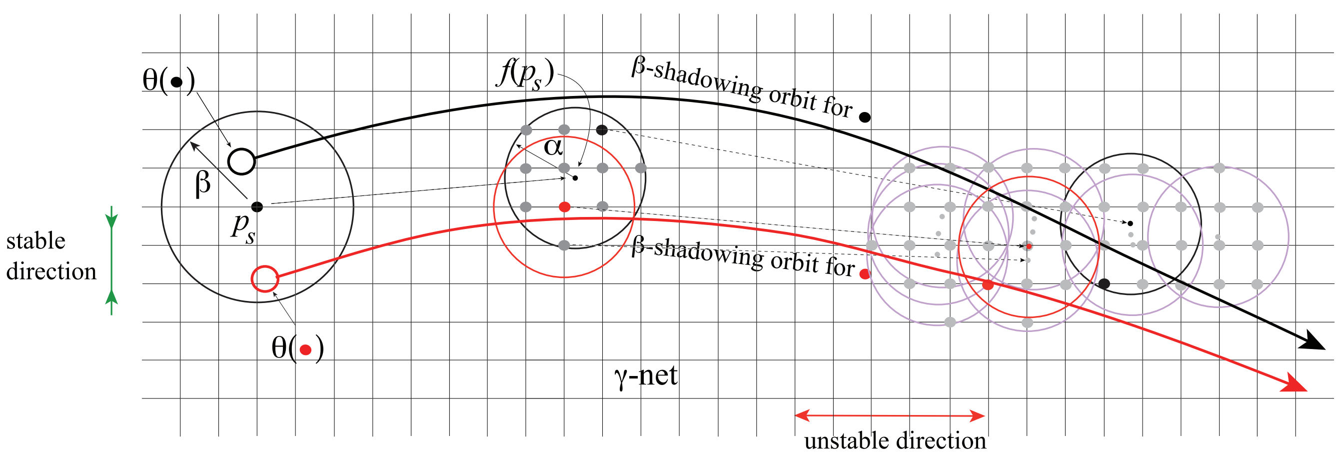

Figure 39.10: Construction of T_s. P consists of all the lattice points. Larger filled dots denote pseudo orbits; continuous curves denote shadowing curves; Open big dots give θ(pseudo orbits). Dotted lines denote f.

The illustration here assumes the forward time evolution, but as noted explicitly in **39.18** we must also consider the backward time evolution. The explanation below is for forward time evolution.

(1) Construct pseudo orbits. As may be guessed from Fig. 39.10 more and more points in P that can make pseudo orbits spread in the unstable direction (expanding direction) as gray dots indicate.

(2) The β-shadowing orbits for these pseudo orbits make a bundle whose width in the stable direction is basically determined by α; without this 'error' the width converges to zero. In the unstable direction the width increases exponentially. However, if the bundle is translated to its initial points (that is, in terms of $\theta(\boldsymbol{q})$), since backward in time there is a severe contraction along the unstable direction, the spread of $\theta(\boldsymbol{q})$ is again determined by α; again without this 'error' the width converges to zero.

39.20 Boundary of T_s

Let us determine the boundaries transversal to the unstable direction. To do so, we evolve the system backward in time. A caricature of what happens is in Fig. 39.11.

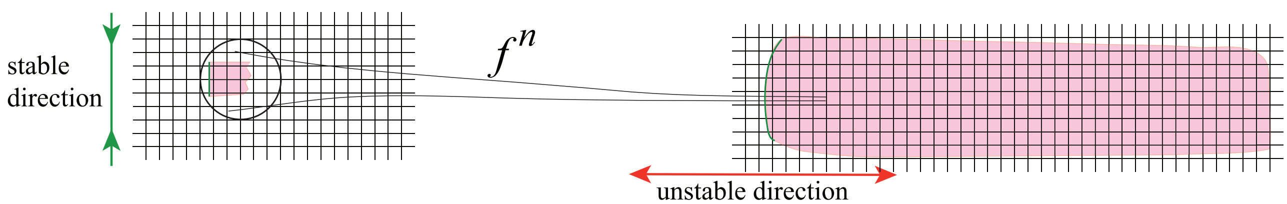

Figure 39.11: The green curve mapped backward by f^n $n \gg 1$ is (almost parallel to the stable manifold.

Imagine you come backward from the right situation to the initial conditions (i.e., the image of θ). There is a tremendous contraction along the red arrow and extremely expanded along the gree arrow, so however curvy the boundary transversal to the unstable direction is, as illustrated by the green line, the boundary becomes parallel to the stable mfd.

You can apply a parallel argument for f^{-n} to conclude that the to and the bottom boundaries are parallel to the unstable manifolds.

Therefore, T_s is bounded by stable and unstable manifolds and obviously it is a proper rectangle in the sense of **39.15**.

39.21 Construction of Markov partition

We have constructed T_ss, but they may have overlap.

As illustrated in Fig. 39.12, repartitioning the obtained T_ss into a set of smaller

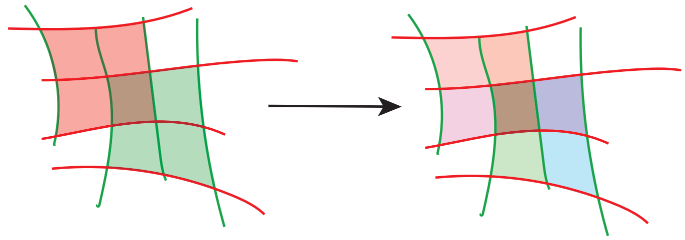

Figure 39.12: Constructing Markov components. Left: constructed T_ss. If there is an overlap we refine partition along the stable and unstable manifolds to mak smaller 'squares' with different colors.

rectangles so that there is no overlap among them, resultant refinement of the partition is a Markov partition. As can be seen from the construction we can make as fine Markov partition as we wish.

39.22 Markov subshift based on Markov partition

Let $\mathcal{R} = \{R_1, \cdots, R_n\}$ be a Markov partition. Basic properties of a Markov partition we use are:

(1) Let $\partial^s \mathcal{R} = \cup i \partial^s R_i$ and $\partial^u \mathcal{R} = \cup i \partial^u R_i$. Then

$$f(\partial^s \mathcal{R}) \subset \partial^s \mathcal{R}, \;\; f^{-1}(\partial^u \mathcal{R}) \subset \partial^u \mathcal{R}. \tag{39.21}$$

See Fig. 39.13.

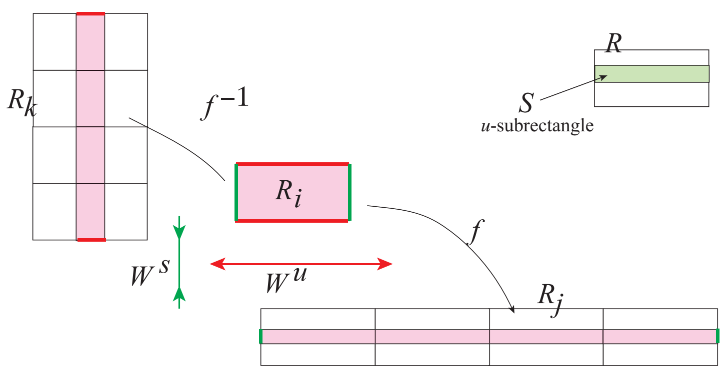

Figure 39.13: (1) illustrated: Boundaries are mapped onto boundaries; u-subrectangle is also illustrated for (2)

(2) A u-subrectangle S of $R \in \mathcal{R}$ is defined as a nonempty subset of R and $W^u(y, S) = W^u(y, R)$ (see Fig. 39.13).

If you can go from R_i to R_j and $S \subset R_i$ is a u-subrectangle, then $f(u) \cap R_j$ is a u-subrectangle of R_j.

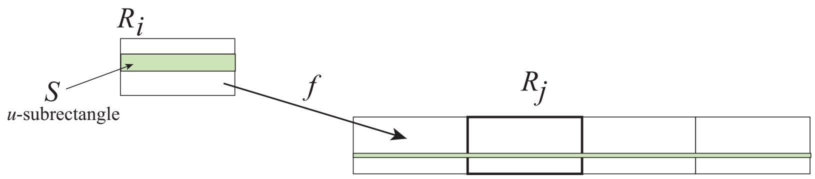

Figure 39.14: (2) illustrated

(3) The coding due to $\mathcal{R}$ is 'almost' one to one. More, precisely, (i) For any $a \in \Sigma_A$ $\pi(a) = \cap_{i \in \mathbb{Z}} f^{-j} R_j$ is a map from Σ_A to Ω_s which is continuous and onto.
(ii) $\pi \circ \sigma = f \circ \pi$ It is one to one on $Y = \Omega_s \setminus \cap_{j \in \mathbb{Z}} f^j(\partial^s \mathcal{R} \cap \partial^u \mathcal{R})$.
This should be clear from Fig. 39.15:

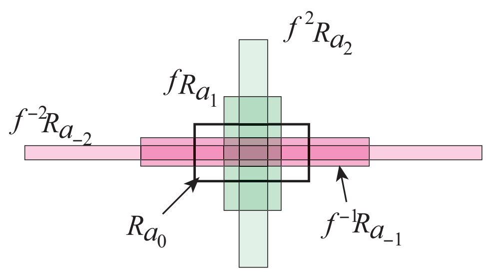

Figure 39.15: Markov coding rule: the points in $f^{-n} R_{a_n} \cap R_{a_0}$ have the code sequence $a_0 \cdots a_n$.