1 Lecture 1: Introductory overview

1.1 What is a dynamical system?

If the states of a system[1] change in time[2] according to a definite rule, the system is called a dynamical system.

'Dynamics' here follows a definite rule; that is, the state at present (or more generally, up to present) determines the future states.[3] Actually, we discuss only two types of dynamical systems:

A discrete-time dynamical system is defined by a map f from a set M into itself. For an 'initial condition $x_0 \in M$[4] the state x_n at time step $n \in$ (say) $\mathbb{N}$[5] is defined by $f^n x_0 \equiv f(f(\cdots f(x_0)\cdots))$ (there are n f's). $M = [0,1]$, $f = 4x(1-x)$ (the *logistic map*) is a typical example (Fig. 1.1 Left).

A continuous dynamical system is defined by a vector field X on M, and the rule is given by a differential equation[6] (Example Fig. 1.1 Right)

$$\dot{x}(t) = X(x(t)). \tag{1.1}$$

1.2 Topological and measure-theoretical dynamical systems

[1]⟪**System**⟫ A 'system' is a part of the universe (or nature) which is more or less with a sort of integrity (for example, an electronic circuit, a dog, the Earth, the solar system, etc.). The word 'system' consists of 'syn' (together) and 'histanai' (stand) in Greek. Thus, its original meaning is "a set of objects supporting each other to make an entity." Therefore, ideally,

(1) we should be able to distinguish things belonging to the system and those not, and

(2) things belonging to the system interact (have relations) with each other through system-specific interactions; their interactions with the things outside the system may be treated separately.

[2]⟪**Time**⟫ What is 'time'? Although I wish you to think this philosophico-physics question at least on and off, in these lectures, we understand 'time' as the one we understand in common sense: it passes homogeneously and has a direction. Carlo Rovelli, *The order of time* (Riverhead Books 2018) is perhaps the best book so far written by a physicist on the topic, although I do not share some key opinions with the author.

[3]The rule usually gives a deterministic result (unique future), but whether a set of rules allows determinism (= unique future from the data at and/or before present) or not must be checked carefully and may depend on contexts.

[4]x_0 is usually a multidimensional vector, but I will not use the vector notation such as $\boldsymbol{x}_0$.

[5]$\mathbb{N} = \{0, 1, 2, \cdots\}$, nonnegative integers. Other standard number set notations will be used freely: $\mathbb{R}$: real numbers; $\mathbb{Q}$: rational numbers; $\mathbb{Z}$: integers; $\mathbb{C}$: complex numbers.

[6]We will discuss whether this can consistently define a time evolution rule later. This is the fundamental problem of ODE.

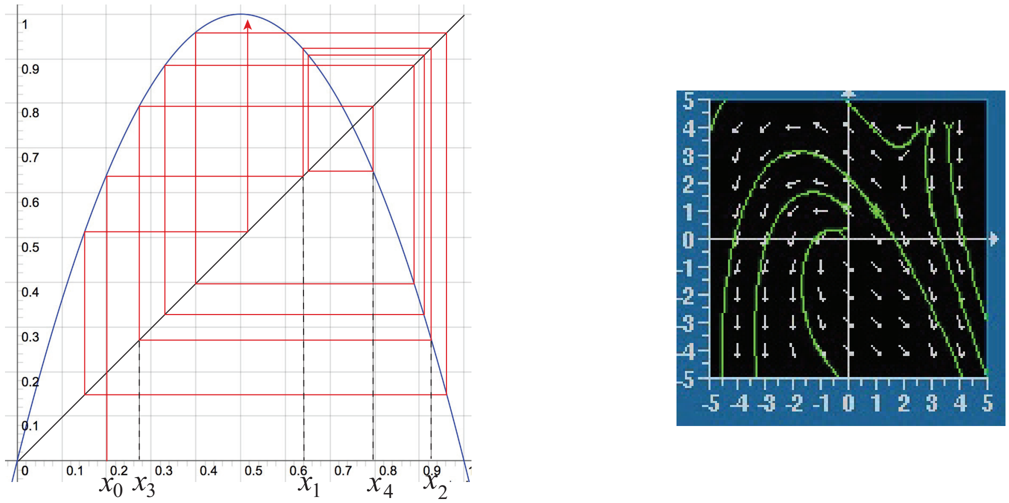

Figure 1.1: Left: The discrete dynamics defined by the logistic map, and how to follow the trajectory $\{x_i\}$. Right: Vector field and its flows for $\dot{x} = x - y$, $\dot{y} = y - x^2$.

Dynamical systems are often classified by their main features we pay attention to, geometrical features of their trajectories (topological dynamical systems) or probabilisitic features (measure-theoretical dynamical systems). When we will discuss the theory of dynamical systems in earnest, we discuss these aspects separately.

1.3 Linear vs nonlinear dynamical systems

The title of the course is 'nonlinear ...,' so we must discuss what 'nonlinear' means.

A system is called a linear system if its observable y depends on the 'control' variable x linearly: the map $y = f(x)$ is linear. f is linear if

$$f(ax_1 + bx_2) = af(x_1) + bf(x_2). \tag{1.2}$$

That is, f is linear, if it satisfies linear scaling $f(ax) = af(x)$ and additivity: $f(x_1 + x_2) = f(x_1) + f(x_2)$. (1.2) is called the superposition principle. If f is not linear, we say the system is nonlinear.[7]

The reader must have heard that nothing strange occurs for linear dynamical systems, but, in contrast, we may encounter strange complicate time evolutions for nonlinear systems. This is largely true, but we will learn that 'strange behaviors (called 'chaos') are caused, in essence, due to (local) 'expansion' in the 'bounded'

[7]A more careful definition may be found in Chapter 1 of YO *The nonlinear world* (Springer, 2012) [henceforth abbreviated as 'TNW'.

phase space. Nonlinearity is required to bound the domain where the states live, if the domain is not naturally bounded (say, $\mathbb{R}^n$). If the domain is intrinsically bounded like an n-torus T^n, chaotic behavior does not require nonlinearity.

The main reason why we are interested in nonlinear dynamical systems is that their behaviors are often unexpected and complicate (e.g., chaos = deterministic unpredictability). However, as we learn, complication is due to local expansion (small errors are magnified) with 'confinement'; if not spatially confined, we would find explosion, which is not usually complicate.

1.4 Nonlinearity means scale interference[8]

Still it is true that nonlinearity is often a key to nontrivial dynamical behavior, because many systems are not geometrically confined. A nonlinear system is a system that is not a linear system as discussed in **1.3**. We know most phenomena in the world are nonlinear,[9] and to characterize this important 'nonlinearity' by the negation of 'linearity' is quite unsatisfactory. We should characterize 'nonlinear systems' more positively rather than through negation of something else.

Thanks to the superposition principle, even if we superpose high-frequency perturbation to a linear system, its behavior (esp., time-coarse-grained behavior) is not affected. However, for a nonlinear system such a guarantee does not exist, because different scales (length and/or time scales) interfere. Typical nonlinear phenomena are often due to such interference. For example, critical phenomena in phase transition are due to the nonlinear coupling of small scale fluctuations producing mesoscale or even macroscale fluctuations. Thus, we may characterize nonlinear systems as systems having scale interferences. We will see that chaos clearly teaches us that the very microscopic scale of the Universe affects our scale sometimes significantly.

1.5 Unknowable affects our lives due to nonlinearity

Since what happens on extreme small scales are never knowable/observable, even if we totally ignore quantum mechanics, our world even on the scale we can directly observe is inevitably riddled with the absolute unknowable.

Perhaps the geometrical study of dynamical systems try to understand what sort of mechansims cause this, and the probabilistic or statistical study of dynamical systems try to understand what is certain despite the unknowable. Thus, statistical and

[8] see Chapter 1 of 'TNW'.

[9] The word 'nonlinear system' is a shorthand for the fact that the aspect of the system we are interested in does not have a description that admits the superposition principle.

topological qualitative studies of dynamical systems are the main goal of the theory of dynamical systems.

1.6 What features should we pay attention to?

As noted in **1.1** we pay attention to topological and probabilistic features of dynamical systems. Although we will discuss some details of concrete examples, we are much more interested in generic (universal) or typical qualitative features of dynamical systems.

Being 'typical' may mean what we can sample from a class of dynamical systems 'casually' without any premeditation. How can we 'formalize' this? This is not so simple, so we will discuss this in a separate section with some mathematical rudiments.

1.7 Where do we encounter dynamical systems?

This may be a rather stupid question, since differential equations appear almost everywhere in physics. Thus, we physicists are very familiar with dynamical systems, or are we? Many elementary examples are linear and analytically simply solvable. However, the study of dynamical systems was initiated by Poincaré to understand unsolvable equations of motion (the three body problem). Lurking in benign-looking dynamical systems are bewilderingly complicate motions. Thus it is not surprising that simply looking systems with complicate unpredictable behaviors appear in, e.g., stellar systems, particle accelerators, plasma confinement, chemical reaction kinetics, population dynamics. We usually do not pay particular attention to such complications in physical systems.

Apart from such relevance to physics, however, it should be simply interesting to know what is possible in apparently not so complicate systems. Thus, application of theory of dynamical systems is not the main concern of the course.[10]

Some old examples are in Figs. 1.2 and 1.4.

[10] "Unlearn this "for," you creators; your virtue itself wants that you do nothing "for" and "in order" and "because." You should plug your ears against these false little words." (F. Nietzsche, *Thus spoke Zarathustra*, On the higher man 11) [Cambridge Texts in the History of Philosophy, edited by A Del Caro and R Pippin].

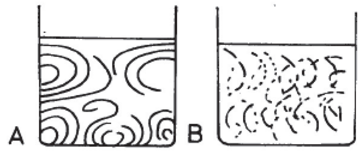

Fig. 5. Sketches of representative patterns. a: ordered phase, b: turbulent phase. In the turbulent phase much more diffuse patterns were often observed.

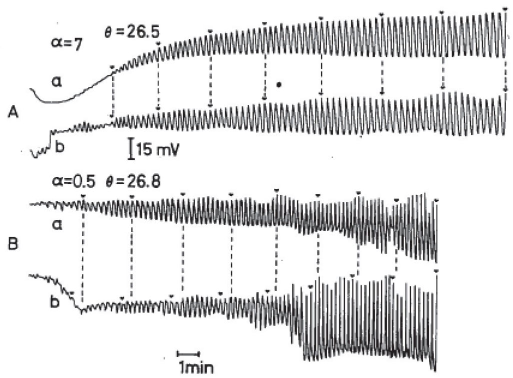

Fig. 6. Comparison of two emf signals observed by two small electrodes separated by a 0.2 mm gap.

Figure 1.2: 'Chemical turbulence' [Fig. 5,6 of Yamazaki et al. JPSJ 46 722 (1978)]

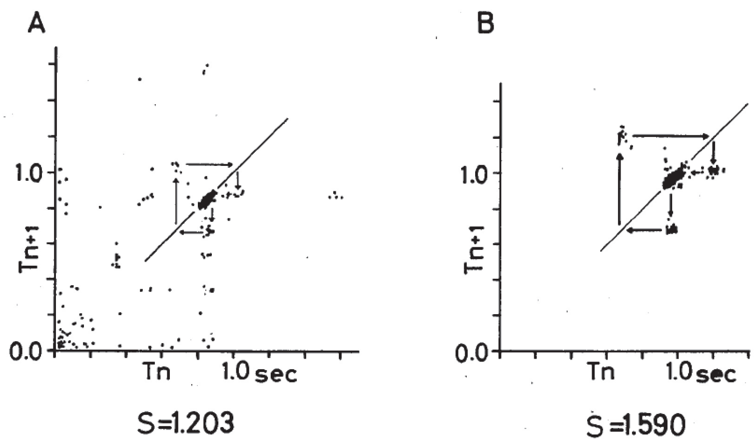

Fig. 8. The interval Lorenz plot for the pulses of a patient suffering from arrythmia. The pattern

Figure 1.3: Arhythmia Lorenz plot [Fig. 5,6 of YO et al JPSJ 48 733 (1980)]

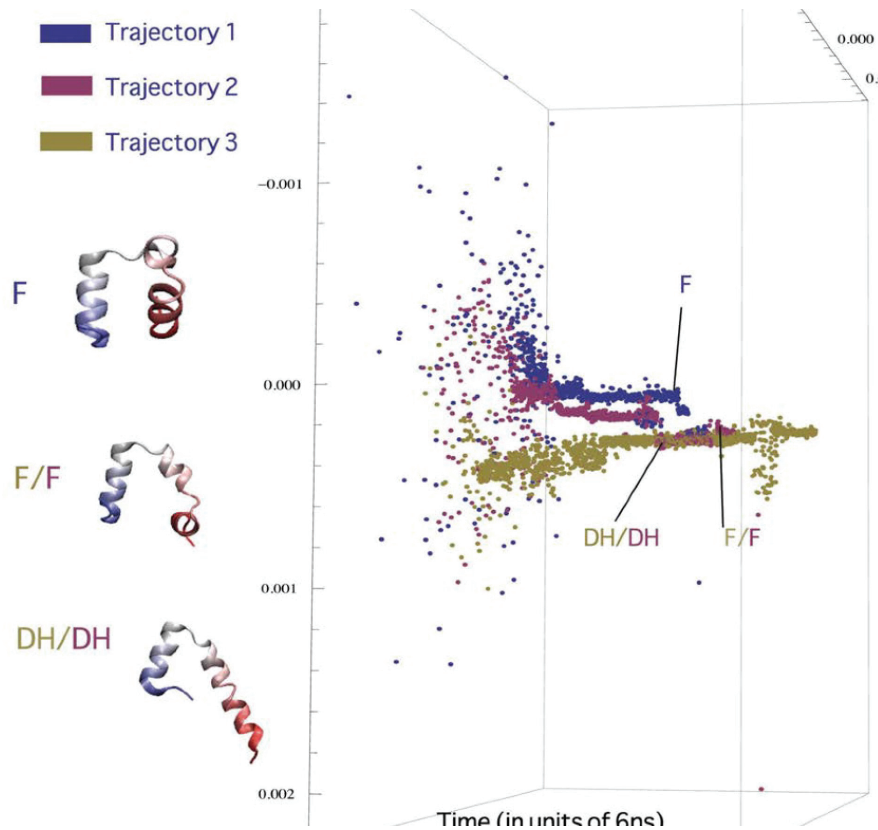

Figure 1.4: Villin head piece full MD[A Rajan Thesis UIUC (2009)]

1.8 Outline of the course

An outline of the course is given. Here, no explanation of technical terms (in italic) will be given (for example, manifold, compact, C^r-vector field, etc.). I know physicists (in the US) hate math,[11] but I wish you to have intuitive grasp of core math concepts.

Basically, after warmup with or without Hamiltonians, we will understand that 'chaos' is algorithmically random behaviors of dynamical systems, whose statistical behaviors we can understand 'statistical mechanically.'

Warmup with ODE

We will begin with a geometrical study of ODE

$$\dot{x} = X(x), \tag{1.3}$$

where $x \in M$, $X \in \mathcal{X}^r(M)$. Here, M is a *n-manifold* (usually *compact*), and $\mathcal{X}^r(M)$ is a C^r*-vector field* on M. Typical discussion topics are:

$*$ When can we say $x(t)$ is unique, or determinism legitimate? X must be *Lipshitz continuous*: $\|X(x) - X(x')\| \le L\|x - x'\|$. Under this condition if $X(x) \neq 0$, in an *neighborhood* of x, X is transformed to a constant field (say, $X = e_1$) [*Rectification theorem*].

$*$ What happens in the $t \to \infty$ limit? We will discuss *attractors*, *invariant manifolds*, etc.

$*$ If $X(x_f) = 0$, x_f is a fixed point (= *critical point* or *singular point*; some examples of flow around it are in Fig. 1.5). Is $x(t) = x_f$ stable against small perturbations? Can we study its stability from the linearized equation around x_f? We will discuss *hyperbolicity* and the fundamental theorem: *Hartman's theorem* that justifies the linear stability analysis.[12]

$*$ A qualitative change of the solution (say, from a stationary to oscillatory behavior) is called a bifurcation. What sort of bifurcations do we generically encounter? When a bifurcation occurs, can we compute the new qualitatively different solution by per-

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[11] Boltzmann said, "What the brain is to man, mathematics is to science" (quoted in K. Sigmund, *Exact thinking in demented times: the Vienna Circle and the epic quest for the foundation of science* (Basic Books, 2017; original in 2015).

[12] Quiz: is the following argument legitimate?
Suppose $X(0) = 0$. Since we are interested in the situation very close to $x = 0$, let us linearize X around $x = 0$ as $\dot{x} = Ax$ (i.e., we compute $A = DX/Dx|_{x=0}$). It happens that all the eigenvalues of A has negative real part. Thus, $x = 0$ is a stable fixed point.

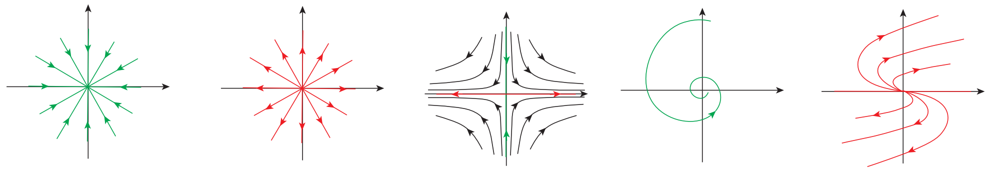

Figure 1.5: Some examples of flows around singular points on $\mathbb{R}^2$.

turbation? We will discuss a renormalization group theory for *singular perturbations* (= perturbations that qualitatively alter the nature of the system).
∗ What kind of singular points can a system have on M? Topological constraints matter (the *Poincaré-Hopf theorem*); *degree theory* can tell you something.

Classical mechanics

Next, we discuss a special class of ODE: the Hamiltonian systems
∗ After reviewing classical mechanics (extremely briefly) *canonical transformation* will be discussed (with related topics as *Lagrange* and *Poisson brackets*, canonical invariants).
∗ Integrable cases will be discussed generally; *Liouville-Arnold's theorem* and *action-angle variables* will be discusssed.
∗ Then we go to the cradle of the theory of dynamical systems: celestial mechanics.
∗ (Bruns and) Poincaré realized perturbation has problems.
∗ However Kolmogorov realized that still for many energies, perturbation series converges [*KAM theorem*]. A prototypical theorem due to Siegel will be proved following Kolmogorov's idea. When the series do not converge, 'chaos' show up as Poincaré realized. An illustration of what happens is in Fig. 1.6

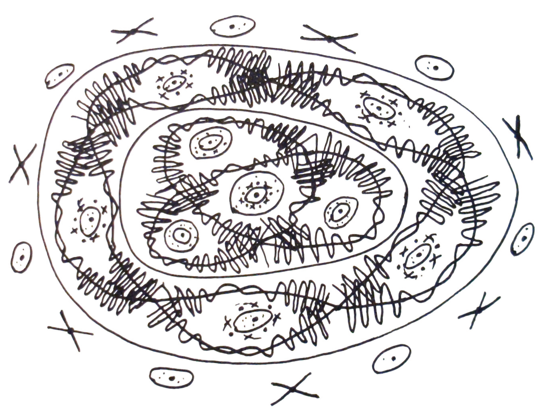

Figure 1.6: Many tori due to integrability are destroyed except for the KAM tori; the destroyed tori exhibit chaotic behavior such as the *Arnold diffusion*. [Fig. 6.7 of A E Jackson, *Perspective of nonlinear dynamics* Vol. 2 (Cambridge, 1990) p56]

∗ This chaotic behavior will be discussed with the aid of the standard map (related to accelerator dynamics).

Up to this point is introductory; although mathematically delicate issues (or at least its delicate nature) should be appreciated (for Hartman, Siegel and Poincaré theorems), technical details will not be emphasized (details are already posted as supplements). However, the flow of the logic should be understood.

An exposition of the 'modern' theory of dynamical systems begins here. Physicists should realize that most of the main ingredients had been finished before 1980 when physicists, esp., in the US started to pay serious attention to the topic. Computers were in large part irrelevant (contrary to the claim of Gleick, whose book I cannot recommend) and many key ingredients were done in USSR.[13]

We must not forget a dictum: What we can show only with computers is not general; what is general should be demonstrable without computers (if we understand it). Needless to say, however, for exploratory work computers are great especially in the hand of those who understand mathematics.

Chaos gallery:
We begin with typical and historical examples to become familiar with the real issues of dynamical systems.
∗ Weather forecasting and *Lorenz system*. The iconic figure must be familiar to most of you:
https://upload.wikimedia.org/wikipedia/commons/1/13/A_Trajectory_Through_Phase_Space_in_a_Lorenz_Attractor.gif
∗ *Strange attractors* and how to observe them (*Takens' reconstruction theorem*).
∗ Interval maps and Theorem: "Period $\neq 2^n$ implies chaos." I will follow the path I took more than 40 years ago.[14]
∗ But what is chaos? We use coding of dynamical behaviors into symbol sequence (*shift dynamical systems*), and discuss its complexity with typical examples: *baker's*

[13] Y. G. Sinai, "Chaos Theory Yesterday, Today and Tomorrow," J. Stat. Phys. **138**, 2 (2010). "My personal experience shows that people in the West consider the development of Chaos Theory differently from their Russian colleagues, mathematicians and physicists." [This seems a great euphemism or sarcasm.] "Many people share the point of view that the beginning of chaos theory dates back to 1959 when the Kolmogorov's paper ["New metric invariant of transitive dynamical systems and automorphisms of Lebesgue spaces," Dokl. Acad. Nauk SSSR **119**, 861 (1958)] on the entropy of dynamical system appeared."

Since my collaborator in this field (Y. Takahashi) was a postdoc of Sinai, naturally I am close to the Russian school.

[14] before 1980 in the US only the people at UCSC [Rob Shaw, Doyne Farmer, Normal Packard and Jim Crutchfield] were active in the field, and we were in correspondence.

transformation and *Bernoulli systems*.
$*$ We must know how to measure complexity or the extent of being chaotic. This requires *Kolmogorov complexity* which requires *Turing machines*. Thus we will discuss the ABC of *algorithmic complexity* and *computability* of physical processes: I wish you to think of the scientific significance of a theoretical result that is not computable.
$*$ Eventually *Brudno's theorem* tells us chaos = algorithmically complex trajectories.

After these elementary examples we go into measure theoretical and topological dynamical systems.

Measure-theoretical dynamical systems
We will prove the Poincaré recurrence theorem and the Birkhoff ergodic theorem. The latter was long misunderstood as the key to statistical mechanics (even long after Boltzmann himself realized this). The *Kolmogorov-Sinai entropy* is a (the?) measure of the extent of chaos, which is related to statistical mechanical entropy. We will discuss the *thermodynamic formalism* for dynamical systems. This allows us to characterize observable invariant measures (i.e., observable or numerical-experimentally detectable stationary states) with a variational principle (mathematically isomorphic to Gibbs variational principle in statistical mechanics).

General theory of dynamical systems
Hopefully, the course will conclude with the typical behaviors of dynamical systems (e.g., *Axiom A systems*), which are *structurally stable* (i.e., stable against small modification of the system). Palis proposed the conjecture (*Palis conjecture*): typical systems have a finite number of (strange) attractors which support the observable invariant measures (called the *Sinai-Ruelle-Bowen measure*). Or in the physicist-friendly words: Any typical and structurally stable finite dimensional dynamical system with its phase space being compact may be understood by using a statistical mechanical device. The demonstration of this conjecture will close a very active phase that started with *Smale's horseshoe*.

1.9 Some 'practical' topics we will discuss

When we observe various time-dependent phenomena in hydrodynamics, cell biology, population dynamics, etc., often we do not have any explicit mathematical description at the beginning. Can we visualize 'strange attractors' directly from the observed time data? This is broadly answered affirmatively by Takens' theorem and related topics. The 'UCSC gang of four' proposed a similar method (see 1.7).[15] Even if clean

[15] N. H. Packard, J. P. Crutchfield, J. D. Farmer, and R. S. Shaw, Geometry from a Time Series PRL 45 712 (1979).

geometrical fatures may not be obtained, still something suggestive may be gleaned (for example about our brains). Even some discrete plots could suggest something, although no simple dynamics behind the data can be guessed.

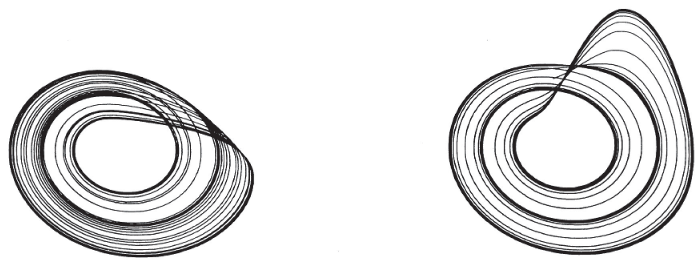

FIG. 1. (x,y) projection of Rossler (Ref. 7).   FIG. 2. $(x,\dot{x})$ reconstruction from the time series.

Figure 1.7: Reconstruction from a time series [Fig. 1,2 of Packard et al., PRL 45 712 (1979)]

Most natural phenomena are high-dimensional phenomena described by partial differential equations and by field theories. However, their asymptotic behaviors may not be very high-dimensional, and at least qualitatively we can reduce them to a low dimensional systems via various attractor theories and analytic methods like the Galerkin method. The famous Lorenz system is obtained by Salzmann in such an attempt.[16]

Such dimensional reduction methods may be of some interest in summarizing large scale MD simulation of biomolecules.

1.10 Video: *Chaos*

The following 9 (artistic) videos from http://www.chaos-math.org/en may be close to the spirit of the course. Most of you feel the pace is too slow and the explanation too elementary, but the movie actually explains sophisticated topics (very quietly esp beyond Chapter 6); the movie exhibits nice European taste.

Chapter 1: Motion and determinism—$\pi\alpha\nu\tau\alpha\ \rho\varepsilon\iota$.
https://www.youtube.com/watch?v=c0gDLEHbYCk&t=4s&frags=pl%2Cwn

Chapter 2: The vector fields—The lego race
https://www.youtube.com/watch?v=_Y68GX2UpQ0&frags=wn

Chapter 3: Mechanics—The apple and the Moon (Newton, universal law of gravity, etc.)

[16]For a systematic approach to the Couette flow is worked out by H. Yahata, Temporal development of the Taylor vortices in a rotating fluid, Prog. Theor. Phys. Suppl. 64 176 (1978).

https://www.youtube.com/watch?v=ZwTGAW0b_bo&frags=wn

Chapter 4: Oscillations—the swing (including Lotka-Volterra; Poincare-Bendixson theorem including the idea to prove it)
https://www.youtube.com/watch?v=uEfB5DG9x9M&frags=wn

Chapter 5: Billiards—Duhem's bull (geodesics on negative curvature surface included, symbolic dynamics a bit)
https://www.youtube.com/watch?v=3u2SJKxJhh8&frags=wn

Chapter 6: Chaos and the horseshoe—Smale in Copacabana (Poincaré map, Smale's horseshoe, symbolic dynamics, structural stability)
https://www.youtube.com/watch?v=ItZLb5xI_1U

Chapter 7: Strange attractors—the butterfly effect (Lorenz system, Lorenz template, symbolic dynamics
https://www.youtube.com/watch?v=aAJkLh76QnM&frags=wn

Chapter 8: Statistics—Lorenz' mill (measure-theoretical aspect, physical model of Lorenz system, sensitivity to initial conditions, SRB measure)
https://www.youtube.com/watch?v=SlwEt5QhAGY&frags=wn

Chapter 9: Chaotic or not—research today (bifurcation diagram, heteroclinic connection, non-generic case, Palis conjecture)
https://www.youtube.com/watch?v=_xfi0NwoqX8

1.11 Classic books still fresh

Recommended books for very serious students are listed here. They are now classic but still almost fresh. Readable? Yes, if you know basic math well (after one semester of my course you know the outline of many key portions of these classics).

∗ V. I. Arnold and A. Avez: *Ergodic Problems of Classical Mechanics* (Advanced Book Classics; Addison-Wesley; Reprint edition 1989; original in French 1967).

∗ J. Moser: *Stable and random motions in dynamical systems* (Annals of Mathematical Studies 77, Princeton UP 1973).

∗ J. Palis, Jr. and W. de Melo: *Geometric theory of dynamical systems* (Springer,1982).

∗ J. Palis and F. Takens: *Hyperbolicity & sensitive chaotic dynamics at homoclinic bifurcations* (Cambridge studies in advanced mathematics 35, Cambridge UP, 1993).

∗ R. E. Bowen: *Equilibrium states and the ergodic theory of Anosov diffeomorphisms* (Springer Lecture Notes in Mathematics 470; Second Ed 2008).

1.12 Do not misunderstand complexity

It is often said that the study of chaos is an important part of complexity study. I totally disagree with this popular view. If you admit that biological systems are complex systems, which is not merely complicated, you must recognize the distinction between the truly complex systems and pseudo complex systems that have been studied under the name of complexity study. If something is easy to (re)produce without any special preparation, that something must not be complex. Chaos is a typical example. As you learn it is easy to produce; perhaps the surprise was that benign-looking simple systems readily exhibit bewilderingly complicated behaviors (as illustrated in Fig. 1.6). In contradistinction, you cannot readily produce life from scratch; we do not even understand how life began at all. Complex systems are systems requiring a lot of prerequisite that we cannot (at leat readily) construct; you have your parents, because you are complex systems. Pasteur realized that complex systems are produced only by complex systems.

Thus, chaos has nothing to do with complex systems. The so-called complex-systems study in physics[17] studied only pseudo-complex systems that can self-organize almost from scratch.

[17]Even in our department this is the case, unfortunately.