28 Lecture 29 Horseshoe

28.1 Horseshoe dynamical system

The horseshoe dynamical system is constructed as follows:
First a map $g : D \to D$ in Fig. 28.1 is constructed that maps the (yellow) square ABCD to a horseshoe overlapping the original square (which we will call R):

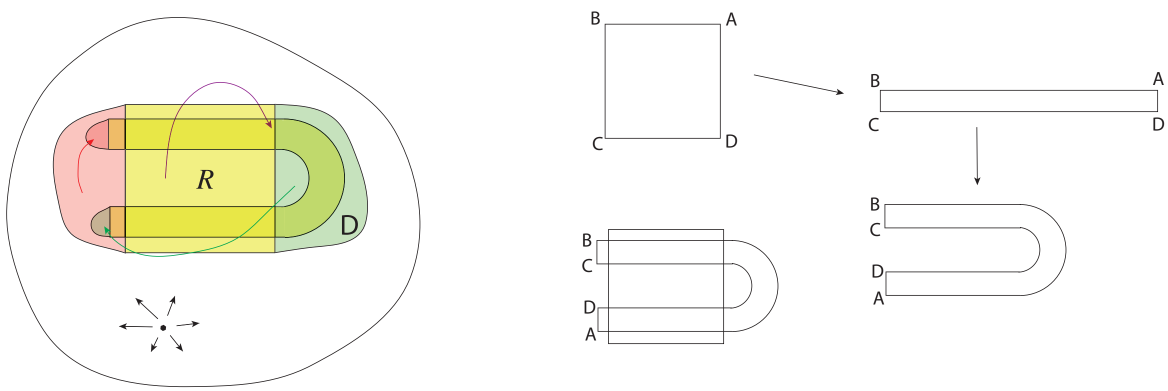

Figure 28.1: Construction of horseshoe

Then, prepare one more disk and complete S^2. Thus as an example of $C^r(S^2, S^2)$ the horseshoe dynamical system has been constructed. Let us write this (η, S^2).

There is a unique sink in the red semidisk in Fig. 28.1.

The inverse map η^{-1} is also a horseshoe map: $\eta^{-1}(R)$ (R = yellow square in Fig. 28.1) has again a horseshoe shape. Can you illustrate this?

28.2 Where do horseshoes appear naturally?

The construction of the horseshoe system might look rather artificial, but horseshoes appear 'everywhere' when we have homoclinic points as illustrated in Fig. 28.2.

28.3 All the nontrivial nonwandering points are in R

By construction, all the points outside R is attracted to the sink of η. If we reverse time, all the points outside R is attracted to the source in Fig. 28.1. Therefore, all

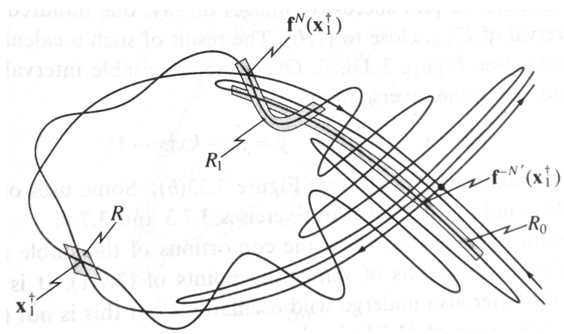

Figure 28.2: Horseshoes can appear naturally when there is a homoclinic point, [Fig. 3.32 of AP]

the other nonwandering points must be in R.

Therefore, all other nonwandering points must be in

$$\Lambda = \cap_{n \in \mathbb{Z}} \eta^n(R). \tag{28.1}$$

28.4 Nontrivial nonwandering set of horseshoe

To understand the structure of Λ let us consider how R is successively mapped onto R. We can understand the preimage of the blue rectangles as follows (Fig. 28.3).

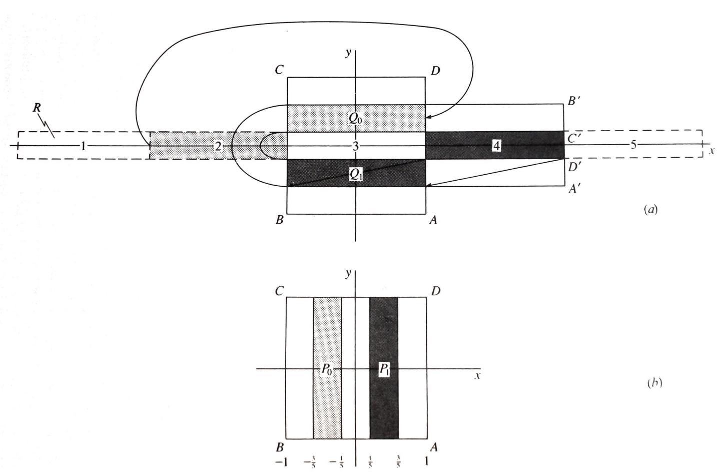

Figure 28.3: η on R. P_a is mapped to Q_a [Fig. 3.13 of AP]

If we repeat mapping we have Fig. 28.4.

Thus, $\vee_{n \in \mathbb{N}} Q^{(n)}$ consists of local unstable manifolds of points in Λ (See Fig. 28.4).

We know η^{-1} is also a horseshoe (90° rotated). Therefore, we can repeat the above argument in the reverse time direction to construct (local) stable manifolds of points

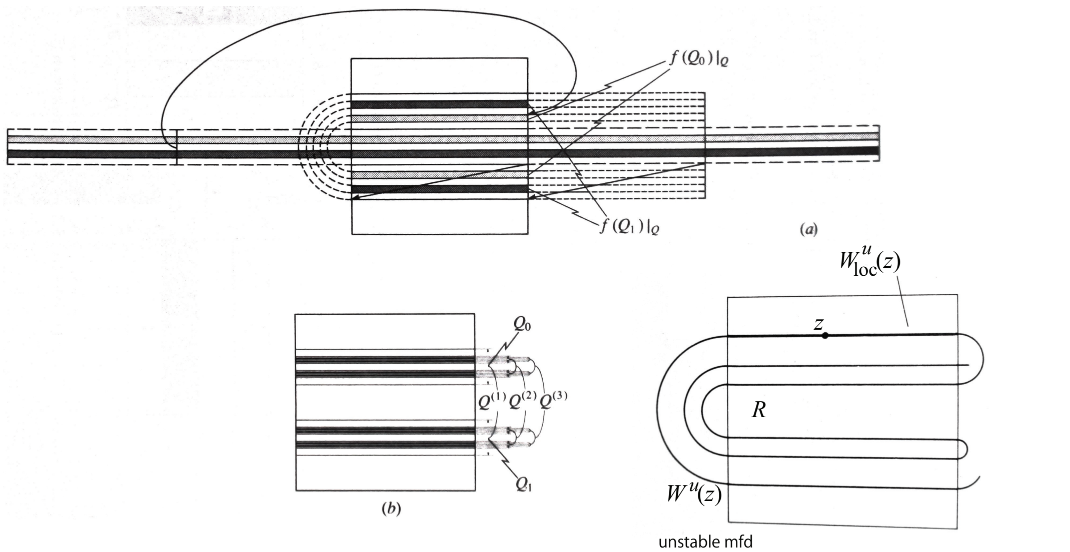

Figure 28.4: η, η^2 and η^3 on R. Here, $Q_0 \cup Q_1$ is $Q^{(1)}$, $\eta(Q_0 \cup Q_1) \cap R$ is $Q^{(2)}$ and $\eta^2(Q_0 \cup Q_1) \cap R$ is $Q^{(3)}$. [Fig. 3.14 of AP]

in Λ; $Q^{(-1)}, Q^{(-2)}, \cdots$, instead (Fig. 28.5)

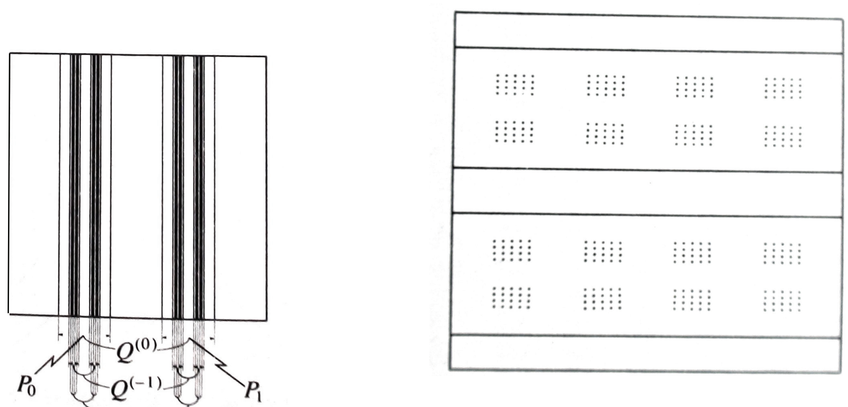

Figure 28.5: Left: η^{-1}, η^{-2} ... on R. As Fig. 28.4 we can make $Q^{(-1)}, Q^{(-2)}, \cdots$. Right: Λ[Fig. 3.15c of AP, 8.5 of Yano]

The invariant set of the horseshoe consists of a source, a sink (in the red region in Fig. 28.1) and a Cantor set Λ.
https://www.youtube.com/watch?v=ItZLb5xI_1U&frags=pl%2Cwn illustrates the horseshoe invariant set (perhaps you feel the pace of explanation a bit too slow).

The horseshoe is structurally stable as can be guessed from the following illustration (Fig. 28.6)

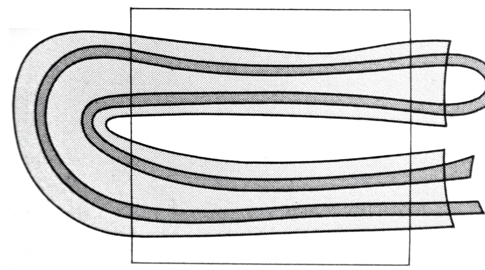

Figure 28.6: The structure of the horseshoe is quite stable [Fig. 8.6 of Yano]

https://www.youtube.com/watch?v=2aeFG5YN_mk&frags=pl%2Cwn Smale talks about his dynamical system study

28.5 Symbolic dynamics of horseshoe

As can be clear from the structure of Λ and its construction, each point in Λ must be coded uniquely in terms of an element in $\{0,1\}^{\mathbb{Z}}$. Thus $\eta|_\Lambda$ is (as a measure-theoretical dynamical system) isomorphic to $B(1/2,1/2)$. Thus the horseshoe system is not Morse-Smale.

Actually $\eta|\Lambda$ is an Anosov system, so it is Ω-stable.