30 Lecture 30 Ergodic theorems

This section is based on I Kubo *Dynamical systems I* (Iwanami 1997) Section 2.2.

30.1 Measure preserving transformation

Let $(M, \mathcal{F}, \mu)$ be a probability space (but we will 'ignore $\mathcal{F}$.' See **29.3**). A map $T : M \to M$ preserves μ if $\mu = \mu \circ T^{-1}$. In this case μ is said to be T-invariant. For any μ-integrable function f

$$\int_M f(Tx)d\mu(x) = \int_M f(x)d\mu(x). \tag{30.1}$$

This can be shown easily if we note[343] for any $A \subset M$[344]

$$\int_M \chi_A(Tx)d\mu(x) = \int_M \chi_A(x)d\mu(x), \tag{30.2}$$

because

$$\int_M \chi_A(Tx)d\mu(x) = \int_M \chi_A(y)d\mu(T^{-1}y) = \int_M \chi_A(y)d\mu(y). \tag{30.3}$$

30.2 Poincare's recurrence theorem

Consider a measure-theoretical dynamical system (T, μ, M).

Theorem. Let $B \subset M$ with $\mu(B) > 0$. Then, μ-almost all $x \in B$ (i.e., μ-$\tilde{\forall} x \in B$) $T^k x \in B$ for infinitely many $k \in \mathbb{N}$.

[Demo] We show that the totality of the points $x \in B$ that do not return to B infinitely many times is μ-measure zero. Since x does not return to B infinitely many times, there must be $n \in \mathbb{N}$ such that $T^k x \in B^c$ (i.e., $x \in T^{-k}B^c$) for $\forall k > n$.

Let $E_n = \cap_{k \geq n} T^{-k}B^c$ (the totality of points that stays in B_c after time n; i.e., points that never return to B after time $n-1$). $T^{-1}E_n$ consists of points going into E_n by one time step, that is, points that do not return after time $n-1$ + one. Thus, $T^{-1}E_n = E_{n+1} \supset E_n$. $E = \cup_{n=0}^{\infty} E_n$ is the totality of the points never returns to B

[343] Integrals are all Lebesgue integrals, so f is considered as a limit of simple functions (= piecewise constant functions).

[344] $A \in \mathcal{F}$, precisely speaking, but as announced, I will not mention such a thing again.

or never in B (that is E_0). Suppose for some n $\mu(E_{n+1} \setminus E_n) > 0$. Then, this holds for all n, because μ is T-invariant:

$$\mu(E_{n+1} \setminus E_n) = \mu(T^{-1}[E_{n+1} \setminus E_n]) = \mu(T^{-1}E_{n+1} \setminus T^{-1}E_n) = \mu(E_{n+2} \setminus E_{N+1}). \quad (30.4)$$

Therefore,

$$\mu(E) = \mu[\cup_{n=1}^{\infty}(E_{n+1} \setminus E_n) + E_0] = \sum_{n=1}^{\infty} \mu(E_{n+1} \setminus E_n) + \mu(E_0) \nearrow \infty. \quad (30.5)$$

Thus, $\mu(E \setminus E_0) = 0$, but E_0 is the set of points never touching B, so $E_0 \cap B = \emptyset$. Therefore,

$$0 = \mu(B \cap (E \setminus E_0)) = \mu(E \cap B). \quad (30.6)$$

There is almost no points in B that cannot return to B infinitely often.

Remark: Note that $Q_n = E_{n+1} \setminus E_n$ is the totality of points in B that returns to B at time n and stays in B ever since. Thus, $T^{-1}Q_n = Q_{n+1}$. All have the same measure and disjoint and in $B^c = M \setminus B$. All Q_n must stay in B^c without overlap, so $\mu(Q_n)$ must be zero. As is clear from this what matters is that μ is normalizable. Whether M is bounded or not (as a metric space) is irrelevant. Thus, this theorem can be used to destroy Boltzmann's logic.

30.3 Zermelo's Wiederkehreinwand[345,346]

This theorem played an important role when Zermelo[347] pointed out Boltzmann's logical error. Boltzmann claimed that (after responding to the criticism Umkehreinwand[348] by Loschmidt just after the Boltzmann equation paper) his approximate dynamical system explains irreversibility: due to approximation irreversibility ensues. Zermelo (1895) pointed out still reversibility occurs indefinitely accurately due to this theorem (published in 1890); recurrence is not due to any property of the dynamics, but merely due to the finiteness of the total mass of the relevant invariant measure. He was confident because of the correct math and the moral support of his boss M. Planck.

[345] 'Wiederkehr' = recurrence, 'Einwand' = objection.

[346] H.-D. Ebbinghaus, *Ernst Zermelo, an approach to his life and work* (Springer, 2007) p15-26 1.4 Boltzmann Controversy.

[347] A road in Freiburg was named in his honor in 2017 (Wikipedia).

[348] 'Umkehr' = turning back.

30.4 Birkhoff's individual ergodic theorem

Theorem. Under the same condition as **30.5**
(i) The time average exists for μ-$\tilde{\forall} x \in M$

$$\overline{f}(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f(T^k x). \tag{30.7}$$

(ii) This convergence is also in L_1.
(iii) For any invariant set B

$$\int_B \overline{f}(x) d\mu(x) = \int_B f(x) d\mu(x). \tag{30.8}$$

Remark: Notice that for this theorem to hold whether μ is ergodic or not does not matter. We need ergodicity to reduce (iii) to our familiar formula (see **30.8**):

$$\text{Time average of } f = \int_M f(x) d\mu(x). \tag{30.9}$$

We demonstrate these below **30.6**-**30.7**. The maximal ergodic theorem **30.5** drastically simplifies the original almost unreadable proof.

30.5 Maximal ergodic theorem

Let $T : M \to M$ preserves a measure μ. For μ-integral function f define the following functiions for $n \in \mathbb{N}$

$$S_n(f; x) = \sum_{k=0}^{n-1} f(T^k x), \;\; S_0(f; x) = 0. \tag{30.10}$$

Notice that

$$S_n(f; Tx) = S_{n+1}(f; x) - f(x). \tag{30.11}$$

Theorem

$$\int_{\{x \mid \sup_{n\ge 0} S_n(f;x)>0\}} f(x) d\mu(x) \ge 0. \tag{30.12}$$

[Demo][349]
We make a 'finite approximation' of $B = \{x \mid \sup_{n\ge0} S_n(f; x) > 0\}$ as $B_n = \{x \mid M_n(x) > 0\}$:

$$M_n(x) = \max\{0, S_1(f; x), \cdots, S_n(f; x)\}. \tag{30.13}$$

[349] All the detailed are filled in, so you should be able to follow the proof without pencil.

Note $S_0(f;x) = 0$. Since (30.11)

$$M_n(Tx) = \max\{0, S_2(f;x) - f(x), \cdots, S_{n+1}(f;x) - f(x)\}. \tag{30.14}$$

We also introduce

$$M_n^*(x) = \max\{S_1(f;x), \cdots, S_n(f;x)\}. \tag{30.15}$$

This means (notice that $S_1(f;x) = f(x)$ from (30.10))

$$M_{n+1}^*(x) = \max\{S_1(f;x), \cdots, S_{n+1}(f;x)\} = \max\{f(x), S_2(f;x), \cdots, S_{n+1}(f;x)\}. \tag{30.16}$$

Since $\max\{x_n + c\} = \max\{x_n\} + c$,

$$M_n(Tx) + f(x) = M_{n+1}^*(x). \tag{30.17}$$

That is, (note that $M_n^*(x)$ is an increasing sequence in n)

$$f(x) = M_{n+1}^*(x) - M_n(Tx) \geq M_n^*(x) - M_n(Tx). \tag{30.18}$$

We make an approximation of the integral (recall $B_n = \{x \,|\, M_n(x) > 0\}$)

$$\int_{B_n} f(x) d\mu(x) \geq \int_{B_n} [M_n^*(x) - M_n(Tx)] d\mu(x). \tag{30.19}$$

If $x \in B_n$, then you can ignore 0 in the definition of $M_n(x)$, so $M_n^*(x) = M_n(x)$. Therefore,

$$\int_{B_n} M_n^*(x) d\mu(x) = \int_{B_n} M_n(x) d\mu(x). \tag{30.20}$$

Therefore, (30.19) reads

$$\int_{B_n} f(x) d\mu(x) \geq \int_{B_n} M_n(x) d\mu(x) - \int_{B_n} M_n(Tx) d\mu(x). \tag{30.21}$$

Since $M_n = 0$ oputside B_n, we have

$$\int_{B_n} f(x) d\mu(x) \geq \int_{M} M_n(x) d\mu(x) - \int_{B_n} M_n(Tx) d\mu(x). \tag{30.22}$$

Since $M_n(Tx) \geq 0$,

$$\int_{B_n} M_n(Tx) d\mu(x) \leq \int_{M} M_n(Tx) d\mu(x). \tag{30.23}$$

Therefore, we have obtained

$$\int_{B_n} f(x)d\mu(x) \geq \int_M M_n(x)d\mu(x) - \int_M M_n(Tx)d\mu(x). \tag{30.24}$$

Now, take the $n \to \infty$ limit, and the right-hand side vanishes to reach (30.12).

30.6 Existence of time average

Notice that (i) in **30.4** is a generalization of the (weak) law of large numbers.

We wish to show that $S_n(f;x)/x$ converges almost μ-surely. If this does not converge lim sup and lim inf must be different. Therefore, there must be reals $a < b$ such that the following set has a positive measure:

$$B(a,b) = \left\{ x \,\middle|\, \liminf \frac{1}{n} S_n < a < b < \limsup \frac{1}{n} S_n \right\}. \tag{30.25}$$

Since $B(a,b)$ is an invariant set with a positive measure, let us confine ourselves to $B(a,b)$ and apply the maximal ergodic theorem to $a - f(x)$ and $f(x) - b$. For example, for $g(x) = a - f(x)$ the maximal ergodic theorem reads:

$$S_n(g;x) = na - \sum_{k=0}^{n-1} f(T^k x). \tag{30.26}$$

and

$$\int_{\{x \,|\, \sup_{n\geq 0} S_n(g;x)>0\}\cap B(a,b)} g(x)d\mu(x) = \int_{\{x \,|\, a>\sup_{n\geq 0} S_n(f;x)/n\}\cap B(a,b)} (a - f(x))d\mu(x) \geq 0. \tag{30.27}$$

However, $\{x \,|\, \sup_{n\geq 0} S_n(h;x) > 0\} \supset B(a,b)$, so this means

$$a\mu(B(a,b)) - \int_{B(a,b)} f(x)d\mu(x) \geq 0. \tag{30.28}$$

Analogously, for $h(x) = f(x) - b$

$$0 \leq \int_{B(a,b)} h(x)d\mu(x) = \int_{B(a,b)} f(x)d\mu(x) - b\mu(B(a,b)). \tag{30.29}$$

Hence,

$$a\mu(B(a,b)) \geq \int_{B(a,b)} f(x)d\mu(x) \geq b\mu(B(a,b)), \tag{30.30}$$

a contradiction.

30.7 L^1 convergence and (iii)

First we show (ii) to guarantee the commutativity of the integration and limit in (30.31).

If f is bounded as $|f(x)| \leq K$, $|S_n/n| \leq K$, so $|\overline{f}(x)| \leq K$. Therefore, we may apply Lebesgue's bounded convergence theorem, implying the L^1-convergence of the time average. For general f we L^1-approximate f with a sequence of bounded functions to complete the proof. Thus (ii) holds.

If g is integrable on M, then (ii) allows

$$\int_M \overline{g}(x)d\mu(x) = \lim_{N\to\infty}\int_M \frac{1}{N}\sum_{K=0}^{N-1} g(T^k x)d\mu(x) = \int_M g(x)d\mu(x), \tag{30.31}$$

Now let $g(x) = \chi_B(x)f(x)$. Since B is invariant, $\overline{g}(x) = \chi_B(x)\overline{f}(x)$, which concludes the demonstration.

Notice that the above theorem holds for any invariant measure.

30.8 Time average and phase average

If μ is an ergodic invariant measure, then the time average agrees with the phase average (or the ensemble average):

$$\int_M f(x)d\mu = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1} f(T^k x). \tag{30.32}$$

Notice that $\overline{f}(x)$ is time independent: $\overline{f}(Tx) = \overline{f}(x)$. Then, $\overline{f}(x)$ must be μ-almost surely constant: Let $A(c) = \{x \,|\, \overline{f}(x) = c\}$. It is an invariant set, so its measure is 0 or 1. (iii) in **30.4** tells us $c =$ the phase average value.

30.9 von Neumann's mean ergodic theorem

Before Birkhoff von Neumann proved the L^2 convergence

$$\lim_{N\to\infty}\int_M \left|\frac{1}{N}\sum_{k=0}^{N-1} f(T^k x) - \overline{f}(x)\right|^2 d\mu = 0. \tag{30.33}$$

No proof will be given, but if f is bounded, then L^1 and L^2 convergence on M is equivalent.

30.10 Weyl's equidistribution theorem

Historically, the first ergodic theorem is the following:

Theorem. For an integrable function f defined on a unit circle

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f(2\pi k\gamma \dot{+} \theta) = \int f(x) d\mu(x), \tag{30.34}$$

where $\dot{+}$ is the sum mod 2π, γ an irrational number and μ the uniform probability measure on the unit circle.

μ is an ergodic invariant measure for rotation. A more direct proof (e.g., using Fourier expansion) may be a good exercise.

30.11 Tragicomical history of ergodicity[350]

The word 'ergode' was used to specify the microcanonical ensemble. Maxwell in his "On Boltzmann's theorem on the average distribution of energy in a system of material points,"[351] he clarified the logic of Boltzmann's attempt (his second paper written at age 22) to derive equipartition of energy and the second law from mechanics, saying "The only assumption which is necessary for the direct proof is that the system, if left to itself in its actual state of motion, will, sooner or later, pass through every phase which is consistent with the equation of energy." Maxwell then considered an ensemble of systems having this property to develop a statistical theory. Actually Boltzmann adopted the idea of 'ensemble' and introduced his 'ergodic ensemble.'

Lord Kelvin on April 27, 1900 gave a talk on the clouds over the dynamical theory of heat and light[352] The second cloud he mentioned was the anomaly of the specific heat ratio $\gamma = C_P/C_V$. "Spectrum analysis showing vast numbers of lines for each gas makes it certain that the numbers of freedoms of the constituents of each molecule is enormously greater than those which we have been counting," γ should

[350]Based, in part, on I. Kubo's article on the history of ergodic theory (1982 September-December

[351]Cambridge Phil. Soc. Trans. 7 547 (1979). This is his last paper.

[352]"Nineteenth century clouds over the dynamical theory of heat and light," Phil. Mag. Series 6, 2: 7, 1-40 (1901). 'Cloud II' starts at Section 12. The outline I give here should be regarded as a hasty one by a very impatient theoretical physicist.

be very close to 1 in contrast to empirical results. Since Maxwell and Boltzmann premised "that the mean kinetic energies with which the Boltzmann-Maxwell doctrine (=equipartition of energy) is concerned are time integrals of energies divided by totals of the times," their hypothesis (ergodic hypothesis) should be questioned. The, he goes on to the study of ergodicity of simple dynamical systems—billiards! He studies an elliptic billiard and says, "It seems not improbable that if the figure deviates by ever so little from being exactly ellipsoidal, Maxwell's condition might be fulfilled." "the meaning of the doctrine is that a single geodesic drawn long enough will not only fulfil Maxwell's condition of passing infinitely near to every point of the surface in all directions." "I have made many efforts to test it for the case in which the closed surface is reduced to a plane with other boundaries than an exact ellipse.' He studied (not convex) polygons and 'experimentally' (by the assistant Mr Anderson with a straight rule) studied the equipartition: after 600 collisions $\langle K_x \rangle = \langle K_y \rangle$ was not fulfilled with an error of 7.5 %. He also studied a converging billiard as well (see Fig. 30.1; even he studied the ice cream cone!).

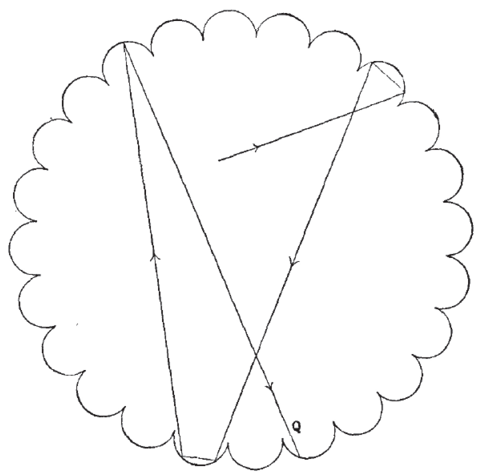

Figure 30.1: Kelvin's 'simulation' for this failed to demonstrate ergodicity

Thus Lord Kelvin extremely seriously cast doubt on ergodicity.

A and T Ehrenfest clearly formulated 'ergodic hypothesis' in their encyclopedia article:[353] according to their definition, 'ergodic' means, as Maxwell said, a trajectory

[353] P. Ehrenfest & T. Ehrenfest (1911) Begriffliche Grundlagen der statistischen Auffassung in der Mechanik, in: Enzyklopdie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Band IV, 2. Teil (F. Klein and C. Mller (eds.). Leipzig: Teubner, pp. 390. Translated as *The conceptual Foundations of the Statistical Approach in Mechanics*, New York: Cornell University Press, 1959. ISBN 0-486-49504-3.

passes every point energetically allowed and 'quasiergodic' means, as formulated by Lord Kelvin, a trajectory passes any neighborhood of any point energetically allowed. Then, 'ergodic hypothesis' means that the time average along an ergodic trajectory is identical to the microcanonical average.

This hypothesis was immediately shot down by Plancherel and by Rosenthal (independently in 1913): there cannot be such an orbit; simply topologically impossible. In those days, physicists realized that Gibbs' statistical mechanics is practically usable, so the interest in ergodicity faded, and the topic came to be considered as a pure mathematical question.

Then, in 1914 Weyl proved his theorem **30.10**, showing that quasiergodicity may be enough. His theorem is about T^n. von Neumann extended this to a general domain[354].[355]

Theorem [Mean ergodic theorem] If f is $L^2(X)$, then a function $\overline{f}(x)$ exists such that

$$\lim_{t\to\infty}\int_X \left|\frac{1}{t}\int_0^t f(T_t x) - \overline{f}(x)\right|^2 d\mu(x) = 0. \tag{30.35}$$

This means

Corollary. μ-almost every x there is an increasing time sequence $\{\tau_n\}$ such that

$$\overline{f}(x) = \lim_{n\to\infty}\frac{1}{\tau_n}\int_0^{\tau_n} f(T_t x)dt. \tag{30.36}$$

This is basically what physicists 'needed.'

On Oct 22, 1931, Birkhoff received von Neumann's letter on the proof of the mean ergodic theorem, and started his study vehemently.[356] He declared that the quasiergodic hypothesis was now replaced by its modern version: measure-theoretical transivity.

Has Birkhoff unraveled or resolved the secret of statistical mechanics? Simply, the question becomes whether the Liouville measure (i.e., the Lebesgue measure on the phase space) is ergodic or not. For almost all systems of interest, this has never been proved.

We now clearly recognize that ergodicity cannot found statistical mechanics; ergodicity has been a big red herring.

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[354] 1932

[355] This possibility was suggested to him by Koopman (in 1930) and by A Weil (in 1931).

[356] You must learn a great lesson from this episode; you should not tell even what you are studying.