29 Lecture 29. Measure-theoretical dynamical systems

29.1 Why measure-theoretical dynamical systems?

We have realized that even for a single dynamical system (that is defined by a definite law or rule) its behavior depends on initial conditions. Thus, even if we are told that the system exhibit periodic movements or apparently random (that is, chaotic) behaviors, we may not be lucky enough to observe them. How lucky can we be? That is a question of probability, and probability is a 'measure' of our confidence.

For example, for a continuous time dynamical system $\mathcal{X}^r(M)$ defined on M, a trajectory starting from $x_0 \in M$ at $t = 0$ $\varphi_t(x_0)$ depends on x_0. Suppose the system exhibits an unstable periodic orbit, you can observe it only when x_0 is on the orbit. What is your chance to observe it? If we throw a dart 'randomly' on M, you would expect that you would never hit the orbit. However, if there is some device channeling the darts on to the orbit, with probability one, you will observe it. If x_0 is in the basin of attraction of a stable periodic orbit and the basin is of positive Riemann volume defined on M, then we may have a finite (not infinitesimal) chance to observe the periodic motion. Even in this case if there is a fixed point to whose basin of attraction all the initial conditions are channelled, there will be no chance to observe the stable periodic orbit.

These trivial considerations clearly tell us that it is very important (especially for physicists) to specify how we can sample the initial condition. This is the question of the sampling measure.

Once the system settles down to a steady state, that is, the system is in a certain attractor, then we may be interested in its average behavior. For example, in the case of a chaotic attractor, we could ask how nearby orbits leave each other on the average (that is, we ask 'how chaotic' the system is). Inevitably we need a measure to average the behavior.

Thus, if we ask about observability and about the average behavior, we need (probability) measures.

29.2 Measure

What is a measure? See Appendix. You should be able to answer the question: What is the area?

29.3 Remark on family of measurable sets

If you read a respectable book on probability you will see a setup of probability space $(M, \mathcal{F}, \mu)$, where μ is a (probability) measure defined on M and $\mathcal{F}$ is a set of all the measurable sets on which μ is defined. Not all the subsets of M can be assumed to have 'probability.' This restriction is a must under the usual axioms of mathematics.

Therefore, when we consider a measure-theoretical dynamical system on M with an invariant measure the 'dynamical rule' (say, the map defining a discrete time dynamical system) must be compatible with (preserved) $\mathcal{F}$. When we say two dynamical systems are isomorphic (= equivalent), then there must be a correspondence (modulo μ-measure zero sets) between the families of measurable sets for these dynamical systems.

However, in this exposition we will not mention about $\mathcal{F}$.

29.4 Invariant measure: introduction

Prepare numerous clones of the dynamical system under consideration, and plot their states in the same phase space. Then, we would see a cloud of points that describe individual clones in the phase space (Fig. 29.1).

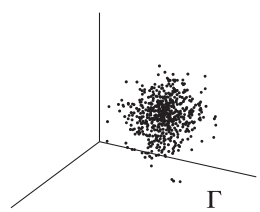

Figure 29.1: Distribution or ensemble. The space is the phase space Γ. Each point represents an instantaneous state of each system. An invariant measure corresponds to a steady state for which the cloud as a whole becomes time-independent, although individual points may keep wandering around.

The distribution described by the cloud, if the total mass is normalized to be unity, may be understood as a probability measure on the phase space. Following the time evolution of the system, the cloud may change its shape.

After a sufficiently long time, often the cloud representing an ensemble ceases to change its shape. Then, we say that the ensemble has reached its steady state. Individual points corresponding to the members of the ensemble may still keep wandering around in the cloud, but the cloud as a whole is balanced and the distribution on the phase space becomes invariant. The (probability) distribution corresponding to this invariant cloud is called an *invariant measure* of the dynamical system.

29.5 Invariant measure of dynamical system: discrete time

Discrete time case: Measure μ on the phase space M is an invariant measure of a dynamical system $T \in C^r(M)$, if

$$\mu = \mu \circ T^{-1} \tag{29.1}$$

holds. That is, for an arbitrary (μ-measurable) subset $A \subset M$

$$\mu(A) = \mu(T^{-1}A) \tag{29.2}$$

holds, where $T^{-1}A$ is the totality of points that comes to A after a unit time step (Fig. 29.2).

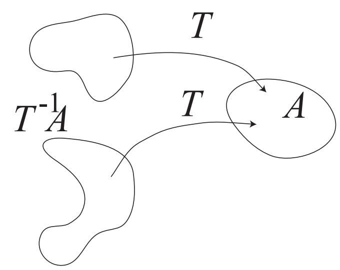

Figure 29.2: The preimage $T^{-1}A$ of A by the time evolution operator T need not be connected (in this figure it consists of two connected components). Since $T(T^{-1}A) = A$, $T^{-1}A$ is the totality of the points that come to A after one time step. As you see from the figure, unless T is invertible, $\mu = \mu \circ T$ does not guarantee the invariance of μ.

For an arbitrary measurable set A, if we know $T^{-1}A$ at present, we know everything we can discuss probabilistically that will occur after one time step. (29.2) expresses the condition that the evaluation of the weights of A and $T^{-1}A$ by μ is consistent with the conservation law of th members of the ensemble.

29.6 Invariant measure of dynamical system: continuous time

Continuous time case: Measure μ on the phase space M is an invariant measure of a dynamical system φ_t defined by a vector field $\mathcal{X}(M)$, if

$$\mu = \mu \circ \varphi_t^{-1} \tag{29.3}$$

holds for all t. That is, for an arbitrary (μ-measurable) subset $A \subset M$

$$\mu(A) = \mu(\varphi_t^{-1}A) \tag{29.4}$$

holds (quite analogous to the discrete time case).

29.7 Invariant measures are not unique for a given dynamical system

Generally speaking, an invariant measure for a given dynamical systems is not unique (as already suggested in **29.1**). For example, $Tx = \{2x\}$ discussed previously has actually uncountably many distinct invariant measures.

For a Hamiltonian system we know the phase volume is invariant, but there are many other invariant measures. Furthermore, not all the invariant measures can give 'natural averages' that agree with averages obtained by continuous even observations.[328]

29.8 Measure-theoretical dynamical system

Let μ be an $T \in C^r(M)$-invariant measure on M. The triplet (T, μ, M) is called a *measure theoretical dynamical system*. This may be interpreted as a mathematical expression of a steady state allowed to the dynamical system $T \in C^r(M)$; given a dynamical system $T \in C^r(M)$, for each invariant measure μ, a distinct measure theoretical dynamical system is constructed.

For a continuous time dynamical system defined by $X \in \mathcal{X}^r(M)$, (X, μ, M) may be understood as a measure-theoretical dynamical system.

29.9 Absolutely continuous invariant measure

If a measure μ satisfies $\mu(A) = 0$ for any Lebesgue-measure zero set[329] A, μ is called an absolutely continuous measure. For a dynamical system with an absolutely continuous invariant measure, chaos may often be observed by numerical experiments.

If a measure is absolutely continuous, its probability density g may be defined as $d\mu = gd\lambda$, where λ is the Lebesgue measure.[330]

Absolutely continuous invariant measures are observable (see **22.3**) (e.g., computationally).

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[328] As we will see later, invariance does not mean ergodicity.

[329] More precisely, we should say 'for any set whose Riemann volume is zero.' Riemann volume is the volume based on the Riemann metric = the usual length. It seems that the volume based on the length is the most natural measure for us.

[330] That is, g is the Radon-Nikodym derivative $d\mu/d\lambda$. See, e.g., Kolmogorov and Fomin cited already.

29.10 Ergodic measure

For any invariant set $A \subset M$ (that is, any set satisfying $A = T^{-1}A$[331]), if $\mu(A) = 0$ or 1, the measure theoretical dynamical system is said to be *ergodic*.

Note that this is neither the property of the measure nor the property of the dynamicarule T.

Roughly speaking, if a dynamical system is ergodic, the trajectory starting from any initial point[332] can eventually go into any μ-measure positive set.

In fact, if there is a positive measure set C to which any trajectory starting from a certain positive measure set D cannot reach, then there must be an invariant set B with $0 < \mu(B) < 1$ such that $B \supset C$ and $B^c \supset D$.

Conversely, if there is an invariant set B such that $0 < \mu(B) < 1$. Then, we can choose a measure positive set C with $B \cap C = \emptyset$ such that any trajectory starting from C never visits B [if possible, $\mu(\cup_{k=1}^{\infty} T^{-k}B \cap C) > 0$ (note that $T^{-k}B \cap C$ means the points now in C and will be in B after k), but $T^{-1}B = B$, so $B \cap C \neq \emptyset$, a contradiction].

Let Γ be a unit circle, and f be a rigid rotation around the center by an angle that is irrational multiple of 2π. Since the uniform distribution μ_U on the circle is rotation-invariant, we can make a measure-theoretical dynamical system (f, μ_U, Γ), which is ergodic. This not so interesting example illustrates why ergodicity is totally insufficient for modeling irreversible phenomena such as relaxation exhibited by usual many-body systems.

29.11 Mixing measure

For any sets[333] A and B (both $\subset M$), if

$$\lim_{n \to \infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B) \tag{29.5}$$

holds, the measure-theoretical dynamical system is said to be *mixing*, where T^{-n} implies to apply T^{-1} n-times. $T^{-n}A$ is the set that agrees with A after n time steps. $T^{-n}A \cap B$ is the totality of the points that is at present in B and will be in A after n time steps. Intuitively, the cloud of points starting from B spread over the phase space evenly if n is sufficiently large, so the probability for the cloud to overlap with A is proportional to the 'statistical weight' of A. It is obvious that initial conditions

[331] This condition must be, precisely speaking, the equality ignoring the μ-measure zero difference.

[332] Precisely speaking, starting from μ-almost all initial points. 'μ-almost all' implies that all except for μ-measure zero sets.

[333] Precisely speaking, any μ-measurable sets. From now on, such statements will not be added.

cannot be used to predict the future. Physically, irreversible processes such as relaxation phenomena occur. In particular, the time-correlation function eventually decays to zero.

Appendix: What is measure[334]

The concept of measure does not appear in elementary calculus, but it is a fundamental and important concept. It is not very difficult to understand, since it is important. Besides, the introduction of the Lebesgue measure by Lebesgue is a good example of conceptual analysis, so let us look at its elementary part. A good introductory book for this topic is the already quoted Kolmogorov-Fomin. It is desirable that those who wish to study fundamental aspects of statistical mechanics and dynamical systems have proper understanding of the subject.

29.12 What is the volume?

For simplicity, let us confine ourselves to the two dimension. Thus, the question is: what is the area? Extension to higher dimensions should not be hard. If the shape of a figure is complicated, whether it has an area could be a problem,[335] so let us begin with an apparently trivial case.

"The area of the rectangle $[0, a] \times [0, b]$ is ab."

Is this really so? If so, why is this true? Isn't it strange that we can ask such a question before defining 'area'? Then, if we wish to be logically conscientious, we must accept the following definition:

Definition. The area of a figure congruent to the rectangle $\langle 0, a\rangle \times \langle 0, b\rangle$ (here, '$\langle$' implies '[' or '(', '$\rangle$' is ']' or ')', that is, we do not care whether the boundary is included or not) is defined as ab.

Notice that the area of a rectangle does not depend on whether its boundary is included or not. This is already included in the definition.

29.13 The area of a fundamental set

A figure made as the direct sum (that is, join without overlap except at edges and vertices) of a finite number of rectangles (whose edges are parallel to the coordinate axes and whose boundaries may or may not be included) is called a fundamental set (Fig. 29.3). It should be obvious that the join and the product (common set) of two fundamental sets are both fundamental sets. The area of a fundamental set is defined as the total sum of the areas of the constituent rectangles.

29.14 How to define the area of more complicated figures; a strategy

For a more complicated figure, a good strategy must be to approximate it by a sequence of fundamental sets allowing increasingly smaller rectangles. Therefore, following Archimedes, we approximate the figure from inside and from outside (that is, the figure is approximated by a sequence of fundamental sets enclosed by the figure and by a sequence of fundamental sets enclosing the figure). If the areas of the inside and the outside approximate sequences

[334] Taken from TNW Chapter 2 Appendix

[335] (Under the usual axioms of mathematics) we encounter figures without areas.

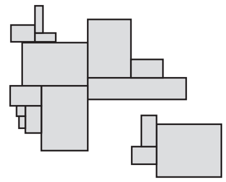

Figure 29.3: Fundamental set: it is a figure made of a finite number of rectangles whose edges are parallel to a certain Cartesian coordinate axes and the rectangles do not overlap except at edges and vertices. Its area is the total sum of the areas of the constituent rectangles.

agree in the limit, it is rational to define the area of the figure by the limit.

Let us start from outside.

29.15 Outer measure

Let A be a given bounded set (that is, a set that may be enclosed in a sufficiently large disk). Using a finite number of (or countably many) rectangles P_k $(k = 1, 2, \cdots)$, we cover A, where the boundaries of the rectangles may or may not be included, appropriately. If $P_i \cap P_j = \emptyset$ $(i \neq j)$ and $\cup P_k \supset A$, $P = \{P_k\}$ is called a finite (or countable) cover of A by rectangles (Fig. 29.4O). Let the area of the rectangle P_k be $m(P_k)$. We define the outer measure $m^*(A)$ of A as follows:

$$m^*(A) \equiv \inf \sum_k m(P_k). \tag{29.6}$$

Here, inf[336] is taken over all the possible finite or countable covers by rectangles.

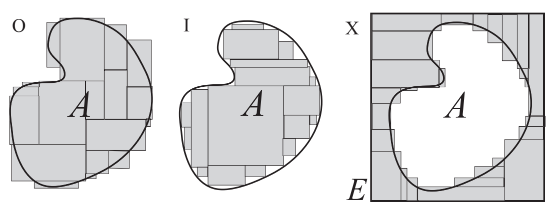

Figure 29.4: Let A be the set enclosed by a closed curve. O denotes a finite cover by rectangles. If there is an area of A, it is smaller than the sum of the areas of these rectangles. The outer measure is defined by approximating the area from outside. In contrast, the inner measure is computed by the approximation shown in I by the rectangles included in the figure A. In the text, by using a large rectangle E containing A, $E \setminus A$ is made and its outer measure is computed with the aid of finite covers; the situation is illustrated in X. The relation between I and X is just the relation between negative and positive films. If the approximation O from outside and the approximation I from inside agree in the limit of refinement, we may say that A has an area. In this case, we say A is measurable, and the agreed area is called the area of A.

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[336] The infimum of a set of numbers is the largest number among all the numbers that are not larger than any number in the set. For example, the infimum of positive numbers is 0. As is illustrated by this example, the infimum of a set need not be an element of the set. When the infimum is included in the set, it is called the minimum of the set. The above example tells us that the minimum need not exist for a given set.

29.16 Inner measure

For simplicity, let us assume that A is a bounded set. Take a sufficiently large rectangle E that can enclose A. Of course, we know the area of E is $m(E)$. The inner measure of A is defined as[337]

$$m_*(A) = m(E) - m^*(E \setminus A). \tag{29.7}$$

It is easy to see that this is equivalent to the approximation from inside (Fig. 29.4I). Clearly, for any bounded set A $m^*(A) \geq m_*(A)$ holds.

29.17 Area of figure, Lebesgue measure

Let A be a bounded set. If $m^*(A) = m_*(A)$, A is said to be a measurable set (in the present case, a set for which its area is definable) and $\mu(A) = m^*(A)$ is called its area (2-dimensional Lebesgue measure).

At last the area is defined. The properties of a fundamental set we have used are the following two:
(i) It is written as a (countable) direct sum of the sets whose areas are defined.
(ii) The family of fundamental sets is closed under $\cap$, $\cup$ and $\setminus$ (we say that the family of the fundamental sets makes a set ring.[338])
An important property of the area is its additivity: If P_i are mutually non-overlapping rectangles, $\mu(\cup P_i) = \sum \mu(P_i)$. Furthermore, the σ-additivity for countably many summands also holds.[339]

Notice that such a summary as that the area is a translationally symmetric σ-additive set-theoretical function which is normalized to give unity for a unit square does not work, because this does not tell us on what family of sets this set-theoretical function is defined.[340] The above summary does not state the operational detail about how to measure the areas of various shapes, so no means to judge is explicitly given what figures can be measurable. Lebesgue's definition of the area outlined above explicitly designates how to obtain the area of a given figure.

29.18 General measure (abstract Lebesgue measure)

The essence of characterization of the area is that there is a family of sets closed under certain 'combination rules' and that there is a σ-additive set-theoretical function on it. Therefore, we start with a σ-additive family $\mathcal{M}$ consisting of subsets of a set X: A family of sets satisfying the following conditions is called a σ-additive family:
(s1) $X, \emptyset \in \mathcal{M}$,
(s2) If $A \in \mathcal{M}$, then $X \setminus A \in \mathcal{M}$,

[337] $A \setminus B$ in the following formula denotes the set of points in A but not in B, that is, $A \cap B^c$.

[338] More precisely, that a family $\mathcal{S}$ of sets makes a ring implies the following two:
(i) $\mathcal{S}$ includes $\emptyset$,
(ii) if $A, B \in \mathcal{S}$, then both $A \cap B$ and $A \cup B$ are included in $\mathcal{S}$.

[339] Indeed, if $A = \cup_{n=1}^{\infty} A_n$ and A_n are mutually exclusive (i.e., for $n \neq m$ $A_n \cap A_m = \emptyset$), for an arbitrary positive integer N $A \supset \cup_{n=1}^{N} A_n$, so $\mu(A) \geq \sum_{n=1}^{N} \mu(A_n)$. Taking the limit $N \to \infty$, we obtain $\mu(A) \geq \sum_{n=1}^{\infty} \mu(A_n)$. On the other hand, for the external measure $m^*(A) \leq \sum_{n=1}^{\infty} m^*(A_n)$, so $\mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n)$.

[340] If we assume that every set has an area, under the usual axiomatic system of mathematics, we are in trouble. See Banach-Tarski's theorem (Discussion 2.4A.1).

(s3) If $A_n \in \mathcal{M}$ $(n = 1, 2, \cdots)$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{M}$.
$(X, \mathcal{M})$ is called a measurable space. A nonnegative and σ-additive set-theoretical function m defined on a measurable set that assigns zero to an empty set is called a measure, and $(X, \mathcal{M}, m)$ is called a measure space. Starting with this measure m, we can define the outer measure on a general set $A \subset X$, mimicking the procedure already discussed above. The inner measure can also be constructed. When these two agree, we can define a set-theoretical function μ as $\mu(A) = m^*(A)$, and we say A is μ-measurable. Thus, we can define μ that corresponds to the Lebesgue measure explained above in the context of the area. μ is called the Lebesgue extension of m (this is called an abstract Lebesgue measure, but often this is also called a Lebesgue measure). This construction of μ is called the completion of m. In summary, if $(X, \mathcal{M}, m)$ is a measure space, we define a new family of subsets of X based on $\mathcal{M}$ as

$$\overline{\mathcal{M}} = \{A \subset X : \exists B_1, B_2 \in \mathcal{M} \text{ where } B_1 \subset A \subset B_2, m(B_2 \setminus B_1) = 0\}, \tag{29.8}$$

and if μ is defined as $\mu(A) \equiv m(B_2)$ for $A \in \overline{\mathcal{M}}$, $(X, \overline{\mathcal{M}}, \mu)$ is a measure space, and is called the completion of $(X, \mathcal{M}, m)$.[341]

The final answer to the question, "What is the area?" is: the area is the completion of the Borel[342] measure, where the Borel measure is the σ-additive translation-symmetric measure that gives unity for a unit square and is defined on the Borel family of sets which is the smallest σ-additive family of sets including all the rectangles. Generally speaking, a measure is something like a weighted volume. However, there is no guarantee that every set has a measure (μ-measurable). It is instructive that a quite important part of the characterization of a concept is allocated to an 'operationally' explicit description (e.g., how to measure, how to compute). Recall that Riemann's definition of the integral was based on this operational spirit, so it can immediately be used to compute integrals numerically.

[341]The completion is unique. In a complete measure space, if A is measure zero ($\mu(A) = 0$), then its subsets are all measure zero. Generally, a measure with this property is called a complete measure. Completion of $(X, \mathcal{M}, m)$ may be understood as the extension of the definition of measure m on the σ-additive family generated by all the sets in $\mathcal{M}$ + all the measure zero set with respect to m.

[342]E. Borel (1871-1956)