34 Lecture 34 KS entropy as isomorphism invariant

34.1 Kolmogorov-Sinai entropy as isomorphism invariant

The Kolmogorov-Sinai entropy was originally proposed as an invariant under isomorphism of measure theoretical dynamical systems. Suppose there are two measure theoretical dynamical systems (T, μ, Γ) and (T', μ', Γ'). Let $\phi : \Gamma \to \Gamma'$ be a one-to-one (except for measure zero sets) correspondent and measure-preserving: $\mu = \mu' \circ \phi$ (that is, the measure of any measurable set measured by the measure μ on Γ and measured by the measure μ' on Γ' after mapping from Γ by ϕ are identical).[379] If the following diagram:

$$\begin{array}{ccc} \Gamma & \xrightarrow{T} & \Gamma \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ \Gamma' & \xrightarrow{T'} & \Gamma' \end{array}$$

is commutative (except on measure zero sets of Γ and Γ'), that is, if $T = \phi^{-1} \circ T' \circ \phi$, these two measure-theoretical dynamical systems are said to be isomorphic.

It is almost obvious that the Kolmogorov-Sinai entropies of isomorphic measure-theoretical dynamical systems are identical, because for a partition $\mathcal{A}$ of Γ $H(\mathcal{A}) = H(\phi\mathcal{A})$ and $\phi \circ T = T' \circ \phi$. We say that the Kolmogorov-Sinai entropy is an isomorphism invariant. If two dynamical systems have different Kolmogorov-Sinai entropies, then they cannot be isomorphic. For example, as can be seen from (34.2) $B(1/2, 1/2)$ and $B(1/3, 2/3)$ cannot be isomorphic.

An isomorphism invariant that takes the same value if and only if dynamical systems are isomorphic is called a complete isomorphism invariant. If we find such an invariant, the classification of dynamical systems according to isomorphism is reduced to the computation of the invariant. Is the Kolmogorov-Sinai entropy such an invariant? When Meshalkin[380] demonstrated the isomorphism between $B(1/4, 1/4, 1/4, 1/4)$ and $B(1/2, 1/8, 1/8, 1/8, 1/8)$ (both have the Kolmogorov-Sinai entropy $2 \log 2$), the affirmative answer was expected (also there was a crucial contribution of Sinai: the weak isomorphism theorem for Bernoulli processes). In 1970[381] Ornstein proved that for Bernoulli processes the Kolmogorov-Sinai entropy is a complete invariant. Fur-

[379] Precisely, we must assume not only that $\phi : \Gamma \to \Gamma'$ is one to one as a map between measurable spaces, but also that measurable subsets of Γ are mapped on those of Γ' by ϕ, and vice versa by ϕ^{-1}.

[380] L. D. Meshalkin, "A case of Bernoulli scheme isomorphism," Dokl. Acad. Sci. USSR, **128**(1) 41 (1959).

[381] [1970: Russel died (1872-), Aswan High Dam, Allende became the President of Chile.]

thermore, completeness was proved for any finite mixing Markov chain with finite Kolmogorov-Sinai entropy.[382,383]

34.2 Ornstein-Weiss' theorem

Ornstein and Weiss proved the following theorem:[384]

Theorem 2.7A.1 Not completely predictable systems (= systems with positive Kolmogorov-Sinai entropy) have Bernoulli flows (continuous dynamical systems whose periodically sampled sequences become Bernoulli processes) as their factor dynamical systems.

Here, "to have A as a factor dynamical system" implies that the original dynamical system behaves as dynamical system A if it is reduced to a certain space (more precisely, there is a homomorphism from the system onto A). They simply equate the chaotic system and the system with positive Kolmogorov-Sinai entropy in the quoted review article.

34.3 Bernoulli system

Let $M = \mathbb{Z}_n^{\mathbb{Z}}$, where $\mathbb{Z}_n = \{0, 1, \cdots, n-1\}$. We can define a shift dynamical system (actually it is a full shift). We introduce a measure μ on M through the measures of the cylinder sets as[385]

$$\mu([\omega_{k+1} = \alpha_1, \cdots, \omega_{k+q} = \alpha_q]) = \prod_{m=1}^{q} p_{\alpha_m}, \tag{34.1}$$

where $\alpha_m \in \mathbb{Z}_n$ and $p_1, \cdots, p_n$ are $p_m \in [0,1]$ with $\sum_{m=1}^{n} p_m = 1$ (i.e., p_i is the probability for a letter in the alphabet $\mathbb{Z}_n$.

(σ, M, μ) is a measure-theoretical dynamical system called the Bernoulli system and is denoted as $B(p_1, \cdots, p_n)$.

$B(1/2, 1/2)$ corresponds to the coin-tossing process, and is isomorphic to baker's transformation (see **27.4**). It is also isomorphic to the horseshoe dynamical system restricted on its nontrivial invariant set (see **28.5**).

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[382] I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, *Ergodic Theory*, (Springer, 1982); D. S. Ornstein, *Ergodic Theory, Randomness, and Dynamical Systems* (Yale University Press, 1974).

[383] However, the cases with zero entropy are different; there are numerous non-isomorphic dynamical systems with zero entropy.

[384] D. S. Ornstein and B. Weiss, "Statistical Properties of Chaotic Systems," Bull. Amer. Math. Soc. **24**, 11 (1991), Theorem 1.4.3.

[385] This can actually uniquely specify μ on M according to Kolmogorov's extension theorem.

34.4 Kolmogorov-Sinai entropy of Bernoulli system

The Kolmogorov-Sinai entropy of $B(p_1, \cdots, p_n)$ is given by

$$h = -\sum_{i=1}^{n} p_i \log p_i. \tag{34.2}$$

We can compute the loss of information by σ easily (see **32.1**). It is indeed given by this formula.

34.5 Central statements of theory of Bernoulli processes[386]

Two central statements of the theory are the Sinai theorem and the Ornstein theorem:
(I) [Sinai's weak isomorphism theorem] Every ergodic process with positive entropy h has any Bernoulli shift of entropy smaller than or equal to h as a measure-theoretic factor.[387]
(II) [Ornstein's isomorphism theorem] Bernoulli shifts with equal entropies are isomorphic.

The Ornstein theorem fails for unilateral shifts (but the Sinai theorem holds, which is less known).

[386] T. DOWNAROWICZ and J. SERAFIN, A short proof of the Ornstein theorem, Ergod. Th. & Dynam. Sys. 32 587 (2012). The authors of this note managed to simplify the Burton, Keane and Serafin method. It is the residual Sinai theorem which is central in our method, and the number of applications of the Baire theorem is reduced to one. We continue to invoke the elementary combinatorial 'marriage lemma.' I give up to illustrate this paper.

[387] ⟪**Factor**⟫ A coarse-grained dynamical system is called a factor of the original dynamical system. Formally, (S, ν, N) is a factor of (T, μ, M), if there is a measurable 'homomorphism' $\varphi : M \to N$ such that
(i) $S \circ \varphi = \varphi \circ T$,
(ii) $\nu(A) = T(\varphi^{-1}(A)0$ for measurable set in N