21 Lecture 21. Strange attractors

21.1 Landau's view of turbulence

Turbulence in fluid dynamics is a phenomenon in which all the space-time scales of apparently random fluctuations simultaneously appear in the fluid motion. It is believed that this is governed by the (incompressible) Navier-Stokes equation, a deterministic system.[209] Naturally, Landau considered the problem as the couplings between numerous destabilized modes (Fourier components of the velocity vector). Let A be a complex mode. It obeys a time-dependent Landau-Ginzburg equation

$$\frac{dA}{dt} = \gamma A - aA|A|^2 + Q, \tag{21.1}$$

where γ changes its sign when the mode linearly destabilizes, a is a positive constant and Q denotes the mode coupling terms. Thus, Landau and many statistical physicists thought the problem is closely related to critical phenomena, only the cascade of the driving proceed in the opposite direction, from large (stirring scale) to small (viscous dissipation scale).[210] The resultant quasi periodic motion with numerous modes of different frequencies is Landau's interpretation of turbulence.

21.2 Ruelle and Takens point of view[211]

Ruelle and Takens pointed out that quasi periodic motion is not the usual 'chaotic' motion we observe in dissipative systems, so Landau's picture must be wrong. They pointed out that if the dimension of dynamics is too low, no complicated motion is generically possible (quoting Peixoto's theorem which will be discussed later). They pointed out that in any neighborhood of a parallel flow on T^4 (i.e., 4 mode quasi periodic motion) there is a flow with a 'strange attractor.'[212] An example may be

[209] We will not discuss this equation. Its derivation from particle mechanics is not mathematically justifiable generally, and we do not know whether the equation is well-posed or not. Furthermore, its mathematical nature is quite sensitive to seemingly benign modifications. For example, if the viscosity is velocity gradient dependent (as physically natural), then the unique existence of the solution of the modified Navier-Stokes equations may be proved almost trivially.

[210] It turned out that the energy flow in the opposite direction (causing intermittency) turned out to be crucial as well.

[211] D Ruelle and F Takens, On the nature of turbulence, CMP 20 167 (1971). D. Ruelle, *Chance* tells us that they could not publish the paper (by rejections), so he decided to publish it to the journal for which he was on the editorial board.

[212] The original proposal is an attractor that is not a manifold, and its cross section is a Cantor set.

constructed as follows.

Make a diffeomorphism that maps a solid 2-torus (donut) into itself as illustrated in Fig. 21.1.

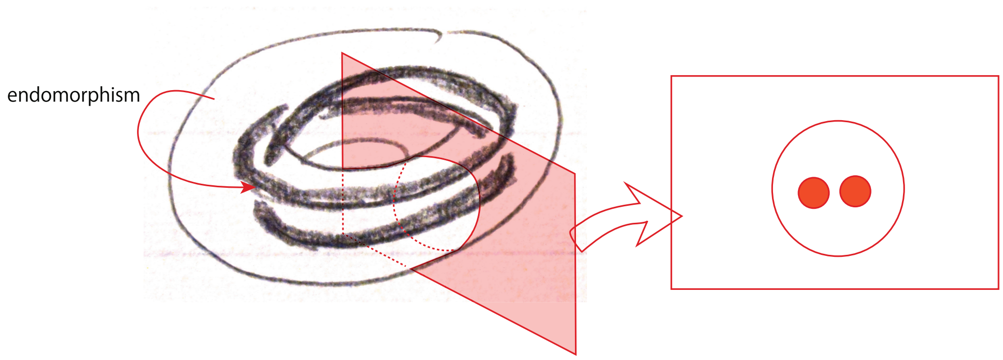

Figure 21.1:

Then, we can suspend (recall Ambrose-Kakutani **17.2**) this diffeomorphism as a flow in 4-space. By a local surgery of the parallel flow on T^4, we can embed this suspended flow into the flow on T^4 smoothly. Notice that this surgery can be as local as one wishes (i.e., the donut in Fig. 21.1 can be indefinitely small). We may conclude that in any neighborhood of T^4 quasi periodic flow is a flow with an attractor that is neither a fixed point nor a periodic orbit.

21.3 3-flow is enough to have "strange attractors"[213]

If a surgery of T^3 parallel flow (3-mode quasi periodic flow) exists, then strange attractors can exist in 3-space flows. Therefore, to show this, the key point is to make a 2D 'analogue' of Fig. 21.1, say, from a disk into itself. Such a map had been constructed by Plykin[214] (Fig. 21.2[215]):

[213] S Newhouse, D Ruelle, and F Takens, Occurrence of strange axiom A attractors near quasi periodic flows on T^m, $m \geq 3$, CMP 64 35 (1978).

[214] R V Plykin Source and sink of A-diffeomorphism of surfaces, Math Sbor 94 233 (1974).

[215] Newhouse et al., proposed different examples as well in their paper.

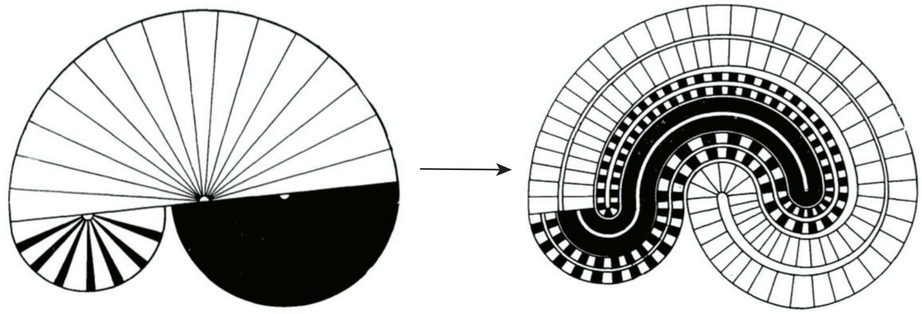

Figure 21.2: Plykin's map from a disk into itself [Fig. 1 of Plykin S Math Sbor 94 233 (1974)]

21.4 Can we really find 'chaos' indefinitely close to 3 or 4 mode quasiperiodic systems?

After reading Ruelle-Takens, I numerically studied whether it is easy to find an example, and did not see any positive result (before 1979). Later Grebogi, Ott and Yorke did an extensive study:[216]
"The results reported here suggest that if arbitrarily small smooth perturbations exist which destroy three-frequency quasiperiodicity, then they must have to be very delicately chosen and are thus unlikely to occur in practice. In particular, we believe that for a fixed typical."

21.5 Strange attractors

Here, a definition by Palis and Takens[217] is given.

For a diffeomorphism $\phi : M \to M$ a positive orbit $\{\phi^n(x)\}_{n\in\mathbb{N}}$ $(x \in M)$ is sensitive or chaotic, if there is a positive constant C (> 0) such that for any $q \in \omega(x)$ and for any $\varepsilon > 0$ $\exists n_1, n_2, n \in \mathbb{N}^+$ such that $\|\phi^{n_1}(x) - q\| < \varepsilon$, $\|\phi^{n_2}(x) - q\| < \varepsilon$ and $\|\phi^{n_1+n}(x) - \phi^{n_2+n}(x)\| > C$.

A compact set $A \subset M$ is a strange attractor, if there is an open set U with a measure zero subset $N \subset U$ such that $\forall x \in U \setminus N$, $\omega(x) = A$ and its positive orbit

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[216] C Grebogi, E Ott and J A Yorke, Are Three-Frequency Quasiperiodic Orbits to Be Expected in Typical Nonlinear Dynamical Systems?, PRL 51 339 (1983)

[217] J Palis and F Takens, *Hyperbolicity & sensitive chaotic dynamics at homoclinic bifurcations* (Cambridge studies in advanced mathematics 35, Cambridge UP, 1993) p8-9. A more general discussion can be found in Section 7.2 of this book.

is chaotic.[218]

Perhaps, however, Bunimovich's definition may be easier to understand, although he does not call it a strange attractor and his definition is measure-theoretic:
(1) An invariant set A is an attractor, if there exists a neighborhood U_0 of A ($\subset U_0$) such that $U_t = T_t U_0 \subset U_0$ for $t > 0$ and $\cap_t U_t = A$.
(2) For any absolutely continuous measure μ_0 on U_0, its time evolved version μ_t weakly converges to an invariant measure λ on A that does not depend on μ_0, and $\{T_t, \lambda, A\}$ is mixing.

[218]That is, almost surely the points in U are attracted to A, and their orbits are chaotic. The exception set N is required because even for hyperbolic attractors dense set of points are attracted to periodic orbits.