2 Lecture 2: Setting the stage

2.1 Two main purposes of this lecture

There are two main topics today. Dynamical systems are defined as maps or flows defined by vector fields on manifolds. Therefore, first, we discuss manifolds and vector fields on them.

For practical physicists, basically we have only to understand what happens in the ordinary n-dimension Euclidean space $\mathbb{E}^n$, because manifolds are patchworks of Euclidean spaces. Thus, the purpose of the first part of this lecture is to introduce concepts mathematically properly, but to tell you how to 'ignore' their technicality. To discuss a dynamical system we need a stage: a manifold.

The second part reflects on what 'common' or 'general' means, because we are interested in 'general pictures' of dynamical systems. We review very basic concepts, openness, denseness, etc. Cantor sets are introduced and then we will discuss what 'dimension' is.

Although you must be able to find (and to understand) real math definitions, as an active physicist it is very important to grasp math concepts and theorems intuitively (and even emotionally).

2.2 Discrete-time dynamical systems

A discrete time dynamical system is a map $f : M \to M$, where M is a compact[18] (often C^r-)manifold (see justbelow).

I hope you know what map is.

The stage (or the totality of the states of the system; the phase space in classical statistical mechanics is an example) of the dynamical system may not be a simple space like $\mathbb{R}^n$ (n-dimensional Euclidean space; $\mathbb{E}^n$ may be a better notation). It could be a 2-torus (T^2), so it cannot be mapped continuously to $\mathbb{R}^2$. Usually, we choose the stage to be a *manifold*.

An n-manifold is a geometrical object that can be constructed by 'smoothly' pasting a n-Cartesian coordinate patches that are 'flexibly deformed' (Fig. 2.1):

A formal definition is in **2.3**. To understand this we need a concept 'diffeomorphism': Let A and B be sets (actually, topological spaces). A map $f : A \to B$ is a C^r-diffeomorphism, if it is continuously r-times differentiable (C^r-map), and its

[18] In a finite dimensional space, 'compact' means that the object is covered with a finite number of open sets. See **3.9**

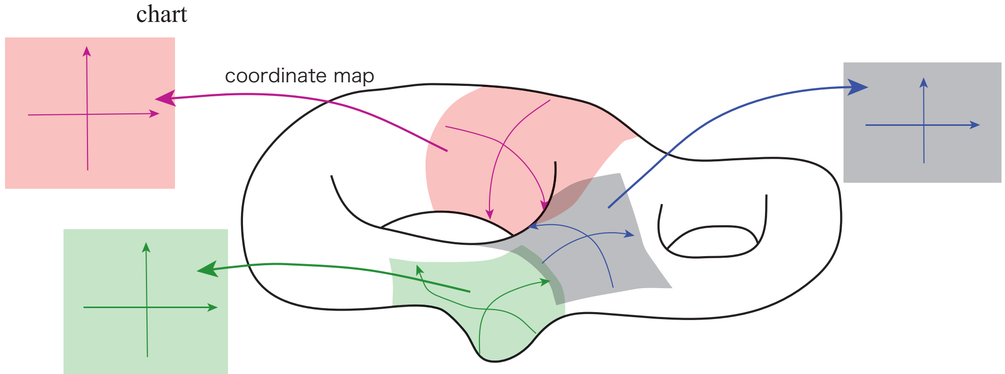

Figure 2.1: Intuitive idea of a manifold; here only some coordinate patches are illustrated; you must fill the whole object with such a cover.

inverse f^{-1} is also a C^r-map. If f and f^{-1} are continuous maps, we say f is a homeomorphism.

Practically, whenever we work on a manifold, we take a chart and then locally work in a Cartesian system. We must seamlessly connect the local results, but that part is usually only technical, so for physicists intuitively the core discussions are over on the chart.

2.3 Manifold

A manifold $M = (\mathcal{M}, \mathcal{U})$ consists of two elements:
(1) Basic set $\mathcal{M}$, which is usually a Hausdorff space.[19]
(2) Local coordinate system $\mathcal{U} = \{U_\alpha, \phi_\alpha\}_{\alpha \in A}$, where A is a subscript set. Here
(i) $U_\alpha \subset \mathcal{M}$ is open and $\cup_\alpha U_\alpha = \mathcal{M}$ (i.e., $\{U_\alpha\}$ is an open cover of $\mathcal{M}$.
(ii) ϕ_α is a homeomorphism (or diffeomorphism)[20] from U_α to an open set in $\mathbb{R}^n$.
(iii) If $U_\alpha \cap U_\beta \neq \emptyset$, then $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is a homeo or diffeomorphism.

(U_α, ϕ_α) is called a chart, and $\mathcal{U}$ is called an atlas.[21] Needless to say, charts in an atlas of a manifold must be consistent as (iii).

[19] ⟪**Hausdorff space**⟫ A topological space X is a Hausdorff space if all distinct points in X are pairwise neighborhood-separable. That is, if $x, y \in X$ and $x \neq y$, there is a neighborhood U of x and that V for y such that $U \cap V = \emptyset$.

[20] This choice depends on how smoothly you wish to set up your manifold.

[21] The largest atlas including all the atlases containing a given chart is called the maximal atlas (or the differentiable structure). It may not be unique even diffeomorphically. S^7 has 28 different differentiable structures.

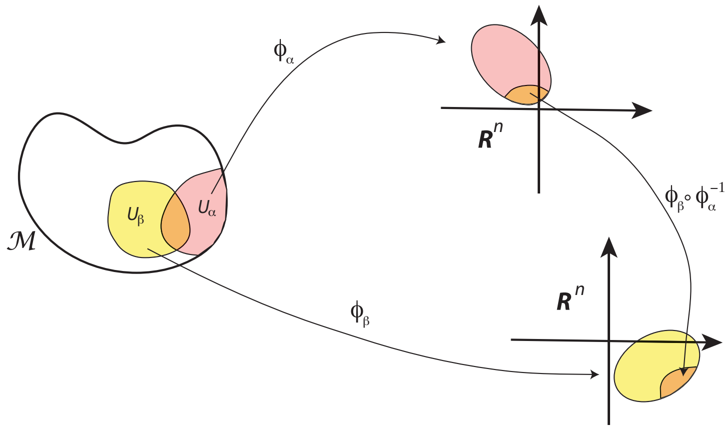

Figure 2.2: Manifold: local coordinate system/charts

2.4 'Topological continuation'

When we discuss mathematics on a manifold, it is almost always the case that we assume some convenient charts and on each chart we discuss what we are interested in. Then, we show that we can continue the conclusion to different charts using the 'smoothness' of the relations between the overlapping charts. The latter steps may be tedious, but intuitively without any trouble. Thus, in these lecture notes, we almost always discuss mathematics on a chart and will not discuss the 'second step,' simply saying 'according to the topological continuation' we claim the statements on the entire manifold. As you realize, this means that we may discuss things on an appropriate Euclidean space.

2.5 Vector field on manifold

We wish to consider an ordinary differential equation (ODE) defined on a manifold at each point $x \in M$. We take a chart (local coordinates) $(x_1, x_2, \cdots, x_n)$. With respect to this coordinates a vector field $X(x)$ at x may be expressed as

$$X(x) = (X_1, X_2, \cdots). \tag{2.1}$$

Here, X_i is the x_i-component of X. See Fig. 2.3.

Mathematicians express X as follows:

$$X(x) = \sum_i X_i \frac{\partial}{\partial x_i}. \tag{2.2}$$

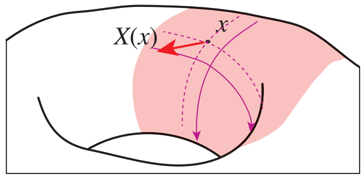

Figure 2.3: Vector field on manifold. The dotted lines correspond to the tangential directions $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$.

Here $\frac{\partial}{\partial x_i}|_x$ points the tangent direction along the local coordinate x_i as illustrated in Fig. 2.3. The vector space spanned by $\{\frac{\partial}{\partial x_i}|_x\}$ is called the tangent space of M at x, and is denoted by the symbol $T_x M$.

The notation (2.2) is very reasonable as you can see from the directional derivative of a function f on M at x along the curve $\xi(t) \in M$. Let the tangent vector for ξ at x be X:

$$\frac{d}{dt}\xi(t) = X = (X_1, X_2, \cdots). \tag{2.3}$$

Then,

$$\frac{d}{dt}f(\xi(t)) = \sum_i X_i \frac{\partial}{\partial x_i} f(x) = Xf(x) = \sum_i X_i \frac{\partial f}{\partial x_i}. \tag{2.4}$$

Besides, the notation (2.2) automatically tells us how to rewrite the vector components when we change a chart from (U, x) to another overlapping chart (V, y):

$$X = \sum_j Y_j \frac{\partial}{\partial y_j} = \sum_j Y_j \sum_i \frac{\partial x_i}{\partial y_j} \frac{\partial}{\partial x_i}. \tag{2.5}$$

Thus, we have

$$X_i = \sum_j Y_j \frac{\partial x_i}{\partial y_j}. \tag{2.6}$$

A more mathematically respectable explanation can be found in 'indented' **2.7-2.9**.

2.6 Continuous-time dynamical systems

An ODE on M may be defined as

$$\frac{d}{dt}x = X. \tag{2.7}$$

This implies

$$\sum_i \dot{x}_i \frac{\partial}{\partial x_i} = \sum_i X_i(x) \frac{\partial}{\partial x_i}. \tag{2.8}$$

that is, as we know in the elementary calculus, $\dot{x}_i = X_i(x)$.

If (3.1) is well-posed (i.e., there exists a unique solution to the Cauchy problem = initial value problem), it can define a continuous-time dynamical system $\phi_t : M \to M$, where ϕ_t is the time evolution operator ($\{\phi_t\}$ is a group if $t \in \mathbb{R}$ or a monoid if $t \in [0, +\infty)$):
(i) $\phi_0 = 1$,[22]
(ii) $\phi_t \circ \phi_s = \phi_{t+s}$.

The totality of the C^r-vector fields on M is written as $\mathcal{X}^r(M)$. As we will know, $X \in \mathcal{X}^r(M)$ defines a continuous-time dynamical system.

Remark We discuss only the autonomous systems for which X never depends on t explicitly. If you wish to discuss a time-dependent vector field $X(t)$, you could introduce a new component z satisfying $\dot{z} = 1$.

2.7 Tangent vector

Let $\xi(t)$ be a (differentiable) curve in a manifold M in the chart (U, ϕ). The coordinate system is denoted by $(x_1, \cdots, x_n)$. Let $f : M \to \mathbb{R}$ be a differentiable function. If you wish to differentiate f along the tangential direction of ξ we can compute

$$\frac{d}{dt} f(\xi(t)) = \sum_i \frac{d\xi_i}{dt} \frac{\partial}{\partial x_i} f \equiv X_\xi f \tag{2.9}$$

where

$$\frac{\partial}{\partial x_i} f = D_i f(\phi^{-1}). \tag{2.10}$$

Here, D_i is the differentiation with respect to the coordinate x_i on the chart.

We use the following notational convention:

$$X_\xi = \frac{d\xi}{dt} = \sum_i \frac{d\xi_i}{dt} \frac{\partial}{\partial x_i}. \tag{2.11}$$

X_ξ denotes a vector in the tangential space of M. If $p = \xi(0)$, then the totality of X_ξ at $t = 0$ spans the tangential space of M at p denoted by T_pM:

$$T_pM = \left\langle \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n} \right\rangle. \tag{2.12}$$

[22] '1' means 'multiplying 1' or the identity operator: $1x = x$.

2.8 Vector bundle

A vector bundle (V, M, π) on a manifold M is a triple of a manifold M, a vector space V and a map $\pi : V \to M$.

$\pi^{-1}(x)$ for $x \in M$ is called the fiber at x.

$s : M \to V$ such that $\pi s(x) = x$.

If s is C^r, the vector field X is said to be a C^r-vector field. The totality of the C^r-vector field on M is written as $\mathcal{X}^r(M)$.

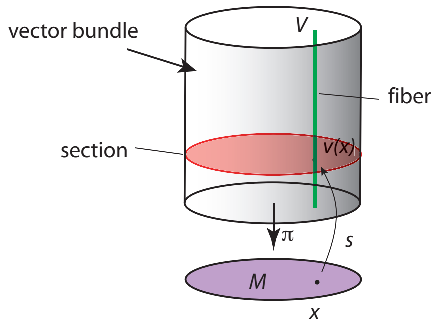

Figure 2.4: Vector bundle

If M is C^∞, then $\mathcal{X}^r(M)$ is a separable Banach space.[23]

2.9 Tangent vector bundle

The vector field on a manifold M whose fiber at $x \in M$ is $T_x M$ is called the tangent vector bundle of M and is denoted as TM. Thus, $\mathcal{X}^r(M)$ consists of smooth sections of TM.

2.10 Pursuit of general pictures

Although particular examples, if representative enough in some sense, are often important to give us deep insights, our main goal is to have an overall general picture of dynamical systems. We study discrete dynamical systems $C^r(M)$ or continuous dynamical systems defined by the corresponding vector fields $\mathcal{X}^r(M)$. We wish to know the common features of dynamical systems, or wish to characterize typical examples in these collections of dynamical systems.

What do we wish to mean by 'general', 'common' or 'typical'?

The most natural or desirable idea is that a general property is a property shared by all the instances in $C^r(M)$, for example. However, in most cases exceptions exist for any apparently very general properties, so 'all' should be relaxed to 'almost all'

[23]A Banach space is a linear space with a complete norm ('complete' = any Cauchy sequence converges). 'Separable' means that the space contains an everywhere dense countable set.

or 'most.' Therefore, how to relax 'all' and still to secure 'many' in a precise sense[24] is our key issue.

In the following we first discuss denseness **2.12** and openness **2.13**. If systems with property A are dense in the totality of dynamical systems (say, $\mathcal{X}^r(M)$), we can always find (need not be easy) a system with property A as close as you wish to a system you pick up. If systems with property A make an open subset of the totality of dynamical systems, then a system with property A is surrounded by systems with property A, so property A is stable against perturbations. However, this does not guarantee 'many,' although it tells us that 'all' (really all, not almost all) around the example.

Then, 'open-denseness' = openness + denseness sounds desirable. Not quite as we will see below.

2.11 Topology[25]

If we wish to say somethings are common, we must be able to classify the objects we are interested in, or at least, we must be able to tell which objects are more similar or less so than other objects. Thus, we need a concept to judge 'closeness' or similarity. You might immediately have in your mind some sort of distances, but a more basic concept is some sort of 'nearness' or being in the same neighborhood. A topology furnishes this concept.

Let τ be a family of subsets of a set X. τ is a topology defined for a set X, if

(i) $\tau \ni \emptyset$ and $\ni X$.
(ii) If $\{G_a \in \tau\}$, then $\cup_a G_a \in \tau$.
(iii) If $\{G_a \in \tau\}$ is a finite collection, then $\cap_a G_a \in \tau$.

(X, τ) is called a topological space, and the elements in τ are called open sets.

Any open set containing $x \in X$ is called a neighborhood of x (in this topological space).

We can define continuous functions between two topological spaces X and Y: $f : X \to Y$ is continuous around $x \in X$, if for any nbh U of $f(x) \in Y$ is a nbh V of $x \in X$ such that if $y \in V \Rightarrow f(y) \in U$.

Thus, geometrical properties that are preserved by continuity are called topo-

[24] Here, 'precise' means, ultimately, 'can be axiomatized'.

[25] For such basic concepts the following books are strongly recommended:
I. M. Singer and J. A. Thorpe, *Lecture Notes on Elementary Topology and Geometry* (Springer Undergraduate Texts in Mathematics 1976)
A. N. Kolmogorov and S. V. Fomin, *Elements of the Theory of Functions and Functional Analysis* (Martino Fine Books, 2012) $9.5 !

logical properties.

2.12 Denseness

Let Ω be a set. A subset $U \subset \Omega$ is dense if every neighborhood of any element in Ω contains an element in U.

As $\mathbb{Q} \subset \mathbb{R}$ shows, denseness can still mean shear minority. If we evenly and randomly sample a point from $[0,1]$ almost surely you are in $\mathbb{Q}^c$. It is a countable set, so its total length is zero (= measure zero).

Thus, a bigger subset than mere dense subset is desirable.

A countable set Q is measure zero (volume is zero; more precisely its Lebesgue measure is zero). Since Q is countable, it is one-to-one correspondent to $\mathbb{N}^+$. Thus we can write $Q = \{a_k\}_{k\in\mathbb{N}^+}$. For example, let us assume $Q \in \mathbb{R}^n$. Then, we choose a ball B_k of volume $\varepsilon/2^{k+1}$ ($\varepsilon > 0$, arbitrary) centered at a_k. Obviously $Q \subset \cup_k B_k$. Therefore, the total volume of Q (denoted as $|Q|$) must be less than the total volume of these balls $B_1, B_2, \cdots$:

$$|Q| \le \sum_k |B_k| = \sum_k \frac{\varepsilon}{2^{k+1}} = \varepsilon. \tag{2.13}$$

Since $\varepsilon > 0$ is arbitrary, $|Q|$ must be smaller than any positive number: 0.

2.13 Openness, structural stability

A subset U is open in Ω, if any element of U has a neighborhood contained in U.

Thus, intuitively speaking, when $x \in U$ is perturbed to $x + \delta x$ with any sufficiently small δx $x + \delta x \in U$. This is a stability of the property. If a dynamical system itself (in contrast to perturbing its initial condition) is perturbed and if still the system 'looks (behaves) similar' to the original system, we say the system is structurally stable.

In a certain sense 'openness' can be too strong or too weak, depending on the situation. It can be too strong, because there can be special direction to destroy the property we are interested in, although in all other directions it is stable against small perturbations. A 'too weak' case is in **2.14**. Also if we recall that irrational numbers are with full measure but not open in $\mathbb{R}$, demanding openness may not be wise to characterize 'general.'

2.14 Open denseness

If a set U is open, then it is measure positive for 'natural sampling measure' as an open set in $[0,1]$ illustrates. Therefore, 'open denseness' could be a candidate of 'naturalness,' and certainly it is used to assert certain generality. However, the compliment of an open dense set could be as close to the full (measure) set as possible as the fat Cantor set illustrates.[26]

However, unfortunately, it is often the case that most properties for dynamical systems are vulnerable to a particular perturbation in a certain 'direction', so 'openness' does not usually hold. Thus, we wish to relax 'open denseness'.

Can we find an open dense set easily? Not so. See **2.19**.

2.15 Cantor set

Cantor discussed a perfect nowhere dense set C (Cantor set in the general sense). 'Perfect' means no point is isolated. That is, for any neighborhood of $x \in C$ is a point in C. 'Nowhere dense' means $[C]^\circ = \emptyset$.

The most famous example of a Cantor set is the one invented by H. J. S. Smith, the set built by removing the open middle thirds of a line segment (see 2.7). Cantor mentioned this as an example, in passing, but it is now usually recognized as 'the Cantor set.'

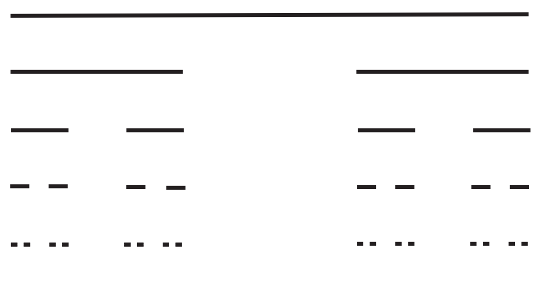

Figure 2.5: 'The' Cantor set due to Smith (1874) constructed by removing the middle third open set

This Cantor set may be analytically expressed as

$$C = \left\{ x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \,\middle|\, a_n \in \{0, 2\} \right\}. \tag{2.14}$$

It is a measure zero uncountable and self-similar set. Notice that the number of gaps is a countable infinity. $[0,1] \setminus C$ is an open dense set and with full measure.

[26]B. R. Gelbaum and J. M. H. Olmsted, *Counterexamples in Analysis* (Holden-Day, Inc. San Francisco, 1964) is a great source book for delicate issues.

2.16 Why Cantor sets are relevant to dynamical systems

Perhaps, you might think Cantor sets are rather contrived artificial sets. It is actually not. As noted before, complicated behavior due to nonlinearity is due to cofinement of the phase space in a finite domain by 'folding.' If the invariant set[27] is not identical with M, then it is very natural to exhibit a self-similar structure (see Fig. 2.8). Therefore, U itself is a Cantor set or a direct product of Cantor set and 'an ordinary set' (interval, circle, etc.).

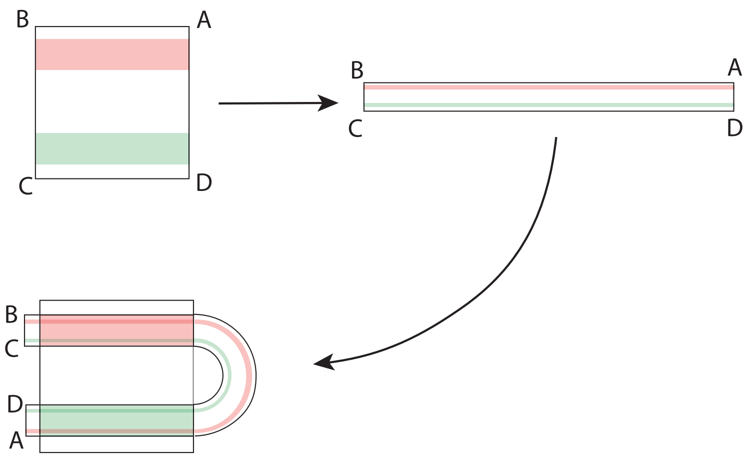

Figure 2.6: Folding can produce self similar Cantor-like invariant set. This is a part of the famous Smale's horseshoe.

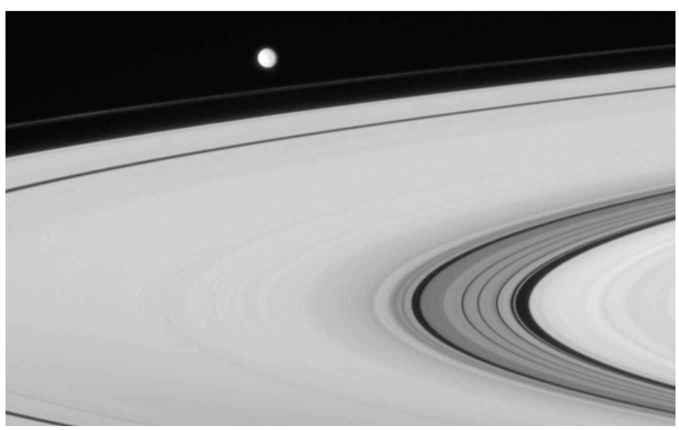

Figure 2.7: Is the ring of Saturn cantor?

[27]We will discuss this in detail, but here you may intuitively understand that a set $U \subset M$ such that $f(U)$ or $\phi_t(U) = U$.

2.17 Cantor set due to 'excessively tall' tent maps

Consider a tent map $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} kx \text{ for } x \le 1/2, \\ k(1-x) \text{ for } x > 1/2. \end{cases} \tag{2.15}$$

The points remaining in $[0, 1]$ make a Cantor set. For $k = 3$ we get the 'standard Cantor set' explained in Fig. 2.7.

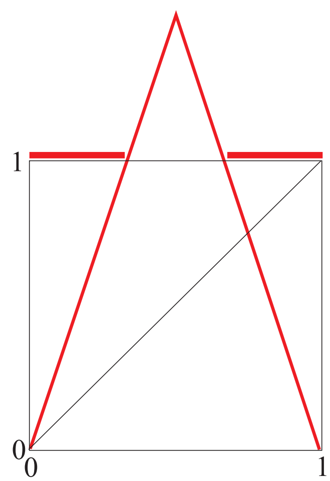

Figure 2.8: Adjusting the slopes of a tent map, we can make a self-similar middle removed Cantor set as an invariant set

2.18 Fat Cantor set

In the standard Cantor set, we remove 1/3 of the interval in a self-similar fashion. What if we remove much smaller interval? If the construction is self similar, the remaining (the resultant) Cantor set is measure zero. What if, then we remove center pieces that are shrinking faster?

Let $\alpha \in (0, 1)$.
(i) Remove $(1/2 - \alpha/4, 1/2 + \alpha/4)$ from $[0, 1]$.
(ii) Then, remove the middle $\alpha/24$ fractions from the remaining two intervals.
(iii) Repeat this procedure ad infinitum (see Fig. 2.9).

The remaining set is a closed set and has no interior and perfect. That is, a Cantor set = nowhere dense perfect set.

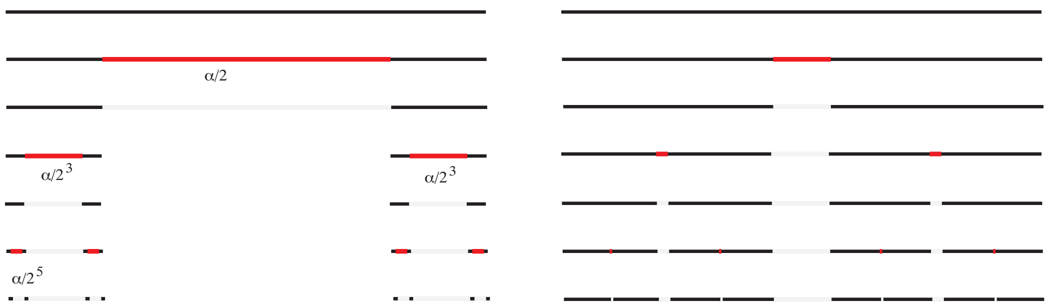

Figure 2.9: Construction of fat cantor set. Remove successively the scaled copies of red chunks. Left the usual measure zero cantor set; Right: a fat cantor set with a positive length (only the values of α are different).

The total length of the resultant cantor set is

$$1-\frac{\alpha}{2}-2\frac{\alpha}{2^3}-2^2\frac{\alpha}{2^5}-\cdots=1-\frac{\alpha}{2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots\right)=1-\frac{\alpha}{2}\frac{1}{1-1/2}=1-\alpha. \tag{2.16}$$

If $\alpha < 1$, then the resultant Cantor set is no where dense but with positive measure, so it is called a *fat Cantor set*. We will actually encounter such a set in chaos.

2.19 Very small open-dense set

Now, it should be clear how to make an open and dense set which is far from the major part of the total set.

Let us construct a fat Cantor set of measure $1-\varepsilon$ for small $\varepsilon > 0$. Then, make its complement in $[0,1]$. It is open and dense, but with measure ε (> 0), which can be as small as you wish. Thus, we have constructed an open dense set that is very hard to sample.

2.20 What is dimension?[28]

We usually say our space is 3D because we need three independent coordinates to specify a point in the space uniquely. However, the space may not be metric (i.e., distances may not be clearly defined; what is space?). Perhaps the simplest idea is that our space is modeled by a 3-manifold. 3-manifold is defined by 3D charts. The latter is defined as a 3D vector space, which we can define mathematically (will not be discussed).

However, the concept of dimension should be 'more primitive'; we should

[28]Detailed reference: Y. B. Pesin: *Dimension theory in dynamical systems* (Chicago Lectures in Mathematics, Chicago UP 1997).

not be required to have a metric, for example.

2.21 Inductive topological dimension[29]

A motivation comes from the fact that a bounded geometric object B its dimension is the dimension of $\partial B + 1$. However, if B is an open set $\partial B = \emptyset$, even though $\partial[B]$ is not empty. Thus, the induction suggested above must be formulated with some care.
The inductive dimension $d_{\rm I}$ is defined as follows:
(i) $d_{\rm I}(\emptyset) = -1$.
(ii) $d_{\rm I}(B) \leq n$ if for all $x \in B$, there is a neighborhood U such that an open set $V \subset U$ such that $x \in V$, $[V] \subset U$ with $d_I(\partial[V]) \leq n - 1$.
(iii) $d_{\rm I}(B) = m$, if m is the smallest number satisfying (ii).

Any totally disconnected sets[30] have dimension zero. Therefore, the union of two dimension zero sets can have a positive dimension. This does not happen for the Lebesgue cover dimension (see **2.22**): $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$.

2.22 Lebesgue covering dimension

Consider an open cover $\mathcal{A}$ of a topological space X. Suppose for any $x \in X$ we can refine $\mathcal{A}$ so that the covering order (= the min number of the open sets in $\mathcal{A}$ covering a point is its covering order) is less than $n+1$. Then the minimum value of n is called the topological dimension of X.[31]

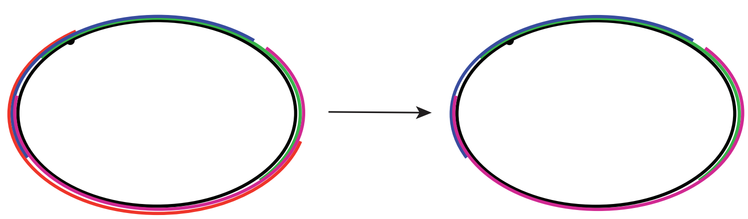

Figure 2.10: How to obtain the Lebesgue covering dimension of S^1

If X and Y are homeomorphic, both have the same topological dimension. Cantor set has dimension zero, because we can alway find a disjoint subcover.

Every compact n-topological space can be embedded in $\mathcal{R}^{2m+1}$ (Whitney's theorem https://en.wikipedia.org/wiki/Whitney_embedding_theorem), which

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[29] There are 'small' and 'large' inductive dimensions, but here only the former is mentioned.

[30] A topological space X is totally disconnected if the connected components in X are the one-point sets.

[31] The illustrations in Wikipedia are wrong (or at best misleading).

will be discussed later.

A problem of this definition is that finding refinements could be daunting.

2.23 Box counting or Minkowski dimension

Let $N_\delta(A)$ be the smallest number of diameter δ sets covering A. If the following limit exists, it is called the box counting (or Minkowski) dimension of A:

$$\dim_M(A) = \lim \frac{\log N_\delta(A)}{-\log \delta}. \tag{2.17}$$

Notice A and $[A]$ have the same $\dim_M$. This is not a very desirable feature, but since it is easy to compute and in many cases it agrees with the following Hausdorff dimension **2.25**. Also an unpleasant example is $\dim_M(\{1/n\}_{n\in\mathbb{N}}) = 1/2$.

2.24 Hausdorff measure

The s-dimensional Hausdorff measure of a set $A \subset \mathbb{R}^n$ is defined by

$$H^s(A) = \lim_{\delta\to 0} H^s_\delta(A), \tag{2.18}$$

where

$$H^s_\delta(A) = \inf_{\mathcal{U}} \left\{ \sum_i \operatorname{diam}(U_i)^s \,\middle|\, U_i \in \mathcal{U}(A) \right\} \tag{2.19}$$

with $\mathcal{U}(A)$ being the open cover of A.

H^s is a decreasing function of s.

2.25 Hausdorff dimension[32]

The Hausdorff dimension of a set A is defined as

$$\dim_H(A) = \sup_s \{s \mid H^s(A) = \infty\} = \inf_s \{s \mid H^s(A) = 0\}. \tag{2.20}$$

The H-dimension of a totally disconnected set is less than 1.

If A is self similar, then $\dim_M(A) = \dim_H(A)$.

[32] B. Simon *Real Analysis* (A comprehensive course in analysis, Part 1.) (AMS 2015) 8.2 is much more detailed.

2.26 Hausdorff dimension of Cantor set

The Hausdorff dimension of the middle third Cantor set is $\log 2/\log 3$. This can be confirmed due to the self-similarity of the Cantor set.

Let us consider the middle one ninth Cantor set. The first step removed 1/9. The next step removes 1/9 of the remaining segments. Thus, $(8/9)^n \to 0$ that is, the resultant set is not fat. As is clear, self-similar Cantor set is measure zero. However, intuition tells us that the middle 1/9th Cantor set should be 'larger' than that of the middle third Cantor set. Thus, the Hausdorff dimension is useful.[33]

2.27 Fractals

A geometric object whose topological dimension is different from its Hausdorff dimension is called a fractal object. A typical examples is the von Koch curve. Its Hausdorff dimensiion is $\log 4/\log 3$.

[33] ⟪**Hausdorff**⟫ Felix Hausdorff (1868-1942) did his work on Hausdorff dimension in 1919. As a World War I veteran, he could avoid the laws against Jews, but by the late 1930s he was dismissed. He became concerned about being shipped to the camps. On the night of Jan 25, 1942, having learned they were to be picked up the next day to be sent to the Endenich camp, Hausdorf, his wife and his wife's sister committed suicide by overdose of barbital [B Simon p49 + Wikipedia).

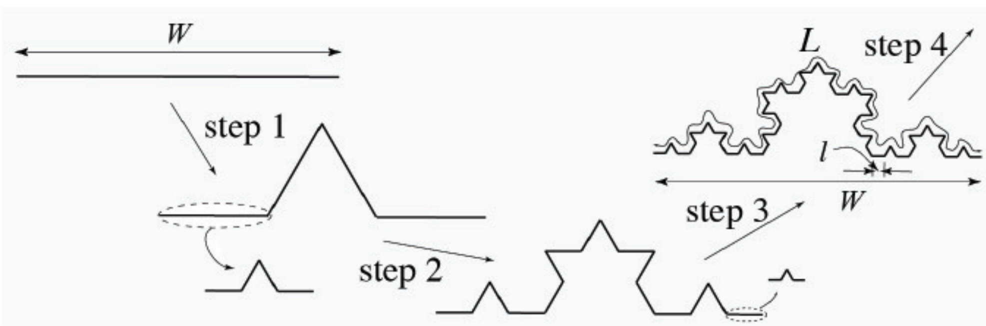

Fig. 3.4 How to construct the von Koch curve. First, take a line segment of length W and make a segment of length 1/3 of the former. Prepare four such segments and make a piecewise linear figure with a triangular mound at the center. This is Step 1. Next, each linear segment of this figure (a representative is encircled with a broken ellipse) is replaced by a 1/3 shrunk copy of the whole figure. This is Step 2. The resultant figure at the bottom of the figure is made of unit segments (monomer units) of length $\ell = W/9$. These length $W/9$ monomer units are all replaced by the small copy of the figure constructed in Step 1 as shown just below the Step 3 arrow. After Step 3 the length of the 'monomer unit' is $W/3^3$. Now, we repeat this procedure *ad infinitum* beyond Step 4, and we will obtain the von Koch curve. In this section, we regard the length of the 'monomer unit' to be finite, so we repeat these steps only finite times. However, since the number of repetition n is large, the length of the monomer unit $\ell = W/3^n$ is invisibly small.

Figure 2.11:

2.28 Residual property
A nowhere dense set E is defined by $\overline{E}^{\circ} = \emptyset$. A meager set is a set described as a countable union of nowhere dense sets. A set A is *almost open* (or Baire set) if there is an open set such that $A \Delta U$ is meager.

A property that holds on Ω except for a meager set is called a residual property. Needless to say, a residual set that is $\Omega\setminus$ a meager set (i.e., the complement of a meager set) is 'smaller' than an open dense set. A residual set can be measure zero (but an open set is always with positive measure).

2.29 Ultimate conjectures on discrete dynamical systems due to Palis [no explanation given here]
Palis conjectured:

> Every diffeomorphism in $\mathrm{Diff}^1(M)$ can be approximated by an Axiom A diffeomorphism or else by one exhibiting a homoclinic bifurcation involving a homoclinic tangency or a cycle of hyperbolic periodic saddles with different indices.

For physicists, the following Milnor-Palis conjecture may be more interesting:[34]

> For a typical smooth dynamical system $f : M \to M$, the global attractor A_f is decomposed into finitely many minimal attractors A_i. Moreover, for almost every point $x \in M$, the ω-limit set $\omega(x)$ is equal to one of the A_i. Typically each minimal attractor supports a unique SRB measure μ that governs behavior of Lebesgue almost all points $x \in M$. The latter means that as $n \to \infty$
>
> $$\frac{1}{n}\sum_{i=0}^{n-1}\phi(f^k x) \to \int \phi(x)d\mu(x) \tag{2.21}$$
>
> for any continuous function $\phi \in C(M)$.

Here 'typical'[35] means:
Some property is considered to be typical if it is satisfied for almost all parameters

[34] in *Abel Prize 2008-2012* by M Lyubich.
[35] The appropriate probabilistic notion (in infinitely dimensional space of systems) goes back to

in a generic one-parameter family of systems.

For one-dimensional unimodal analytical maps the conjecture was proved.[36]

For example, it is still conceivable that from probabilistic point of view, the Newhouse phenomenon is negligible.

[36]Lyubich, M.: Dynamics of quadratic polynomials, Ill. Parapuzzle and SBR measure. 11 Géométric complexe et systémes dynamiques. Asterisque Volume in Honor of Douady's 60th Birthday, vol. 261, pp. 173- 200 (2000); Lyubich, M.: Almost every real quadratic map is either regular or stochastic. Ann. Math. 156, 1-78 (2002); Avila, A., Lyubich, M., de Melo, W.: Regular or stochastic dynamics in real analytic families of unimodal maps, Inv Math 154 451 (2003).