38 Peixoto's theorem

We consider compact 2-differentiable mfd M. We introduce a C^1-topology in $\mathcal{X}(M)$. As you read from the Wikipedia article of Peixoto[413] Peixoto's theorem was a springboard for the study of dynamical systems, Smale was interested in Peixoto's work, and defined the Morse-Smale system. Then to cover more generic dynamics, he introduced the horseshoe dynamical system and then the Axiom A system.

In this section, we taste the original proof of Peixoto's theorem.

38.1 ε-homeomorphism

A homeomorphism that does not move points more than ε is called an ε-homeomorphism.

[413]Usually, M M Peixoto, "Structural stability on two-dimensional manifolds" Topology, 1 101 (1962) is cited, but a large chunk of the proof is in M C Peixoto and M M Peixoto, "Structural stability in the plane with enlarged boundary conditions" Ann Acad Bras Sci 31 135 (1959) [MCP (1921-1960, the first Brazilian woman to receive a doctorate in mathematics) is his wife]. Read Wikipedia M M Peixoto (1920-):
"Once, while talking with his mentor, Solomon Lefschetz, Mauricio Peixoto commented that no one cared about structural stability of dynamical systems and that was the main problem in working with it. But to Peixoto's surprise Lefschetz's answer was no less than "No Mauricio, this is no trouble, this is your luck. Try to work as hard and as fast as you can on this subject because the day will come when you will not understand a single word of what they will be saying about structural stability; this happened to me in topology." Lefschetz's support was very important to Peixoto at the time. In 1957, Peixoto went to research the subject with Lefschetz at the Princeton University, where he spent uncountable hours talking to the Russian professor about Mathematics and other subjects. Despite of the great age difference (Peixoto was 36 years old and Lefschetz 73), they became good friends.

With Lefschetz incentive, Peixoto wrote his first paper on structural stability, that would be later published on the Annals of Mathematics, of which Lefschetz was editor. In 1958, they went to the International Mathematical Congress, in Edinburgh, Scotland, where Lefschetz introduced Peixoto to the Russian mathematician Lev Pontryagin, whose work on dynamical systems was used by Peixoto as a basis for his studies. Pontryagin, though, showed no interest whatsoever in Peixoto's work.

Back to Princeton, Peixoto met Steve Smale, the mathematician that would later become a reference in dynamical systems. Smale was interested in Peixoto's work and realized he could extend his own based on it. Their contact intensified and, when Peixoto came back to Brazil, the American mathematician spent six months at the Instituto Nacional de Matemática Pura e Aplicada (Institute of Pure and Applied Mathematics or IMPA) at Rio de Janeiro. Through Smale, Peixoto would meet the French mathematician René Thom, who would help Peixoto to formulate his theorem, that was finalized during Thom's visit to IMPA."

38.2 Structural stable vector field

$X \in \mathcal{X}(M)$ is structurally stable, if there is a nbh U of X such that any $T \in U$ is homeomorphic to X.

The original definition by Pontryagin and Andronov required ε-homeomorphy instead of the simple homeomorphy. Peixoto demonstrated that homeomorphy implies ε-homeomorphy at least for $\mathcal{X}(M)$ with M being 2-mfd.

The 'smallness of perturbation is usually in C^1-topology, since C^0-small perturbations are "too strong and may destroy any singularity or closed orbit."

38.3 Peixoto's theorem

The *Topology* paper contains two theorems:

THEOREM 1. In order that the vector field X (at least C^1) be structurally stable on M it is necessary and sufficient that the following conditions be satisfied:
(1) there is only a finite number of singularities, all hyperbolic;
(2) the α and ω-limit sets of every trajectory can only be singularities or closed orbits;
(3) no trajectory connects saddle points;
(4) there is only a finite number of closed orbits, all hyperbolic.

THEOREM 2. The set Σ of all structurally stable systems is open and dense in $\mathcal{X}(M)$.

Here, the openness can be shown in any dimension, so the 2D specialty is the denseness.

38.4 How about higher dimensions?

Does something like **38.3** hold for $d \geq 3$? The Morse-Smale system **38.5** was introduced to consider this problem.

Theorem[414] Morse-Smale systems are dense in Diff(M) for any d.

Theorem[415] For Diff(M) for compact M the Morse-Smale systems is structurally stable.

However, there are structurally stable non-MS systems, the converse is not true.

Incidentally,

Theorem[416] In Diff$^r(M)$ structurally stable systems are C^0-dense.

[414] Palis, Topology 8 385 (1969).
[415] Palis-Smale
[416] Shub

38.5 Morse-Smale systems[417]

A dynamical system is called a Morse-Smale system, if
(MS1) Ω is finite (so Ω consists of periodic orbits).[418]
(MS2) Periodic points are all hyperbolic.
(MS3) If $p, q \in \Omega$, then $W^s(p) \pitchfork W^u(q)$.[419]

Gradient dynamical systems are Morse-Smale.[420]

With this terminology
Peixoto's theorem A compact 2-mfd C^1-vector field is structural stable iff it is Morse-Smale.

38.6 Outline of the proof of 38.3

$\Leftarrow$
In this proof actually an ε-homeomorphism is constructed to relate perturbed systems. Thus, Theorem 1 implies that structural stability implies ε-structural stability.

To show vector fields allowing the flow satisfying (1)-(4) are structurally stable, we must construct homeomorphisms between the original and the perturbed fields. This is rather tedious and is shown in the AABS paper quoted above.

$\Rightarrow$
Starting from any vector field X, with a series of approximation lemmas it is shown that X is approximated by Y (i.e., there is a vector field Y in any nbh of X) that satisfies (1)-(4).

Thus, if X is structurally stable, then it must be homeomorphic to its perturbed version, but a perturbed version must satisfy (1)-(4), so X itself must have satisfied (1)-(4), because these properties are homeomorphism invariant.

Theorem 2 is shown almost simultaneously: Structural stable vector fields make an open set (in any dimension, actually), but the proof of $\Rightarrow$ implies such vector fields are dense (exists in any neighborhood of any vector field on a compact 2-manifold).

---

[417] M Shub: http://www.scholarpedia.org/article/Morse-Smale_systems.
[418] Thus, we may say $\cup_j W^s(P_j) = M$, $\cup_j W^u(P_j) = M$.
[419] Because of (MS1) transversality in this case means: no saddle connection nor homoclinic tangency. That is, $W^s(p) \pitchfork W^u(q)$ means they are not in contact except at separatrices.
[420] See also K. R. Meyer Energy Functions for Morse Smale Systems, Am J Math 90 1031 (1968).

38.7 Outline of the proof of $\Leftarrow$[421]

To explain this several definitions must be introduced:

Definition. A region on which the vector field is homeomorphic to one of the following (i)-(iii) in Fig. 38.1 is called 'parallel region.' Note that parallel regions are arcwisely connected.

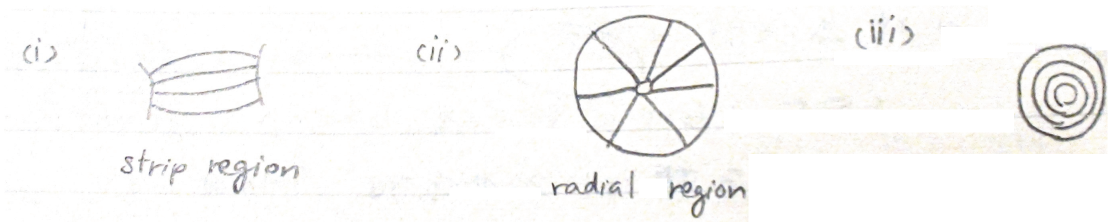

Figure 38.1: Parallel regions

Definition. A trajectory is called an ordinary trajectory, if it is in (i) the strip region. Other trajectories are called separatrices; limit cycles, homo and heteroclinic orbits, etc., are the examples.

Definition. A region $M \setminus \{\text{separatrices}\}$ is called the canonical region. Perhaps it is better to define separatrix set as the complement of the totality of parallel regions in M.

(i) Canonical regions are classified into five types (I)-(V).

(ii) On each canonical region, flows are structurally stable. This is shown by constructing homeomorphisms.

(iii) From these homeos a homeo for M is constructed (which is actually ε-homeo).

38.8 There are five types of canonical regions

Trajectories in a single canonical region R share α and ω limits (Fig. 38.2). Note that ∂R consists of separatrices, so when a band of parallel trajectories is extended none of the trajectories inside the band cannot reach saddles. Therefore, all the members of the bands must share the same destination.

Figure 38.2: Connecting two trajectories on R. The connecting curve σ must always be in R, so it is singly connected.

Thus, R has one α and one ω limit set on its boundary. Each boundary

[421] In MCP+MMP

connecting these limit points cannot have more than one saddle because of **38.3**(3), so we have only the following 5 types of Rs (Fig. 38.3):

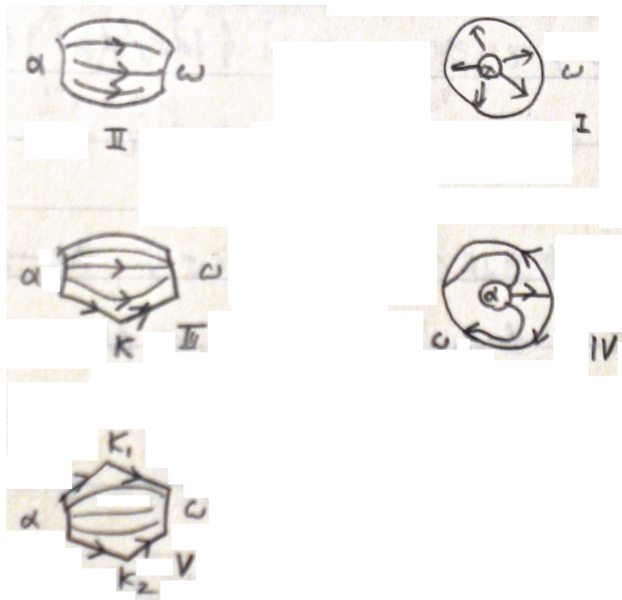

Figure 38.3: Five types of canonical regions

Their stability against perturbation is obvious.

38.9 Structural stability of separatrix set

Let X (separatrix set Λ) and $\tilde{X}$ (separatrix set $\tilde{\Lambda}$) are vector fields within δ in the C^1-norm. Then, Λ and $\tilde{\Lambda}$ are one-to-one correspondent:
(i) Corresponding separatrices are of the same type (homeomorphic),
(ii) If a subset of Λ is the boundary of a canonical region R, then for the corresponding canonical region $\tilde{R}$ its boundary is a homeomorphic subset of $\tilde{\Lambda}$.

This follows from **38.3**(1)-(4) and the usual discussion of the stability of hyperbolic structures (+, if you wish, degree-theoretical arguments).

38.10 Strategy to prove the structural stability of *X*

To show the structural stability we must construct a homeomorphism between X and $\tilde{X}$. The strategy is as follows. Remove small discs D_i containing fixed points and tubular neighborhoods S_i of limit cycles from M to make N.
(i) For each canonical region R on $R^* = R \cap N$ a homeomorphism is constructed.
(ii) Homeomorphisms are constructed on D_i and S_i.

(iii) Connect all these homeo to make a homeo for X.

Notice that both X and $\tilde{X}$ live on the same manifold M and All these local sets, Rs, Ds and Ss for X are perturbed (deformed a bit) to become the counterparts of $\tilde{X}$. Since D and S are built around hyperbolic limit sets, they are stable under perturbation and, e.g., for a D, there is its counterpart $\tilde{D}$. Thus, you can imagine that the structures due to X and their perturbed counterparts are overlapping on M; deformations and displacements are small.

Therefore, first, local homeomorphisms are constructed on these local sets. This is **38.11** and **38.12**.

38.11 Construction of homeo on canonical regions

We wish to show (i) in **38.10**.

For type V region R^* (Fig. 38.3), we wish to do the following (Fig. 38.4). We make a ε-homeomorphism for R^* and $\tilde{R}^*$ by constructing φ and φ' to a square.[422] The homeo we need is just $\varphi \circ \varphi'^{-1}$.

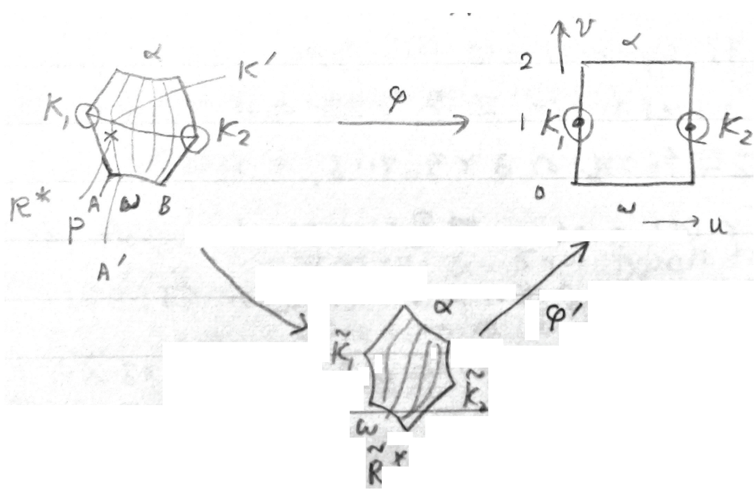

Figure 38.4: Construction of homeo on type V region R^*

In this case, the smooth parallel portion has no problem, since X and $\tilde{X}$ are close. The only (slightly) unclear situations are around the saddle points K_1 and K_2. This is clear if we note that if we take sufficiently small disk D_i around K_i, then the lengths L of the trajectories of $\tilde{X}$ in D_i can be as small as we

[422] Precisely speaking, between R and the square, homeo can be constructed, but not ε-homeo. Thus, we should say that we use the square to construct homeos, and then we must adjust them so that $\varphi \circ \varphi'^{-1}$ is an ε-homeo.

wish. Therefore, displacement of the trajectories due to perturbation must be smaller than the sum of $D - \tilde{D}$ displacement + L, so we can make this as small as we wish.

Types IV and III are considered as 'degenerate' version of type V as illustrated in Fig. 38.5.[423]

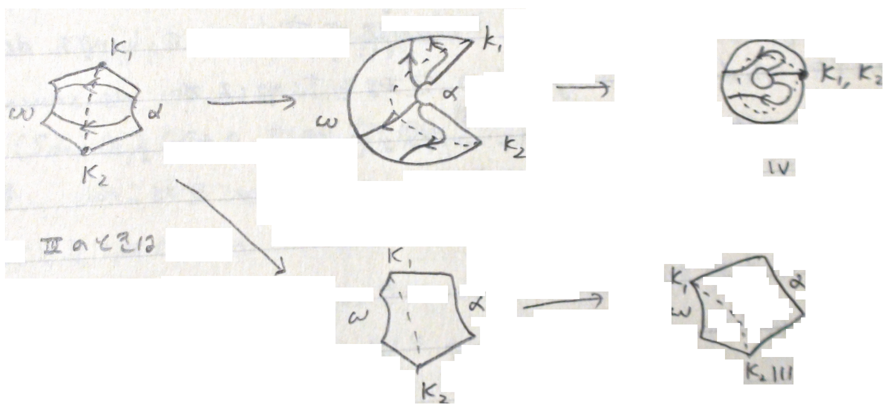

Figure 38.5: Construction of homeo on type IV region and III (bottom) is reduced to that of type V.

Type II should be understood as a portion of V.

For Type I, instead of a square in the type V case, we make maps to a circular annulus.

All these maps are ε-homeo.

38.12 Construction of homeo on nbh of separatrices

Since saddle points are not in the nonwandering set of the system in our case, we have only to consider fixed points for D_i. The situation is exactly the case of Type I.

For limit cycles we can introduce 'polar coordinates' specified by the crossing position on σ and the distance along the curve from σ in the tubular nbh as illustrated in Fig. 38.6. Using such coordinates, we can construct a homeo.

38.13 Completion of homeomorphism

We construct a global homeo by patching the homeos on R^*, D_i and S_i. To do so we must glue these along the boundaries, so we need appropriate local deformations. These deformations can be homeos, so we can adjust all the homeos

[423]If you wish to be precise, for IV you introduce a coordinate system with a periodic boundary condition along the dotted curve direction, and glue side edges (containing Ks) continuously.

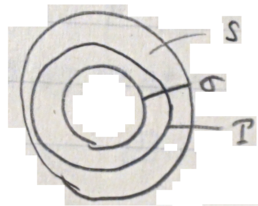

Figure 38.6: Coordinate system near a limit cycle

with these local deformation homeos and patch them continuously over M to construct a global homeo.

Historically, in this way, Peixoto's theorem restricted on a disk was proved first.[424]

38.14 Outline of the proof of ⇒

As is noted in **38.6**, ⇒ is proved by a chain of approximation lemmas.

Start from any X.

(A) X may be approximated by (= X has in its sufficiently small nbh) Y_1 that satisfies **38.3**(1), i.e., with finitely many hyperbolic fixed points. Note that we do not pay any attention to other members of the non-wandering set Ω of X. Thus, te resultant Y_1 may have a 'horrible' invariant set μ.[425]

Definition. A closed invariant set whose genuine subset is not a closed invariant set is called a minimal set. A minimal set that is neither a fixed point nor a limit cycle is called a nontrivial minimal set. [Note its closedness.]

(B) Y_1 may be approximated by Y_1' without nontrivial minimal sets. Here, minimal sets are converted to saddle connections and periodic orbits.

(C) Y_1' may be approximated by Y_2 with fixed points (sink, source and saddle), closed orbits and saddle connections as nonwandering sets.

(D) Y_2 may be approximated by Y_3 satisfying **38.3**(1)-(3): We can sever all the saddle connections, but perhaps new periodic trajectories are created. Note that periodic

[424] M M Peixoto, On structural stability, Ann Math 69, 199 (1959).

[425] You might say on 2-mfd not much complicated dynamics is possible. We know chaos requires three dimensions, and putting trajectories in 2-mfd is like putting noodles on a tray. Indeed on T^2 there is no flow with positive KS entropy. However, Denjoy constructed a highly non trivial invariant set whose 'cross section' is related to a Cantor set. If the genus g of 2-mfd increases (g is the number of holes: $g(S^2) = 0$, $g(T^2) = 1$, etc. Adding a handle increases g by one), then M is riddled with holes, so perhaps trajectories could come back through holes and mingle with the staying trajectories in a nontrivial fashion. Thus, it is safe to assume such μ exists in the flow due to X (and so in Y_1).

trajectories need not be limit cycles.
(E) Y_3 may be approximated by Y_4 satisfying **38.3**(1)-(4). This step uses Whitney's embedding theorem and Weierstrass' polynomial approximation of continuous functions.

38.15 Any vector field may be approximated by a vector field with finitely many hyperbolic fixed points

We can make all the singularities as simple by arbitrarily small perturbations. Simple singularities are isolated (and since M is compact, there are only finitely many of them), so we may handle them separately. A nonhyperbolic fixed point can be converted by sufficiently small deformation to a hyperbolic fixed point. Thus, X may be approximated by a vector field Y_1 satisfying **38.3**(1).

38.16 Perturbatively nontrivial minimal sets can be killed: outline

Suppose Y_1 has a nontrivial minimal set μ. Since there are only finitely many saddle points, we consider the following two cases (recall that usually dynamical systems are defined on the time range $(-\infty, +\infty)$)
(A) There is no trajectory connecting μ and a saddle.[426]
(B) There is a trajectory connecting μ and a saddle.

For case (A) we can show (**38.17**) that for any $p \in \mu$ there is a coordinate nbh such that all the trajectories leaving it return to it,[427] and the trajectory going through p can be converted into a periodic orbit. For case (B) such a trajectory can be converted into a saddle connection (**38.18**).

After these perturbations, if nontrivial minimal sets still remain, we repeat the procedure. It will be shown (**38.19**) that only finitely many repetition is required, so Y_1 can be perturbed with an arbitrarily small perturbation into Y_1' without nontrivial minimal set.

38.17 Case A: no saddle connection

For (A) in **38.16** Y_1 has a closed orbit passing through μ. This closed orbit does not bound any cell (see Fig. 38.7) [Closing lemma].

[426]As we will see, this happens only if $M = T^2$ pf a Klein bottle, so the system actually has no singularity at all.
[427]Since μ is non trivial, there is a trajectory returning to any nbh of p infinite times.

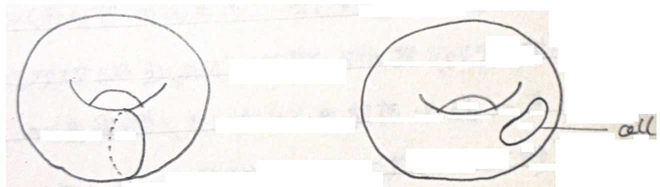

Figure 38.7: Two kinds of closed orbits on 2-mfd.

To show this take a local coordinate square 'abcd' around a point P in μ (Fig. 38.8).

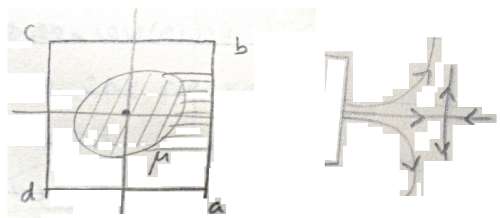

Figure 38.8: Local coordinates for μ; Right illustrates the impossible case for (A)

Since μ is minimal, at least one trajectory leaving from edge ab must come back to μ through edge cd.[428] The boundary trajectory between the trajectories coming back to μ and those not coming back must go to a saddle point (See Fig. 38.8 Right).[429] However, our assumption is that such μ-saddle connection does not exist. Therefore, all the trajectories leaving μ must come back to μ (infinitely many times to different points on edge cd), since μ is nontrivial.

Also these orbits cannot encircle a cell (see Fig. 38.7 Right); μ is a closed set, so its boundary cannot be a cycle; otherwise, since μ is a closed set, this boundary cycle belongs to μ, destroying its minimal nature.

Since trajectories do not cross each other, the ordering of the trajectories is preserved (or reversed). Consider the perturbation illustrated in Fig. 38.9. We can create a limit cycle going through P.

Needless to say, this may not totally erase μ; some potion(s) may survive as nontrivial minimal sets. Thus, we need to repeat the procedure.

[428]The local square around P is so chosen that the trajectories foliate the square. This is possible, because there must be a trajectory coming back to any nbh of P infinitely many times, and M is a 2-mfd, so near P the returning portion of the trajectory must be almost parallel.

[429]A formal proof is lengthy. Lemma 3 in the original.

Figure 38.9: Closing lemma

38.18 Case B: when μ-saddle connection exists

For (B) in **38.16** the μ-saddle point connection can be converted to a saddle-saddle connection.

Actually, we have only the case of Fig. 38.10 Left. Suppose there is a μ-saddle connection in the future direction (the right-side edge situation in Fig. 38.10 Left). Then, above and below the connection, the trajectories have different fate. However, these trajectories come back to this coordinate square R. Then, reverse the time, we must say there must be a saddle on the left side just as illustrated in the figure. This saddle γ_1 cannot be the same saddle as γ_2., because if so the trajectories must cross the unstable manifold of γ_1.

We may assume that $\omega(\gamma_1) = \alpha(\gamma_2) = p$. If not, we have a situation like Fig. 38.10 Right, and the curly bracketed portion may be subjected to a surgery as performed in case (A).

If this surgery fails to connect γ_1 and γ_2, then the boundary between the curly bracketed portion and its outside must go to a saddle point as illustrated in Fig. 38.8 Right. Thus, we get the situation as Fig. 38.10 Left. When γ_1 and γ_2 are sufficiently close, we can short-circuit them with a small perturbation to create a saddle connection.

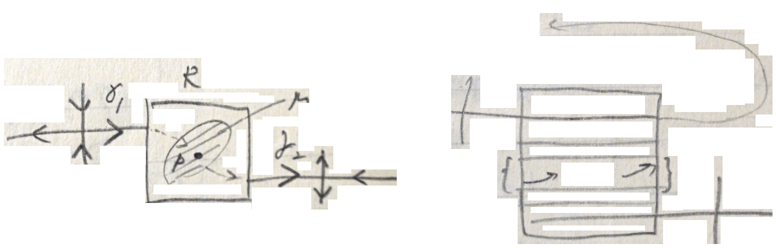

Figure 38.10: Saddle connection case

38.19 Finite repetition of surgeries totally kill nontrivial minimal sets

Since the total number of saddle point in Y_1 is finite, so the number of surgeries in (B) must be finite.

If (A) does not end with finitely many surgeries, we have very many closed orbits. Since M is a 2-mfd, with a finitely many cuts M can be expanded into a 2-disk.[430] Since trajectories cannot cross closed orbits, all the orbits must be on this disk. Suppose we still have some μ on this disk, since all the saddle points have been used up to make saddle connections, it produces an orbit encircling a cell, an impossibility as already discussed in **38.17**.

Thus, the procedure explained above ends with finite repetitions and an arbitrarily small perturbation can convert X into a vector field with finitely many hyperbolic fixed points without any nontrivial minimal set (but with too many periodic trajectories and possibly bands of periodic trajectories (as centers)).

38.20 Almost homoclinic orbits are converted to homoclinic orbits

(C) has almost been demonstrated, but there can be a situation where an almost homoclinic situation occurs for a saddle (Fig. 38.11), because the situation does not produce a nontrivial minimal set.

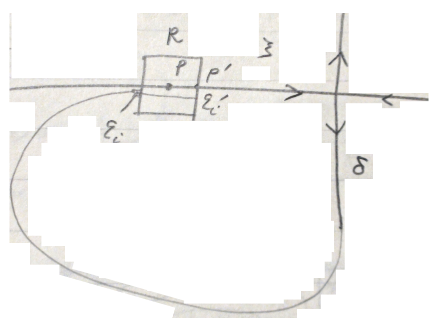

Figure 38.11: With a little perturbation a homoclinic orbit may be created.

If we look at R, since there is trajectory coming back to any nbh of P, so we can convert this into a homoclinic trajectory. Since the number of saddle points is finite, just as discussed in **38.19**, we have only to repeat such a procedure finite times and the perturbation to Y_2 is complete.

38.21 Perturbation can remove all the saddle connections

Graphically, we can exhibit the situation as in Fig. 38.12 [Actually such a diagram is called a graph].

[430] Genus g 2-mfd may be cut open to a disk (or a polygon) with $2g$ cuts, just as T^2 is converted to a square by 2 cuts.

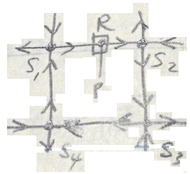

Figure 38.12: Graphic representation of saddle connections

There are two situations:
(1) S_1S_2 is not a part of a large graph.
(2) S_1S_2 is a part of a large graph

For (1) the following perturbation does not produce a saddle connection anew.

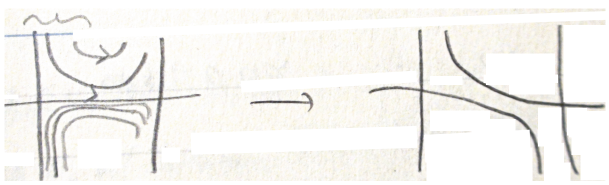

Figure 38.13: Surgery of saddle connections

However, in case (2) a new saddle connection might be formed (Fig. 38.14), but, in this case, if perturbed further n (pushed down further in the figure), the saddle connection would be reconnected to a periodic orbit. Since there are only finitely many saddle points, this can be done by arbitrarily small perturbation, so we may virtually ignore such cases.

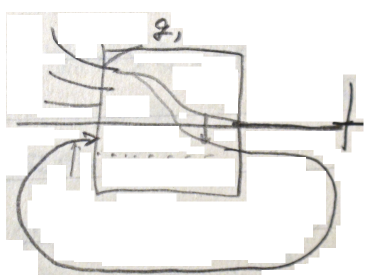

Figure 38.14: Possible emergence of a new saddle connection, but this can be killed by the small perturbation indicated by an arrow

In any case, we can repeat the above procedure finite times. All the saddle connections will be gone. Thus Y_3 has been produced from X. That is, except for the condition for periodic trajectories we are done.

38.22 Periodic orbits are made hyperbolic by perturbation

To make periodic orbits hyperbolic, we apply the perturbation illustrated in Fig. 38.15 to box R:

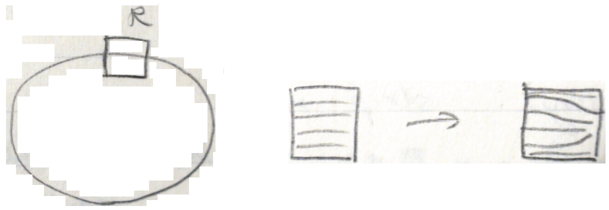

Figure 38.15: Make orbits hyperbolic

Also marginal periodic orbits can be understood as a degenerate version of two hyperbolic orbits. Thus, We can make all the separatrices hyperbolic (the stabilized Y_3). However, still there can be infinitely many closed orbits (like a center).

Peixoto embedded the stabilized Y_3 into $\mathbb{R}^5$ using Whitney's embedding theorem, and then made a polynomial approximation of the vector field (using Weierstrass' theorem). That is, he made an analytic vector field, which has only isolated periodic trajectories and periodic bands. Then, he constructed a vector field homeomorphic to this polynomial approximation on M. The resultant vector field is no more analytic, but the topological features are all preserved. Thus, the vector field satisfies (1) and (3) in **38.3** and isolated periodic trajectories/bands. However, nontrivial minimal sets μ may reappear by the approximation procedure.

If this further satisfies (2), that is, no new μ shows up, then there are two cases:
(A) All the orbits are closed.
(B) Otherwise.

For (A) if M is not a result of gluing a square (i.e., T^2 or Klein's bottle), then we have something like Fig. 38.16.

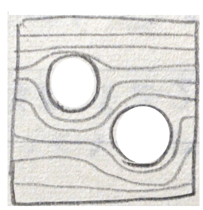

Figure 38.16: Expanded M should not have a hole; the holes mean these peripheries are glued to make a handle.

This implies there are saddle points, contradicting the assumption that there are only closed orbits. If T^2, for example, we can apply 'bunching' as exhibited in Fig. 38.15 to make a limit cycle.

For (B) even if we have a band of closed orbits (no returning orbits), the boundary of the band (black dots in Fig. 38.17) goes to a saddle (or comes from a saddle) just as in Fig. 38.8 Right, so this contradicts **38.3**(3), because Y_3 has already been constructed to satisfy this and polynomial approximation does not alter this.[431]

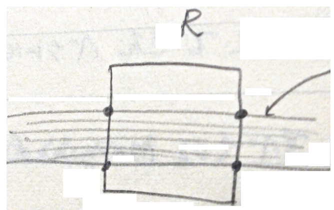

Figure 38.17: Killing the bands

If (2) is not satisfied, that is, there is nontrivial minimal set μ, then repeat the above argument (**38.16**). If, after this repetition, the number of orbits is finite, we are done. If not, we repeat the polynomial approximation, because polynomial fields allow only isolated or band of periodic orbits. Since the polynomial approximation maintains all the simple closed orbits, the newly added simple trajectories during the repetitions are maintained by each polynomial procedure step. However, as discussed in **38.19**, we can repeat the procedure only finitely many times, and μ will not show up eventually. We are done.

[431] need polynomial field check