18 Lecture 18. Introductory examples of chaos

18.1 K. Ito's model of great earthquakes

Island arcs such as the Japanese Archipelago consist of blocks spaced by faults. Each block is being pulled down by a subducting ocean plate and accumulates strain energy (Fig. 18.1). When this energy reaches a threshold, the block breaks (a great

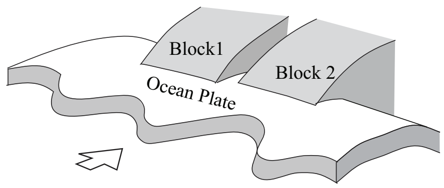

Figure 18.1: The great earthquake model. The subducting ocean plate pulls two blocks down. The blocks accumulate strain energy and at some point they break apart from the plate and earthquakes occur. Then, the blocks return to the original positions.

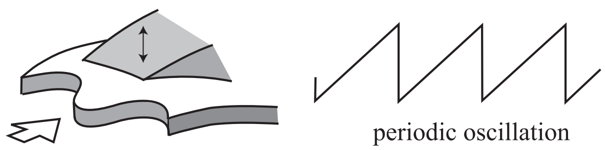

Figure 18.2: If there is only one block, it is a simple model of the relaxation oscillator.

earthquake occurs) and the stored strain energy is quickly reset to its lowest value; the model chooses this value as the energy origin.

If there is no interaction among blocks, earthquakes occur periodically (Fig. 18.2). This is probably the simplest model of the relaxation oscillator. See a traditional example https://www.youtube.com/watch?v=JU0mNSRtHMM&frags=pl%2Cwn.

If there are more than one blocks, the earthquake occurring in one block must affect the surrounding blocks. Ito thought that the earthquake produces cracks in the neighboring blocks that decelerate the accumulation rate of the strain energy.

The simplest nontrivial case is the two-block case. The following rules of time evolution can model the idea depicted above. Let u_i be the strain energy stored in block i ($i = 1, 2$).

(1) The rate of increase of the strain energy is initially b, which is assumed to be a positive number.

(2) If u_i reaches 1, an earthquake occurs in block i. Subsequently,

(2a) u_i is reset to 0, and the rate of increase of strain energy is also reset to b.

(2b) The rate of increase of the strain energy in the other block without the occurrence of the earthquake becomes b^{-1}; if the rate is already with this value, it is maintained as b^{-1}.

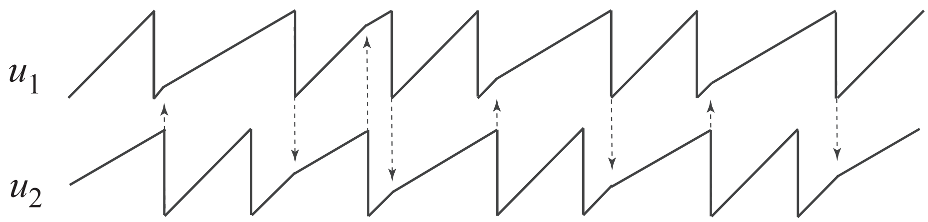

Figure 18.3: A typical behavior of stored energies u_1 and u_2 in two coupled blocks 1 and 2, respectively. This is the case of mutual hindrance $b > 1$. The earthquake in one block decelerates the energy increase in the other block. Vertical arrows denote hindering effects on strain energy accumulation. (Except for the beginning stage of the system behavior, the rates of energy increase of the blocks never coincide.)

A typical behavior under the above rule is in Fig. 18.3. If the rate of increase of the strain energy $b > 1$, we see apparently complicated behavior.

18.2 Trajectories on T^2

The time evolution of the two blocks can be illustrated as a trajectory on $T^2 = [0,1] \times [0,1]$ (with periodic boundary conditions), if we plot u_1 on the horizontal axis and u_2 at the same moment on the vertical axis (Fig. 18.4).

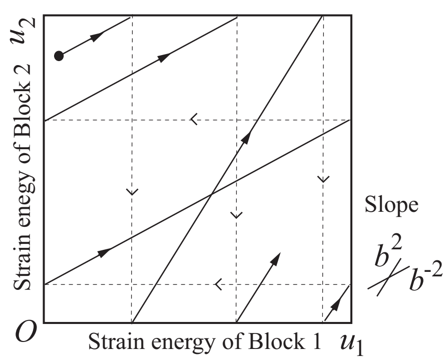

Figure 18.4: The typical time dependence of the stored energy in the blocks may be depicted as a trajectory on $[0,1] \times [0,1]$. Horizontal jumps correspond to the earthquakes in block 1 and the vertical jumps those in block 2.

The rule of the game may be illustrated as in Fig. 18.5. We can write the rules in formulas, but no new insight is obtained by doing so. For example, a portion of the trajectory with slope less than 1 on the square implies that the energy increase rate in block 1 is larger than that for block 2.

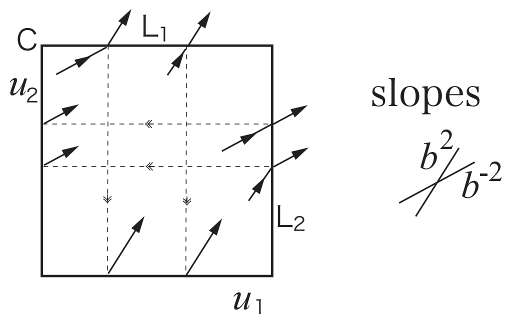

Figure 18.5: The rules for the motion: after a jump how the slope changes is specified. If the point reaches L_1 (the top edge; an earthquake in block 2), it jumps down to the point following the broken line, and then start running with slope b^2 (because u_2 increases at rate b and u_1 at rate b^{-1}); If the point reaches L_2 (the right edge), it jumps to the left according to the broken line and start running with slope b^{-2} ($b > 1$ assumed). If the opposite edges of the square are glued, we can make a torus, and we obtain a continuous trajectory on the torus (as seen in Fig. 18.4).

18.3 Trajectories on universal covering space of $\boldsymbol{T}^2$

Instead of gluing the opposite edges of the square to make the torus, we may tessellate many copies of the square to make the so-called universal covering space.[173] In this space (i.e., instead of returning to itself, moving on to next tiles) we can clearly see what is going on (Fig. 18.6).

In Fig. 18.6 Right three trajectories starting from closely located initial points are depicted.[174] Earthquakes occur in block 1 (resp. block 2) when the trajectory crosses the vertical (resp. horizontal) lines. If $b > 1$, two initially close trajectories come apart exponentially in time. The great earthquake model is with $b > 1$, so according to this model long time prediction of earthquakes is impossible.

18.4 As a vector field on two $\boldsymbol{T}^2$

[173]To obtain elementary knowledge of topology I. M. Singer and J. A. Thorpe, *Lecture Notes on Elementary Topology and Geometry* (Undergraduate Texts in Mathematics) (Springer 1976; original 1967) is the best.

[174]The distance between the two adjacent points increases as $e^{t\lambda}$ as a function of time on the average, where λ is called the Lyapunov exponent (will be discussed in a later lecture). In the present example, it can be computed as λ'/τ, where λ' is the exponent defined for the corresponding discrete system in Fig. 18.11 and τ is the average sojourn time on a single tile. To estimate λ' may not be easy, but it is obviously positive as can be trivially seen from the reduced map in Fig. 18.12.

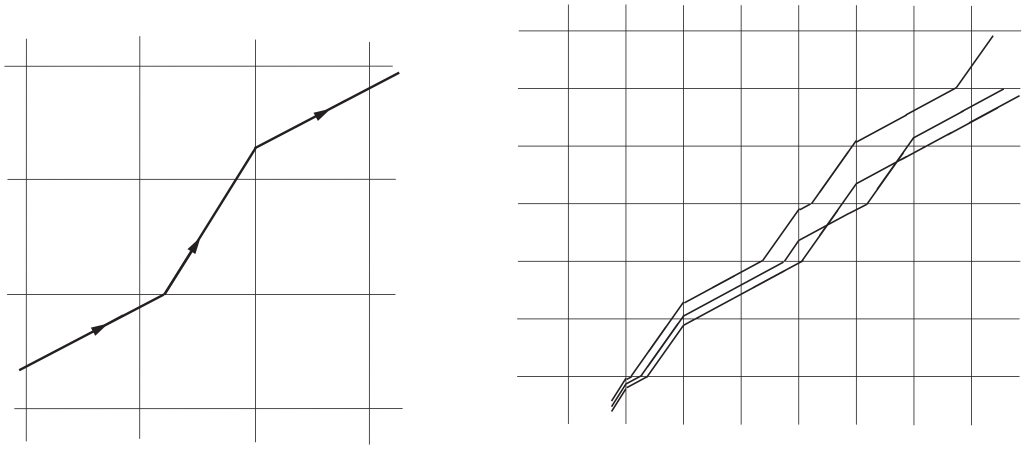

Figure 18.6: **Left**: A typical trajectory on the so-called universal covering space that is made by tessellating the squares instead of forming a torus. The crossings with the vertical lines correspond to earthquakes in block 1 and those with the horizontal lines to those in block 2. **Right**: Locally, trajectories depart from each other exponentially in time. (Around the lattice points something complicated may happen, but this is a global problem.) (To prove that the trajectories separate from each other exponentially on the average, the easiest way is to use the one-dimensional map shown Fig. 18.12.)

If Fig. 18.4 is understood as a description of a two-dimensional motion, we must conclude that the trajectory follows the vector field v_1 in Fig. 18.7 at one time, and then v_2 at another. The history up to the point determines on which flow (vector field) the trajectory follows. A two-valued vector field dependent on the history is not a very convenient object.

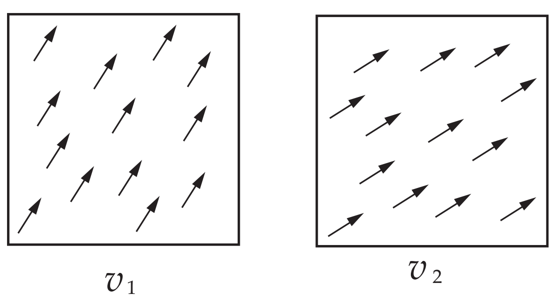

Figure 18.7: In Fig. 18.4 at a particular point on the torus the trajectory runs in one of the two directions depending on the history up to the point. That is, the trajectory runs on the torus according to one of the two vector fields, v_1 or v_2, depending on the history.

To avoid this we prepare two tori, on one of which is v_1 and on the other v_2, and then connect these two tori according to the connection rule consistent with Fig. 18.5 as shown in Fig. 18.8.

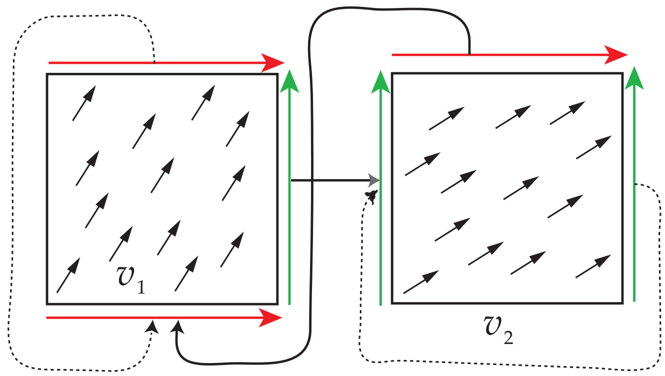

Figure 18.8: The univalent vector field independent of the history on the connected two copies of the original torus. The two vector fields prepared in Fig. 18.7 are connected according to the rules consistent with the trajectory connection rules in Fig. 18.5(i). How to glue the tori is shown by straight colored arrows with large arrowheads (the directions denote correct orientations to glue).

18.5 Why unpredictability?

As can be seen from Fig. 18.8, something special happens when a crossover between two tori occurs. To understand what happens there, it is convenient to glue only the edges connected by broken arrow curves in Fig. 18.8 as A to B in Fig. 18.9 instead of completing the two tori. The trajectories coming into the connection edge from left in Fig. 18.8 (along the short horizontal arrow) have smaller angles with the connection edge than the ones going out to right as clearly illustrated in Fig. 18.9C, so we see the trajectories are spread and then are inserted into the trajectories on the right cylinder. That is, the event happening at the connection is akin to card shuffling. Thus, we understand intuitively why the system 'produces randomness' despite its deterministic nature.

It is clear that the apparently random behavior observed in Fig. 18.3 is caused by the intrinsic nature of this *deterministic dynamical system* (= a system whose behavior is uniquely determined if its past is known) and that external noise has nothing to do with it. If we knew the trajectory precisely without any error, we would not lose any information even if there is an expansive tendency of the trajectory bundle. However, we can never know very small scales. This unknowable is amplified by the expansion of the trajectory bundle and then is fixed into the system behavior by the insertion occurring at the connection to the other cylinder, as illustrated in Fig. 18.9C. Consequently, we feel the behavior of this system random;[175] Recall the KS

[175] As can be seen from this example, nonlinearity is not needed to expand small scales. Nonlin-

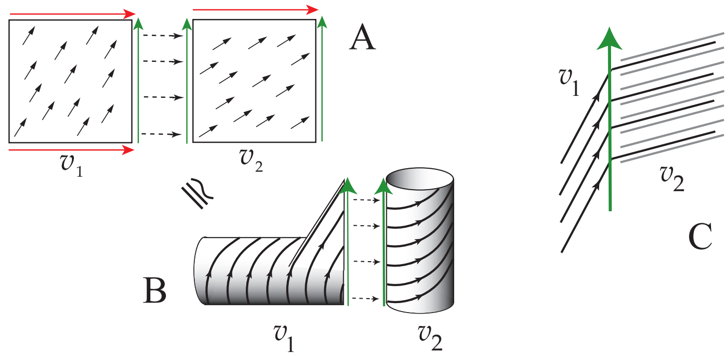

Figure 18.9: Let us pay attention to the connection from v_1 to v_2 only (the broken arrows in A, which is almost a copy of Fig. 18.8). Gluing only one pair of edges in A, we obtain B. The 'screw' part of B corresponds to the inside of the squares in A, where the trajectories run parallelly. In order to make easy to observe the crossover from the left to the right square (or torus), a cut is introduced in the left cylinder that does not affect the parallel trajectories and the edge to be glued is made straight. What happens at the connection between v_1 and v_2 is equivalent to spreading out the trajectory spacings and then inserting trajectories into the right cylinder as illustrated in C; it reminds us of shuffling cards.

entropy computation in the Sinai billiard in **??**.

18.6 Another illustration of randomization

If we deform the system slightly further, we can map it to a system that must be familiar to chaos aficionados. If we squish the cylinders in Fig. 18.9B on which the motion is 'spiral,' we obtain Fig. 18.10.

The figure reminds us of the famous *Lorenz model* that will be discussed in the two subsequent lectures.

18.7 Correspondence to discrete time system

A trajectory in the universal covering space is piecewise linear as we have seen in Fig. 18.6, so we must be able to record a trajectory only by recording its breaking points. We have only to pay attention to the points where the trajectory crosses the lattice

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earity is, however, usually needed to contain the system within a finite range despite local linear expansion. Therefore, if the phase space is intrinsically compact, then we might be able to say nonlinearity is not absolutely necessary for chaos. Actually, a typical chaotic system is given by linear maps from a torus onto itself. The decisive paper on this topic is R. L. Adler and B. Weiss, "Similarity of automorphisms of the torus," Memoir. Am. Math. Soc. **98** (1970).

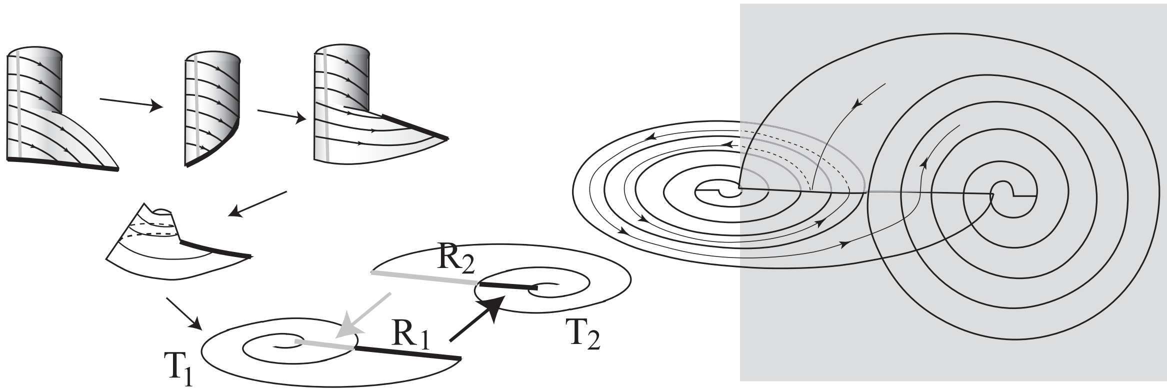

Figure 18.10: Further topological acrobat. The portions where the cylinders are connected are the same as Fig. 18.9B, but the ways to extend the trajectory ends are different. After running the spiral portion, trajectories are inserted into the other spiral. If we dovetail these two sheets with spiral trajectories, we obtain the rightmost figure. T_1, and T_2 correspond to the tori and R_1 and R_2 are connecting edges, where insertion and fixation of the expansion outcomes occur. The difference from Fig. 18.9B is only that the spiral on the cylinders becomes that on the disks, so, similarly as before, we can see how expanded microscopic details influence decisively the world around our scale.

lines in the universal covering space, and convert the continuous time system to a discrete time system (however, it is not a simple discretization such as observing a system periodically with stroboscopic light). More explicitly, in Fig. 18.5(i) (copied in Fig. 18.11) if we record the distance x measured along the edges from the corner C when the trajectory crosses L_1 or L_2, we can map a trajectory to a sequence $\{x_i\}$. This correspondence is one to one.[176] That is, from $\{x_i\}$ we can reconstruct the original continuous trajectory, because we know the speed of the point along the trajectory.

If the sequence $\{x_i\}$ can be reconstructed by a certain recursive rule, the system may be described more simply. Here, a 'recursive rule' means a rule that can give the next time state in terms of the current state (corresponding to the equation of motion in classical mechanics). Actually, this sequence $\{x_i\}$ is determined as a solution to an initial value problem of a (nonlinear) difference equation

$$x_{n+1} = \phi(x_n), \tag{18.1}$$

where $\phi : [0, 2] \to [0, 2]$ is given by the graph in Fig. 18.11

This discrete model can be, as explained in Fig. 18.12, folded into a unimodal piecewise linear map from $[0, 1]$ into itself.[177] Folding implies identifying two points

[176] Here we ignore the 'loose ends' of the trajectories between the lattice crossings.

[177] Unimodal piecewise linear maps are thoroughly studied in the following papers: Sh. Ito, S.

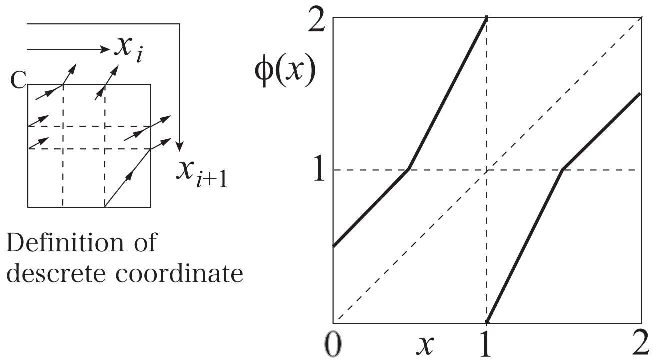

Figure 18.11: If the successive lattice crossing positions of a trajectory is described in terms of the distance measured along the edges of the square from the top left corner C, we obtain a one-dimensional map $[0,2] \to [0,2]$: $x_{i+1} = \phi(x_i)$.

in $[0,2]$. Therefore, the original behavior cannot be reconstructed from the reduced system. However, the correspondence is simple, so we can learn various things about the original system from the simplified system.

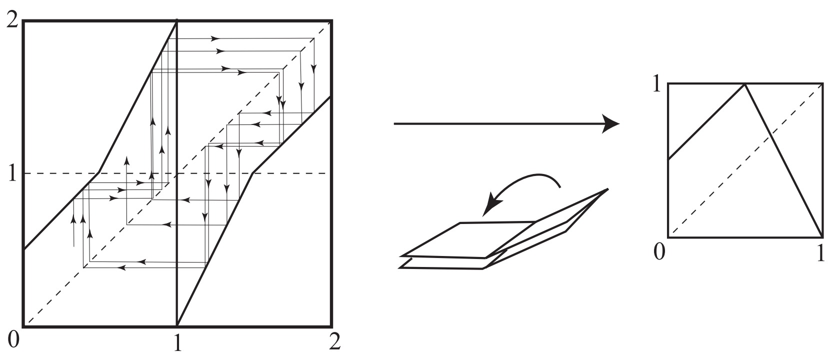

Figure 18.12: A reduced discrete map may be constructed by 'folding' the original map. Such unimodal piecewise linear maps are mathematically thoroughly understood. On the left the way to chase a discrete history is illustrated

Tanaka and H. Nakada, "On unimodal linear transformations and chaos I, II," Tokyo J. Math. **2**, 221, 241 (1979).