33 Lecture 33. Lyapunov characteristic number

33.1 Lyapunov indices

As seen in examples above and from the Brin-Katok theorem, exponential separation of nearby trajectories may be regarded as a characteristic feature of chaos. Consider two trajectories starting from x and a nearby point $x + \varepsilon v$, where ε is a small positive number and v the directional vector. They tend to separate exponentially as time increases. At time t let us write the separation distance between these two trajectories as $\exp(t\lambda(x, v))$. The exponent $\lambda(x, v)$ is called the Lyapunov exponent (or Lyapunov characteristic exponent; A. M. Lyapunov 1857-1918) for the vector v at $x \in \Gamma$. Its precise definition is

$$\lambda(x, v) = \limsup_{n\to\infty} \frac{1}{n} \log \|D\phi^n(x)v\|, \tag{33.1}$$

where D is differentiation with respect to x[375] and $\| \ \|$ is the norm in the tangent vector space of Γ. The map T defining the dynamical system is written as ϕ to avoid confusion in the present note. At each x $\lambda(x, v)$ as a function of v takes q ($\leq$ the dimension of Γ) distinct values:

$$\lambda^{(1)}(x) > \cdots > \lambda^{(q)}(x). \tag{33.2}$$

Its existence is guaranteed by the following theorem:

33.2 Oseledec's theorem[376]

At each $x \in \Gamma$ the tangent vector space $T_x\Gamma$ ($\simeq \mathbb{R}^n$) of Γ may be decomposed into a direct sum of the form

$$T_x\Gamma = \bigoplus_{i=1}^{q(x)} H_i(x) \tag{33.3}$$

and for $v \in H_j(x)$

$$\limsup_{n\to\infty} \frac{1}{n} \log \|D\phi^n(x)v\| = \lambda_j(x). \tag{33.4}$$

[375] This gives a Jacobi matrix in general.

[376] For a proof, see, for example, A. Katok and B. Hasselblat, *Introduction to the Modern Theory of Dynamical Systems* (Cambridge University Press, 1996) p665. The original theorem is about the general cocycle of a dynamical system and is called Oseledec's multiplicative ergodic theorem, but here it is quoted only in the form directly relevant to our topic.

If the dynamical system is ergodic, this does not depend on x.

33.3 Numerical calculation of Lyapunov spectrum[377]

If we choose an arbitrary vector v to compute (33.1), then almost surely we get λ_1 as can be clear from the illustration Fig. 33.1.

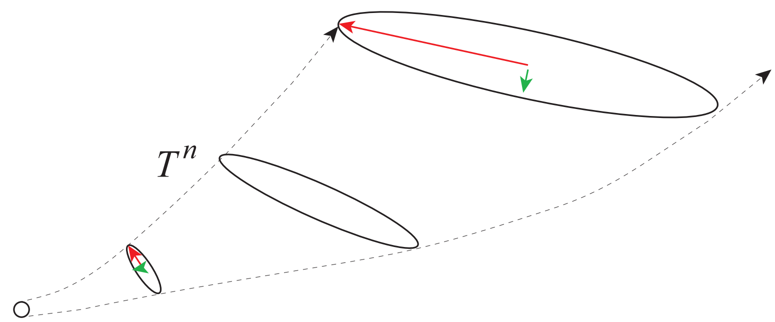

Figure 33.1: Fastest-growing direction wins. In this example, the green direction is also exponentially growing, but the ratio of the lengths of the red and green arrows increase exponentially.

If we could consider the dynamics in the tangent space perpendicular to the fastest-growing direction, then we should obtain the second fastest λ_2. However, the fastest-growing direction rotates as illustrated in 33.1. A natural approach is as follows. Let us assume M to be an n-manifold. Start from a basis $\langle e_1, e_2, \cdots, e_n\rangle$ (of $T_{x_0}M$). For example, the time evolution of the vector looks like the following succession of linear transformations

$$D\phi^n(x_0)e_1 = \phi'(\phi^{n-1}(x_0))\cdots\phi'(\phi(x_0))\phi'(x_0)e_1. \tag{33.5}$$

The expansion rate for this time span for this initial vector is given by

$$\frac{1}{n}\log\|D\phi^n(x_0)e_1\|. \tag{33.6}$$

After several (n) time steps, we may order the length of these vectors, and order them according to the length as $e^{i_1}, \cdots, e_{i_n}$. Then, apply the Gram-Schmidt orthonormalization to make a new ON basis $\langle e_1^{(1)}, e_2^{(1)}, \cdots, e_n^{(1)}\rangle$. Then, repeat the procedure and

[377]The original idea is due to Ippei SHIMADA and Tomomasa NAGASHIMA, "A Numerical Approach to Ergodic Problem of Dissipative Dynamical Systems," Prog. Theor. Phys. 61 1605 (1979). The exposition here is not for actual calculations. For an actual calculation see for example, C. Skokos, The Lyapunov characteristic exponents and their computation, arXiv:0882v2 (Jan 26, (2009).

compute the expansion rate as follows:

$$\frac{1}{N}\left[\frac{1}{n}\log\|D\phi^n(x_0)e_k\| + \frac{1}{n}\log\|D\phi^n(\phi^n(x_0))e_k^{(1)}\| + \cdots + \frac{1}{n}\log\|D\phi^n(\phi^{((N-1)n)}(x_0))e_k^{(N)}\|\right]. \tag{33.7}$$

33.4 Volume expansion rate calculation

In **33.3** each vector direction is calculated separately. Suppose in Fig. 33.1 we calculate the area spanned by the red and green vectors. Its area exponentially grows as $e^{N(\lambda_1+\lambda_2)}$. This means that calculating the volume of the fastest growing k-(rectangular) parallelepiped, we can compute $\lambda_1 + \lambda_2 + \cdots + \lambda_k$ (the sum up to the kth largest LCN).

33.5 Some notable properties of LCN

Suppose a continuous dynamical system has a bounded phase space (i.e., bounded M) and there is no fixed point, then there must be 0 LCN. This is obvious, because the direction tangent to the orbit cannot grow exponentially and its speed is bounded from below.

For a Hamiltonian dynamical system, the phase volume is conserved, so **33.4** implies that the total sum of LCN must vanish.

Moreover, if lam (> 0) is a Lyapunov number, then so is $-\lambda$. This can be shown with the aid of the canonical invariance of

$$\int dq^r dp_r dq^s dp_s \cdots dq^t dp_t. \tag{33.8}$$

33.6 Pesin's equality[378]

For $T \in \mathrm{Diff}^2(M)$ a ergodic measure theoretical dynamical system (T, μ, M) (if any). Then, the following illustration tells us the following equality:

$$h_\mu(T) = \sum_{+} \lambda, \tag{33.9}$$

[378] Ya. B. Pesin, "Characteristic Lyapunov exponents and smooth ergodic theory," Russ. Math Surveys 32Z(4) 55 (1977). Section 5. This proves $h \geq \sum_+ \lambda$; the proof is technical and not enlightening; besides, the inequality $\leq$ uses Mather's paper.

where $\sum_+$ means the sum over all the positive LCN.

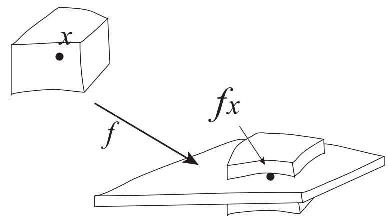

Figure 33.2: Pesin illustrated: the cube edges describe the local unstable and stable directions. Note that these directions are determined by the derivatives of the field. Since the map is C^2, they are smooth so locally expanding directions nicely agree with the original 'blocks' (elements of partitions). Although we may not be allowed to make Markov partitions, but still the partitions are geometrically nice and we can determine the expansion rate in the unstable directions.

33.7 Ruelle's inequality

Generally, if the dynamical system is less smooth, then in Fig. 33.2 nice overlap is not guaranteed, so the expansion direction of the mapped 'cube (the slab in the figure) may no more overlap nicely but only obliquely. Then, the expansion may not as effectively 'dilute' information. Therefore, generally,

$$h_\mu(\phi) \le \sum_+ \lambda \tag{33.10}$$

which is called Ruelle's inequality.[379]

33.8 Kingman's subadditive ergodic theorem[380]

Suppose $X_{m,n}$ $(0 \le m < n)$ satisfy:
(i) $X_{0,n} \le X_{0,m} + X_{m,n}$.
(ii) For each k, $\{X_{nk,(n+1)k}, n \ge 1\}$ is a stationary sequence.
(iii) The distribution of $\{X_{m,m+k}, k \ge 1\}$ does not depend on m.
(iv) $EX_{0,1}^+ < \infty$ and for each n, $EX_{0,n} \ge \gamma_0 n$, where $\gamma_0 > -\infty$.
Then,

[379] D. Ruelle, "An inequality for the entropy of differentiable maps," Bol. Soc. Brasil. Mat. **9**, 83 (1978).

[380] based on R. Durrett, *Probability: Theory and examples* (Wadsworth & Brooks/Cole 1991). This is the best advanced introduction to probability.

(a) $\lim_{n\to\infty} EX_{0,n}/n = \inf_m EX_{0,m}/m \equiv \gamma$.
(b) $X = \lim_{n\to\infty} X_{0,n}/n$ exists a.s. and in L^1, so $EX = \gamma$.
(c) If all the stationary sequences in (ii) are ergodic, then $X = \gamma$ a.s.

(a) is clear from Fekete's inequality.

Remark. If we set $X_{n,m} = f(T^{m+1}) + \cdots f(T^n x)$, then (i)-(iv) hold. The outcome is the usual ergodic theorem, but we use this theorem to prove the subadditive version.

33.9 Estimate of $\limsup X_{0,n}/n$

(i) implies, for $n = km + l$ ($l \in [0, m)$),

$$X_{0,n} \le X_{0,km} + X_{km,n}, \quad X_{0,km} \le X_{0,(k-1)m} + X_{(k-1)m,km}, \tag{33.11}$$

so

$$X_{0,n} \le X_{0,(k-1)m} + X_{(k-1)m,km} + X_{km,n}. \tag{33.12}$$

Since

$$X_{0,(k-1)m} \le X_{0,(k-2)m} + X_{(k-2)m,(k-1)m}, \tag{33.13}$$

etc., repeated use of such inequalities to (33.12) gives

$$X_{0,n} \le X_{0,m} + X_{m,2m} + \cdots + X_{(k-1)m,km} + X_{km,n}. \tag{33.14}$$

Therefore,

$$\frac{X_{0,n}}{n} \le \left(\frac{k}{km+l}\right) \frac{X_{0,m} + X_{m,2m} + \cdots + X_{(k-1)m,km}}{k} + \frac{X_{km,n}}{n}. \tag{33.15}$$

Now, we use the ergodic theorem: there must be a limit

$$\frac{X_{0,m} + X_{m,2m} + \cdots + X_{(k-1)m,km}}{k} \to A_m. \tag{33.16}$$

Thus, assuming nice behaviors of the system,

$$\overline{X} \equiv \limsup_{n\to\infty} \frac{X_{0,n}}{n} \le \frac{A_m}{m}. \tag{33.17}$$

If we may assume the system to be ergodic, then (a) implies

$$\int d\mu\, \overline{X} \le \gamma. \tag{33.18}$$

33.10 Almost sure convergence of $\lim X_{0,n}/n$

Let

$$\underline{X} \equiv \liminf_{n\to\infty} \frac{X_{0,n}}{n} \tag{33.19}$$

We wish to show

$$\int d\mu\, \underline{X} \geq \gamma. \tag{33.20}$$

This with (33.18) implies

$$\int d\mu\, (\underline{X} - \overline{X}) \geq 0. \tag{33.21}$$

That is, $\underline{X} = \overline{X}$ a.s., the a.s. convergence of $X_{0,n}/n$.

Let

$$\underline{X}_m = \liminf_{n\to\infty} \frac{X_{m,m+n}}{n}, \tag{33.22}$$

but this agrees with $\underline{X}$ (ergodicity).