

19 Lecture 19. Lorenz system: Introduction

Warning

After switching the OS (to Mojave) perhaps security has become tighter, and I cannot run many of the copied demos (due to ‘unsupported plug-in’).

Introductory video

The following video is an leisurely introduction to the Lorenz system (Chaos Chapter 7), giving a nice overview of what we should understand mathematically:

<https://www.youtube.com/watch?v=aAJkLh76QnM&frags=pl%2Cwn>

We look at the key portions of Lorenz’s historic paper, “Deterministic nonperiodic flow” (1963); here the basic equation derived by Saltzman is outlined [19.2](#). Lorenz numerically established his system is intrinsically nonperiodic. We watch various behaviors of the Lorenz equation. Notice that to reduce the original system to a simpler system (e.g., time discrete dynamical system) is crucial (Fig. [19.3](#)). You see very similar structures we already encountered in a simpler Ito’s system in the preceding Lecture. Then, we see a similar system describing magnetic reversal of the earth (the Rikitake model [19.7](#)).

19.1 Lorenz’s motivation

Certain hydrodynamical systems exhibit steady-state flow patterns, while others oscillate in a regular periodic fashion. Still others vary in an irregular, seemingly haphazard manner, and, even when observed for long periods of time, do not appear to repeat their previous history.

This is the first paragraph of Introduction to “Deterministic nonperiodic flow” by E N Lorenz.^{[186](#)}

The short-range weather forecaster, however, is forced willy-nilly to predict the details of the large scale turbulent eddies—the cyclones and anticyclones which continually arrange themselves into new patterns. Thus there are occasions when more than the statistics of irregular flow are of very real concern.

¹⁸⁶J Atmos Sci., 20, 130 (1963).

As a system to study Lorenz chose a fluid layer heated from below (the Rayleigh-Benard convection problem):

Rayleigh (1916) studied the flow occurring in a layer of fluid of uniform depth H , when the temperature difference between the upper and lower surfaces is maintained at a constant value ΔT . Such a system possesses a steady-state solution in which there is no motion, and the temperature varies linearly with depth. If this solution is unstable, convection should develop: In the case where all motions are parallel to the xz -plane, and no variations in the direction of the y -axis occur, the governing equations may be written (see Saltzman, 1962):

$$\frac{\partial}{\partial t} \nabla^2 \psi = -\frac{\partial(\psi, \nabla^s \psi)}{\partial(x, z)} + \nu \nabla^4 \psi + g\alpha \frac{\partial \theta}{\partial x}. \quad (19.1)$$

$$\frac{\partial}{\partial t} \theta = -\frac{\partial(\psi, \theta)}{\partial(x, z)} + \frac{\Delta T}{H} \frac{\partial \psi}{\partial z} + \kappa \nabla^2 \psi. \quad (19.2)$$

Here ψ is a stream function for the two-dimensional motion, θ is the departure of temperature from that occurring in the state of no convection, and the constants g, α, ν , and κ denote, respectively, the acceleration of gravity, the coefficient of thermal expansion, the kinematic viscosity, and the thermal conductivity.

Here, let us try to understand Saltzman's equation from basic physics.

19.2 Saltzman's equation¹⁸⁷

For the system considered in 19.1 the basic equations are the incompressible Navier-Stokes equation

$$\frac{\partial}{\partial t} \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P + g\alpha T \mathbf{e}_z + \nu \nabla^2 \mathbf{v} \quad (19.3)$$

with $\nabla \cdot \mathbf{v} = 0$ and the equation governing the thermal advection for T (temperature departure)

$$\frac{\partial}{\partial t} T + (\mathbf{v} \cdot \nabla) T = \kappa \nabla^2 T. \quad (19.4)$$

From (19.3) we can derive the vorticity equation (vorticity $\omega = \text{curl } \mathbf{v}$) as

$$\frac{\partial \omega}{\partial t} + (\mathbf{v} \cdot \nabla) \omega = (\omega \cdot \nabla) \mathbf{v} + g\alpha \frac{\partial T}{\partial x} + \nu \nabla^2 \omega. \quad (19.5)$$

¹⁸⁷B. Saltzman, "Finite amplitude free convection as an initial value problem-I," J. Atmos. Sci., 19, 329 (1962). From the Acknowledgement of Lorenz's paper, "The writer is indebted to Dr. Barry Saltzman for bringing to his attention the existence of nonperiodic solutions of the convection equations."

Here, we assume the flow is translationally symmetric along the y -axis, so $\mathbf{v} = (u, w)$ is a 2-vector. The incompressibility means there is ψ (streaming function) such that $\text{curl } \psi = (u, w)$ ¹⁸⁸

$$u = -\frac{\partial\psi}{\partial z}, w = \frac{\partial\psi}{\partial x}. \quad (19.6)$$

The vortex equation (19.5) reads, since

$$\omega = (0, \partial_x w - \partial_z u, 0) = (0, \nabla^2 \psi, 0), \quad \mathbf{v} \cdot \nabla(A, B) = -\frac{\partial\psi}{\partial z} \frac{\partial A}{\partial x} + \frac{\partial\psi}{\partial x} \frac{\partial B}{\partial z} = \frac{\partial(A, B)}{\partial(x, z)}, \quad (19.7)$$

$$\frac{\partial}{\partial t} \nabla^2 \psi - \frac{\partial(\psi, \nabla^2 \psi)}{\partial(x, z)} = g\alpha \frac{\partial T}{\partial x} + \nu \nabla^4 \psi. \quad (19.8)$$

Now, $T = \theta + \frac{\Delta T}{H} z$ is introduced to (19.8) and (19.4), and we get the equations in 19.1.

19.3 ‘Derivation’ of Lorenz equation

Saltzman derived a set of ODE by expanding ψ and θ in Fourier series. Solving the ODE (quoted from Lorenz, *ibid.*):

He then obtained time-dependent solutions by numerical integration. In certain cases all except three of the dependent variables eventually tended to zero, and these three variables underwent irregular, apparently nonperiodic fluctuations.

Lorenz then thought that if only the three surviving coefficients are kept from the start by ‘drastically’ truncating the expansion as

$$a(1 + a^2)^{-1} \kappa^{-1} \psi = X \sqrt{2} \sin(\pi a x / H) \sin(\pi z / H), \quad (19.9)$$

$$\pi R_c^{-1} \Delta T^{-1} \theta = Y \sqrt{2} \cos(\pi a x / H) \sin(\pi z / H) - Z \sin(2\pi z / H), \quad (19.10)$$

the same results should be obtained. The outcome is the Lorenz equation:

$$\dot{X} = -\sigma X + \sigma Y, \quad (19.11)$$

$$\dot{Y} = rX - Y - XZ, \quad (19.12)$$

$$\dot{Z} = XY - bz, \quad (19.13)$$

¹⁸⁸Let $\omega = -udz + wdx$. Then, incompressibility means $d\omega = 0$, so the converse of Poincaré’s lemma (= closedness means exactness) means there is ψ such that $\omega = d\psi$.

where r, σ, τ are positive constants and ‘dot’ means the derivative wrt the dimensionless time τ .¹⁸⁹

Remark: How ‘qualitatively’ good is (19.12) for understanding the original system? This is not a very relevant question in the present conceptual study, but if you can solve a PDE as a set of a small number of ODEs, it is numerically still very advantageous. This question is related to various ‘finite dimensional attractors’ of PDE systems. See (??) as a general approach.

19.4 What Lorenz established

The phase volume of the space spanned by (X, Y, Z) shrinks (i.e., $\text{div } \mathbf{v} < 0$):

$$\frac{\partial \dot{X}}{\partial X} + \frac{\partial \dot{Y}}{\partial Y} + \frac{\partial \dot{Z}}{\partial Z} = -(\sigma + b + 1). \quad (19.14)$$

The origin is a fixed point. If $r < 1$ it is a sink. Note that $r = 1$ means $R_a = R_c$ where the advection starts. If $r > 1$ it is a hyperbolic saddle with one unstable direction.

For $r > 1$ there are two more fixed points: $X = Y = \pm \sqrt{b(r-1)}, Z = r - 1$. If $\sigma < b + 1$ they are still sinks (one stable direction + two more stable but spiraling plane), corresponding to steady convections. If $\sigma > b + 1$, then it has a plane of instability (Fig. 19.1).

Fig. 19.1 What Lorenz observed numerically

Fig. 1. Numerical solution of the convection equations. Graph of Y as a function of time for the first 1000 iterations (upper curve), second 1000 iterations (middle curve), and third 1000 iterations (lower curve).

Fig. 2. Numerical solution of the convection equations. Projections on the XY -plane and the YZ -plane in phase space of the segment of the trajectory extending from iteration 1400 to iteration 1900. Numerals “14,” “15,” etc., denote positions at iterations 1400, 1500, etc. States of steady convection are denoted by C and C' .

Fig. 3. Isopleths of X as a function of Y and Z (thin solid curves), and isopleths of the lower of two values of X , where two values occur (dashed curves), for approximate surfaces formed by all points on limiting trajectories. Heavy solid curve, and extensions as dotted curves, indicate natural boundaries of surfaces.

Let us see the actual solution.

<https://media.pearsoncmg.com/aw/ide/idefiles/media/JavaTools/lrnzdscv>.

¹⁸⁹ $\tau = \pi^2(1 + a^2)\kappa t/H$, $\sigma = \nu/\kappa$ (the Prandtl number), $r = R_a/R_c$ and $b = 4/(1 + a^2)$. $R_a = g\alpha H^3 \Delta T/\nu\kappa$ is the Rayleigh number, and $R_c = \pi^4 a^{-2}(1 + a^2)^2$ is the critical Rayleigh number.

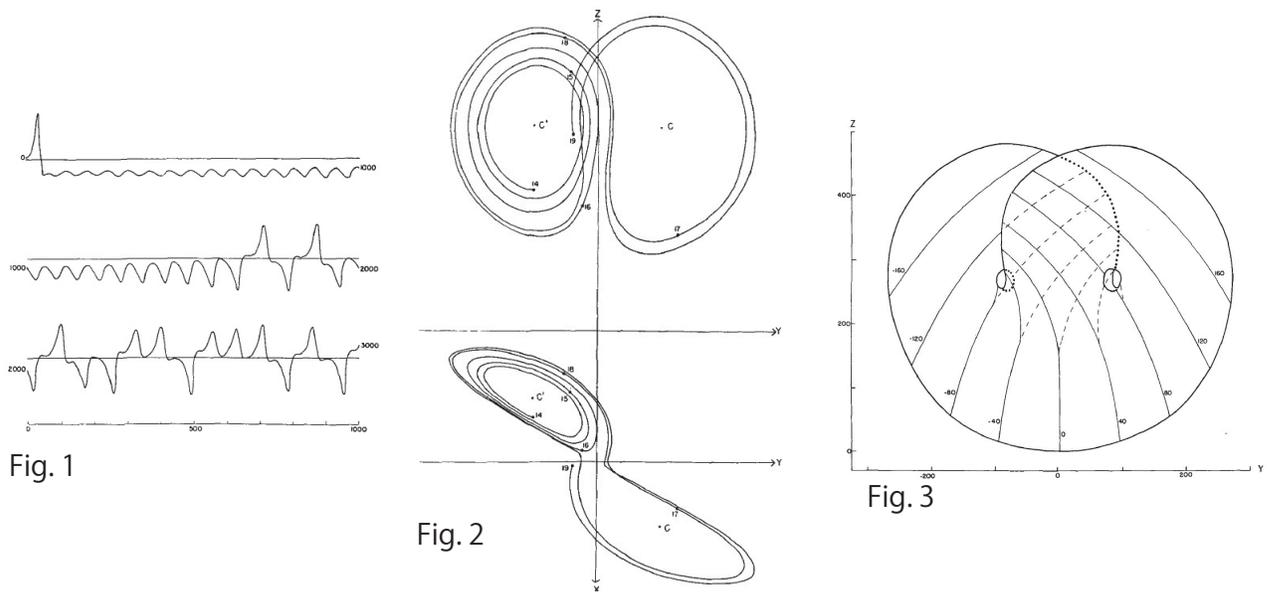


Figure 19.1: What Lorenz found numerically [Fig. 1-3 of Lorenz JAS 20 130 (1963).]

[html](#)

The initial demo uses the initial conditions in the paper. You can see the solutions sensitively depend on the initial choices. After that you can choose the initial x by tapping the panel (Fig. 19.2).

As to the global feature about Fig. 3 in Fig. 19.1 Lorenz said:

Thus, within the limits of accuracy of the printed values, the trajectory is confined to a pair of surfaces which appear to merge in the lower portion of Fig. 3. The spiral about C lies in the upper surface, while the spiral about C' lies in the lower surface. Thus it is possible for the trajectory to pass back and forth from one spiral to the other without intersecting itself.

He observes further

Returning to Fig. 2, we find that the trajectory apparently leaves one spiral only after exceeding some critical distance from the center. Moreover, the extent to which this distance is exceeded appears to determine the point at which the next spiral is entered; this in turn seems to determine the number of circuits to be executed before changing spirals again.

Therefore, he collected successive local max values $\{M_n\}$ of Z and plotted M_{n+1} against M_n (Fig. 19.3).

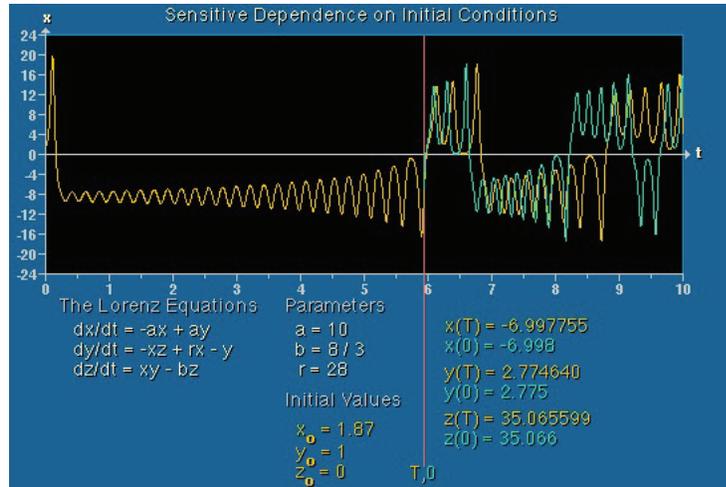


Figure 19.2:

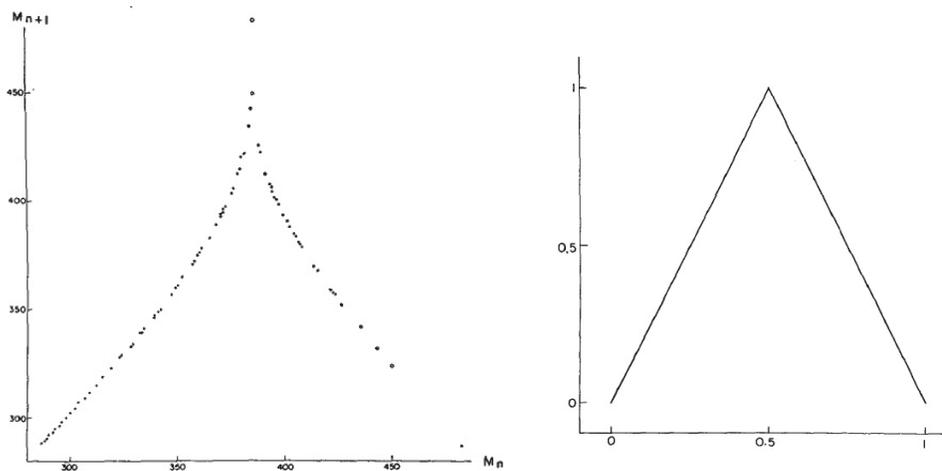


Figure 19.3: Lorenz map [Fig. 4,5 of Lorenz JAS 20 130 (1963).]

Fig. 19.3 Lorenz map

Left: Corresponding values of relative maximum of Z (abscissa) and subsequent relative maximum of Z (ordinate) occurring during the first 6000 iterations. The right figure (the tent map) is proposed to understand the system conceptually.

Lorenz discussed the sensitive dependence of the system to disturbance with the aid of the tent map.

In Conclusion of his paper Lorenz writes:

When our results concerning the instability of nonperiodic flow are applied

to the atmosphere, which is ostensibly nonperiodic, they indicate that prediction of the sufficiently distant future is impossible by any method, unless the present conditions are known exactly. In view of the inevitable inaccuracy and incompleteness of weather observations, precise very-longrange forecasting would seem to be non-existent.

Let us watch a live Lorenz map. You must be a bit patient:
Lorenz map:

<https://media.pearsoncmg.com/aw/ide/idefiles/media/JavaTools/lrnzzmax.html> (Fig. 19.4).

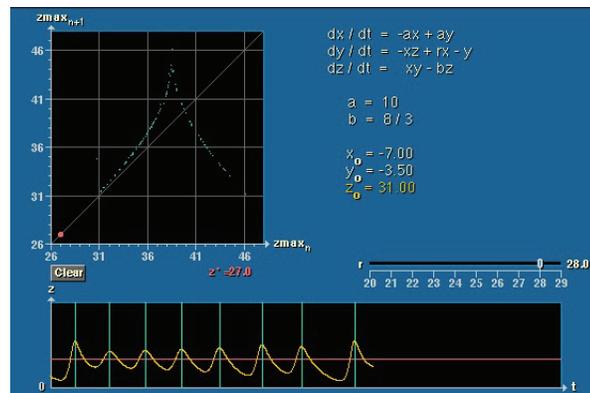


Figure 19.4:

The r dependence of the phase portrait discussed a bit above may be glimpsed from the following panels:

<https://media.pearsoncmg.com/aw/ide/idefiles/media/JavaTools/lrnzphgrd.html> (Fig. 19.5).

19.5 Illustration of Lorenz system behaviors

Now, you can scan r to watch what happens (see Fig. 19.6L):

<https://media.pearsoncmg.com/aw/ide/idefiles/media/JavaTools/lrnzphsp.html>

For wider range of r (XZ -view is popular):

<https://media.pearsoncmg.com/aw/ide/idefiles/media/JavaTools/lrnzr320.html>

You can slide the r scale to choose its value, and then tap the panel. The initial

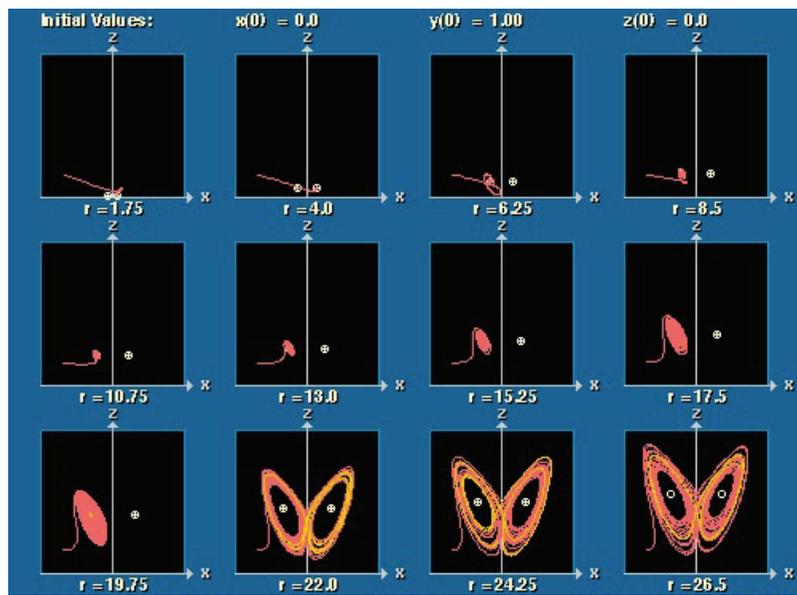


Figure 19.5: “ ω -limit set(s) may be guessed. You can choose the initial condition on the panel position is where you tap the panel.

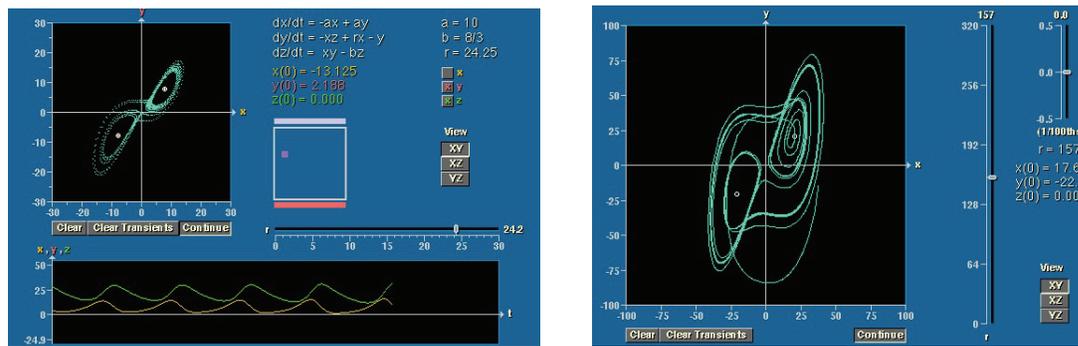


Figure 19.6: Right: wider range.

The following is an interactive visualization in which you can change r , σ and b :

<http://www.malinc.se/m/Lorenz.php>

You can watch basically all the behaviors in Fig. 19.5. To see how the system attractors evolve you may have to be very patient.

A 3D trajectory is visualized with a stereoplot:

<https://www.youtube.com/watch?v=-tpRZCnoi0>

Fate of small change in the initial conditions:

<https://www.youtube.com/watch?v=FYE4JKAXSfY> 3 point animation (esp. after 50 sec)

You can zoom in:

https://www.ibiblio.org/e-notes/webgl/lorenz_model.html

19.6 Physical model mimicking Lorenz

Watch Chaos Chapter 8 beyond about 6 min:

<https://www.youtube.com/watch?v=SlwEt5QhAGY&frags=wn>

A schematic model is here:

<https://www.youtube.com/watch?v=M-94UxoZD2M&frags=pl%2Cwn>

19.7 Earth dynamo reversal

The magnetic field of the earth reverses from time to time, and gave a decisive proof of plate tectonics.

A simple model to explain this reversal is the Rikitake model:

Rikitake, T., Oscillations of a system of disk dynamos, Proc. Cambr. Phil. Soc., 54 89 (1958).

Tsuneji Rikitake, Nonsteady Geomagnetic Dynamo Models, Geophys J Internat 35 277 (1973).

$$\dot{x} = -\nu x + zy, \quad (19.15)$$

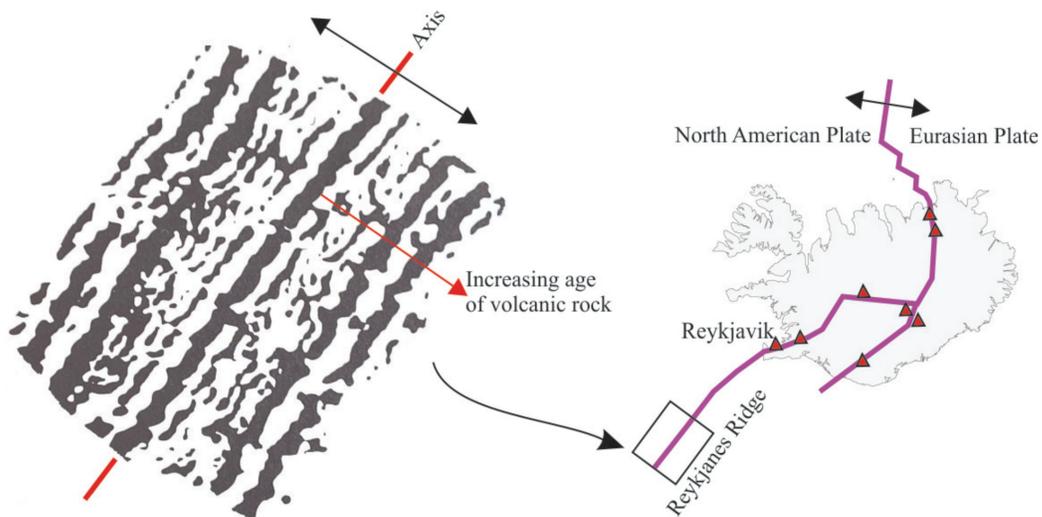
$$\dot{y} = -\nu y + (z - a)x, \quad (19.16)$$

$$\dot{z} = 1 - xy. \quad (19.17)$$

This system exhibits just the Lorenz-like attractors.

<http://demonstrations.wolfram.com/RikitakeModelOfGeomagneticReversal/>

What is a lesson to learn as a researcher? Whatever a model or a system you study is, study it thoroughly with conceptual questions.



Mid-ocean spreading along Reykjanes Ridge, depicted by magnetic 'stripes'. Black patterns are normal magnetic polarity (compass points North), and white are reversed polarity. The age of successive magnetic reversals increases with distance from the active ridge. The inset map (right) shows the complexity of the spreading ridge where it transects Iceland. Red triangles are active volcanoes on Iceland.

Image credit: Reykjanes magnetic pattern from Heirtzler, J.R., Le Pichon, X. and Baron, J.G. (1966). Magnetic anomalies over the Reykjanes Ridge. *Deep Sea Research* 13: 427-433.

Figure 19.7: magnetic reversals recorded near Iceland