14 Lecture 14. Celestial mechanics

14.1 Two-body summary

Everybody should be very familiar with the two celestial body dynamics. Here I mention some topics that may not appear in elementary expositions.

⟪**Bertrand's theorem**⟫ Suppose U is a spherically symmetric potential. For a motion with non-zero angular momentum close to a circle to have a closed orbit, U must be harmonic or gravitational.[140]

⟪**Collision and extension**⟫ If the angular momentum is zero, the particles can collide. The orbit may be, however, uniquely extended beyond collision.[141]

14.2 Necessary condition for stability

The general n-body problem consists of n point masses (m_1, r_1), $\cdots$, (m_n, r_n) attracting one another according to the law of gravity. The total kinetic energy is

$$K = \frac{1}{2}\sum_i m_i \dot{r}_i^2, \tag{14.1}$$

and the potential energy U is

$$U = -\sum_{i<j} \frac{m_i m_j}{|r_i - r_j|}. \tag{14.2}$$

We describe the system from the inertial frame and the origin of the position coordinates is the center of mass.

We say the system is stable, if
(a) No collision: $|r_i - r_j| > 0$ for all i and j for all t.
(b) No escape: there is $c > 0$ such that $|r_i| < c$ for all i and t.

The fate of a three body system is very hard to predict (Fig. 14.1)

14.3 Jacobi's necessary condition for non-escape[142]

If there is no escape nor collisions, the total energy of the system must be negative.

[140] S. A. Chin, A truly elementary proof of Bertrand's theorem, arXiv:1411.7057 [physics.class-ph] (2014).
[141] Arnold III p56
[142] Arnold III p59

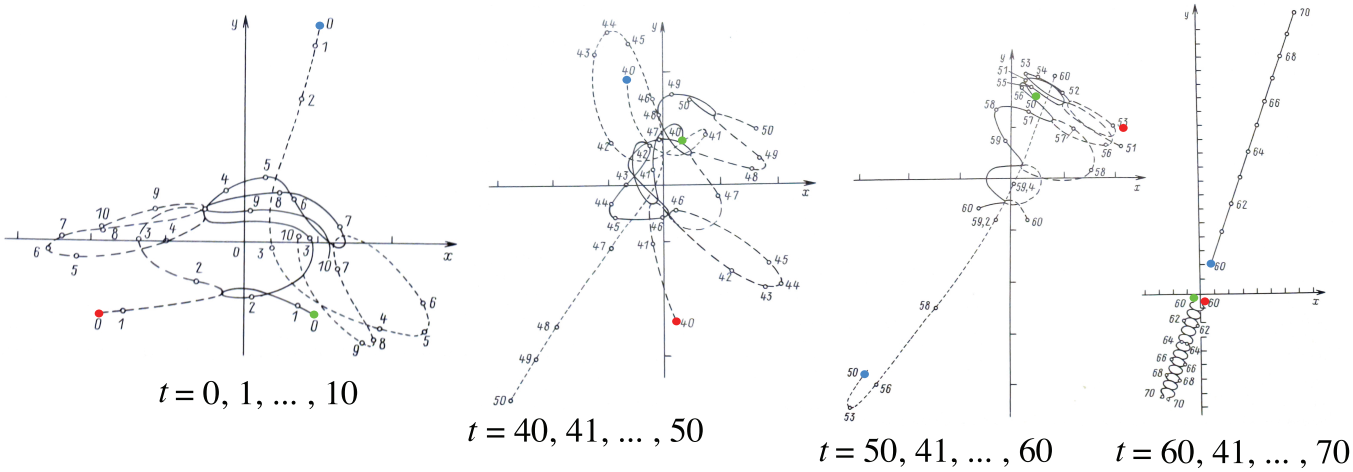

Figure 14.1: Fate of a three body system [Fig. 12-15 of Arnold III]

Remark For $n > 2$, this is not sufficient as seen in Fig. 14.1.
[Demo]
We use Lagrange's formula:*[143] for $I = \sum m_i r_i^2$ (the moment of inertia) and the total energy E

$$\ddot{I} = 4E - 2U. \tag{14.3}$$

Since $U < 0$, $E \geq 0$ implies $\ddot{I} > 0$, so $I(t)$ must be convex. Therefore, it cannot be

---

[143]

$$\ddot{I} = \sum_i m_i(2\dot{r}_i^2 + 2r_i\ddot{r}_i) = 4T - 2\sum_i r_i \sum_{j\neq i} \frac{m_i m_j}{|r_i - r_j|^3}(r_i - r_j)$$

(Note that $\nabla(1/|x|) = -(1/|x|^2)(x/|x|)$, because $\nabla|x|^2 = 2|x|\nabla|x| = 2x$). Therefore,

$$\begin{aligned}\ddot{I} &= 4T - \sum_i r_i \sum_j \frac{m_i m_j}{|r_i - r_j|^3}(r_i - r_j) - \sum_j r_j \sum_i \frac{m_i m_j}{|r_i - r_j|^3}(r_j - r_i) \\ &= 4T - 2\sum_{i,j} \frac{m_i m_j}{|r_i - r_j|^3}(r_i - r_j)^2 = 4T + 2U = 4E - 2U.\end{aligned}$$

bounded from above. Note that*[144]

$$I \sum_i m_i = \sum_{i<j} m_i m_j |r_i - r_j|^2 + \left(\sum_i m_i r_i \right)^2 . \tag{14.4}$$

The center of mass is fixed (say, 0), so the unbounded I means the increase of the mutual distance without bound.

If there is no collision and U is bounded from below, the virial theorem[145] may be applied to the time average of U: $\langle U \rangle = 2E$. Therefore, $E < 0$.

14.4 Collisions[146]

If there is a simultaneous n-body collision I vanishes. If $I(t) \to 0$ as $t \to t_0$, the the total angular momentum must be zero.

For binary collisions the motion can be smoothly extended beyond collisions. This means that for a three body problem with nonzero angular momentum, the motion is well defined for all t.[147]

14.5 Three-body problem and its reduction

For the $n = 3$ case (the three-body problem case) the original equation of motion is $3 \times 3 \times 2 = 18$ first order equations. The problem can be reduced to a problem of 6 first order differential equations (Lagrange 1772).
(i) The location and the velocity of the center of mass are cyclic coordinates (= the coordinate that do not explicitly appear in the equations), so we may regard them

---

[144]

$$I \sum_i m_i = \sum_{i,j} m_i m_j r_i^2 = \frac{1}{2} \sum_{i,j} m_i m_j (r_i^2 + r_j^2) = \frac{1}{2} \sum_{i,j} m_i m_j [(r_i - r_j)^2 + 2 r_i r_j].$$

[145] ⟪**Virial theorem**⟫ Notice that the long-time average of the derivative of a bounded function f(t) vanishes: $(1/T)\int_0^T f'(s)ds \to 0$. Consider pq. If the phase space for the system is bounded, then the time average of $d(pq)/dt$ vanishes. Therefore, for U which is a homogenous function of degree -1

$$p\dot{q} + q\dot{p} = 2T - q\frac{\partial U}{\partial q} = 2T + U = 2E - U.$$

Therefore, $\langle U \rangle = 2E$.

[146] Arnold III p59

[147] details can be found on p60-61 of Arnold III.

to be known. Thus 12 equations remain.
(ii) The total angular momentum is conserved. Thus 9 equations remain.
(iii) We perform the elimination of the nodes. The rotation as a whole may be regarded as known; actually this is done with (ii). Thus, 8 equations remain.
(iv) The total energy is conserved: 7 equations now.
(v) Now, eliminate t from the equation. The resultant set contains 6 first order equations.

The actual procedures and formulas are explained in detail on p343-347 of Whittaker.

14.6 Restricted problem of three body

If the third body (planetoid P) has an infinitesimal mass moving in the plane of the motion of the other two bodies S (Sun) and J (Jupiter) under their influence, the three-body problem is called the restricted problem of three bodies. The Hamiltonian governing the motion of P is given by

$$H = \frac{1}{2}(U^2 + V^2) - \frac{m_1}{\mathrm{SP}} - \frac{m_2}{\mathrm{JP}}. \tag{14.5}$$

SP ad JP are time-dependent, so H is not a constant of motion.

We introduce a moving coordinate system Fig. 14.2 whose origin is the CM of S and J and whose x-axis is from O to J. O-x is chosen to span the plane on which S and J always sit. Let n be the angular speed of SJ.

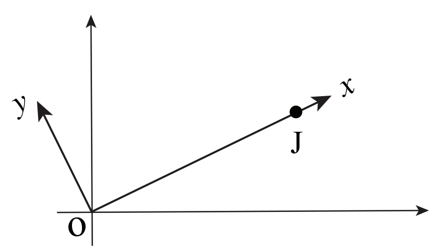

Figure 14.2: Moving coordinates for the restricted problem

Then, the original position coordinates (X, Y) of P may be related to (x, y) as

$$X = x\cos nt - y\sin nt \tag{14.6}$$
$$Y = x\sin nt + y\cos nt. \tag{14.7}$$

Now, we wish to canonical transform the coordinate system from the inertial (X, Y) (the old (q, p)) to the moving (x, y) (the new (Q, P)). We use $W = pq - F$, where F

is the ‘most’ standard one introduced in **13.1**

$$dW = PdQ - Kdt + qdp + Hdt = udx + vdy + XdU + YdV + (H-K)dt. \quad (14.8)$$

Since dW is exact, we can integrate this as $W = pq$ (i.e., in terms of the old variables). Expressing q in terms of the new coordinates, we get

$$W = U(x\cos nt - y\sin nt) + V(x\sin nt + y\cos nt). \quad (14.9)$$

Note that

$$X = \frac{\partial W}{\partial U},\ Y = \frac{\partial W}{\partial V},\ u = \frac{\partial W}{\partial x},\ v = \frac{\partial W}{\partial y}. \quad (14.10)$$

Since our new coordinates are time-dependent, the new Hamiltonian ahas an extra term:*[148]

$$\frac{\partial W}{\partial t} = H - K,\ \text{or}\ K = H - \frac{\partial W}{\partial t} = \frac{1}{2}(u^2+v^2) + n(uy - vx) - F, \quad (14.11)$$

where

$$F = \frac{m_1}{\mathrm{SP}} + \frac{m_2}{\mathrm{JP}} \quad (14.12)$$

is a function of x and y only, so note that K is time independent. $K = \text{const}$ is an integral and called the Jacobian integral. If the sum of the masses of S and J to be chosen unity, we may rewrite F as

$$F = \frac{1-\mu}{\mathrm{SP}} + \frac{\mu}{\mathrm{JP}}. \quad (14.13)$$

If $\mu = 0$, we ignore the effect of J. Thus, μ could be regarded as a perturbation parameter.

A nice restricted three-body simulation video is:
https://www.youtube.com/watch?v=jarcgP1rRWs

148

$$\frac{\partial W}{\partial t} = U(-nx\sin nt - ny\cos nt) + V(nx\cos nt - ny\sin nt).$$

Also we have

$$u = \frac{\partial W}{\partial x} = U\cos nt + V\sin nt,\ v = \frac{\partial W}{\partial y} = -U\sin nt + V\cos nt.\ (*)$$

Therefore,

$$\frac{\partial W}{\partial t} = x(-nU\sin nt + nV\cos nt) + y(-nU\cos nt - nV\sin nt) = n(xv - yu).$$

Obviously, $U^2 + V^2 = u^2 + v^2$, since $(*)$ is an orthogonal transformation.

14.7 Bruns' theorem

Theorem [Bruns 1887] The classical integrals (energy, momentum and angular momentum) are the only independent algebraic integrals of the problem of three bodies.

A proof can be found on p359-377 of Whittaker. The proof fully utilizes the peculiarity of the three-body problem.

14.8 Poincare's theorem 'denying' integrability of perturbed systems

Suppose we have a completely integrable system, whose Hamiltonian is given in terms of action variables only as $H_0(I)$ (with non-resonance condition satisfied, i.e., the Hessian of $H_0(I)$ is non-singular). The perturbation term $H_1(\theta, I)$ is analytic in I and the angle variable θ. The total Hamiltonian reads

$$H(\theta, I, \mu) = H_0(I) + \mu H_1(\theta, I). \tag{14.14}$$

Then, under a natural condition, there is no first integral of motion other than H itself that are analytic in μ.

Poincare proved in 1889 this theorem for the system with two degrees of freedom (as in the restricted three-body problem).

Suppose there is a first integral of motion Φ

$$\Phi = \Phi_0 + \mu\Phi_1 + \mu^2\Phi_2 + \cdots. \tag{14.15}$$

The key observation is that Φ_0 must be a function of H_0, which follows from the observation that Φ_0 is a function of I only. If Φ depends on μ smoothly, perhaps it is not a wild guess that this structure must be preserved for Φ (i.e., Φ must be a function of H only). A proof is given in **14.10-14.12**.[149]

14.9 Significance of Poincare's negative result

Bruns proved[150] in 1887 that for the three-body problem algebraic integrals of motion are exhausted by the classical integrals of motion (3 CM coordinates$-$ t $\times$ velocity, 3 angular momentum components, 3 momentum components and energy), so no simplification further than accomplished by Lagrange long ago (see **14.5**). Then,

[149] based on H. Yoshida 5.1 in Y. Ohnuki and H. Yoshida, *Mechanics* (Iwanami 1994).

[150] E. T. Whittaker, *A treatise on the analytical dynamics of particles and rigid bodies* (Cambridge, 1937 [4th edition]) p358-377.

Poincare proved even for the restricted simpler problem there is no generally smooth (wrt μ) integration possible. Thus, almost all the people lost hope in solving the celestial mechanics in a closed form.

However, do not forget that Poincare did not show that for fixed values of μ analytic Φ exists (a less smooth Φ may exist).

14.10 Φ_0 is a function of I only

Let us Fourier-expand Φ_0:

$$\Phi_0(\theta, I) = \sum_{k\in\mathbb{Z}^2} \phi_k(I)e^{ik\cdot\theta}. \tag{14.16}$$

The Fourier expansion of $[H_0(I), \Phi_0]_{PB} = 0$ reads

$$\frac{\partial}{\partial I}H_0(I)\cdot\frac{\partial}{\partial\theta}\Phi_0 = \nabla H_0(I)\cdot\sum_{k\in\mathbb{Z}^2}(ik)\phi_k(I)e^{ik\cdot\theta} = 0. \tag{14.17}$$

Therefore, for all k

$$\nabla H_0(I)\cdot k\phi_k(I) = 0. \tag{14.18}$$

This must be true for any I, so its derivative wrt I must vanish. Thus,

$$\mathrm{Hess}(H_0(I))k\phi_k(I) + \nabla H_0(I)\cdot k\nabla\phi_k(I) = 0. \tag{14.19}$$

If $\phi_k(I) \neq 0$, then (14.18) implies $\nabla H_0(I)\cdot k = 0$, so $\mathrm{Hess}(H_0(I))k$ must vanish. But since the Hessian is non-singular, this is true only for $k = 0$. Thus, only $\phi_0(I) \neq 0$. That is, Φ_0 cannot depend on θ.

14.11 Φ_0 is a function of H_0 only

This is demonstrated under a complicated condition whose significance is unclear to me. So let us assume this. That is (recall that our system has 2 degrees of freedom),

$$\frac{\partial(H_0, \Phi_0)}{\partial(I_1, I_2)} = 0. \tag{14.20}$$

This implies[151] the existence of a function ψ such that $\Phi_0 = \psi\circ H_0$.

[151] This conclusion follows from (14.20) if everything is smooth enough; of course, we have assume everything is holomorphic, so the argument is OK. Since Jacobian is the volume ratio of $dI_1 d_I 2$ and $dH_0 d\Phi_0$, (14.20) implies dH_0 and $d\Phi_0$ are parallel.

14.12 Conclusion of proof: Φ is a function of H only

Take ψ in **14.11** and make $\Phi - \psi(H)$. This is $\Phi_0 - \psi(H_0) = 0$ for $\mu = 0$. Therefore, we may write

$$\Phi(\theta, I; \mu) - \psi(H(\theta, I; \mu)) = \mu\Phi^{(1)}(\theta, I; \mu), \tag{14.21}$$

which is an integral of motion as well. Expanding this,

$$\Phi^{(1)} = \Phi_0^{(1)} + \mu\Phi_1^{(1)} + \cdots, \tag{14.22}$$

we see that $\Phi_0^{(1)}$ is a function of H_0 (a function of I only is a function of H_0):

$$\Phi_0^{(1)} = \psi^{(1)}(H_0), \tag{14.23}$$

so we can repeat that argument as

$$\Phi^{(1)} - \psi^{(1)}(H) = \mu\Phi^{(2)}(\theta, I; \mu). \tag{14.24}$$

Thus,

$$\begin{aligned}
\Phi &= \psi(H) + \mu\Phi^{(1)} && (14.25)\\
&= \psi(H) + \mu(\psi^{(1)}(H) + \mu\Phi^{(2)}) && (14.26)\\
&= \psi(H) + \mu\psi^{(1)}(H) + \mu^2\Phi^{(2)} && (14.27)\\
&\cdots && (14.28)\\
&= \psi(H) + \mu\psi^{(1)}(H) + \mu^2\psi^{(2)}(H) + \cdots && (14.29)
\end{aligned}$$

This implies that if Φ can be obtained perturbatively, it is not an independent invariant of motion.

14.13 Asteroids and gaps

Due to Jupiter they could not form a planet (inside the so-called frost line). The total mass is about 4% of Moon and 40% of the total mass is concentrated in Ceres and Vesta.
Introductory video (this is for kids, but good)
    https://www.youtube.com/watch?v=iy19nHTVLEY
All known asteroids
    https://www.youtube.com/watch?v=vfvo-Ujb_qk

There are several major gaps in the distribution of asteroids called Kirkwood gaps (discovered 1866 by D Kirkwood 1814-1895). He correctly explained their origin in terms of motional resonance with Jupiter (e.g., 3:1 at 2.5 AU; see Fig. 14.3).

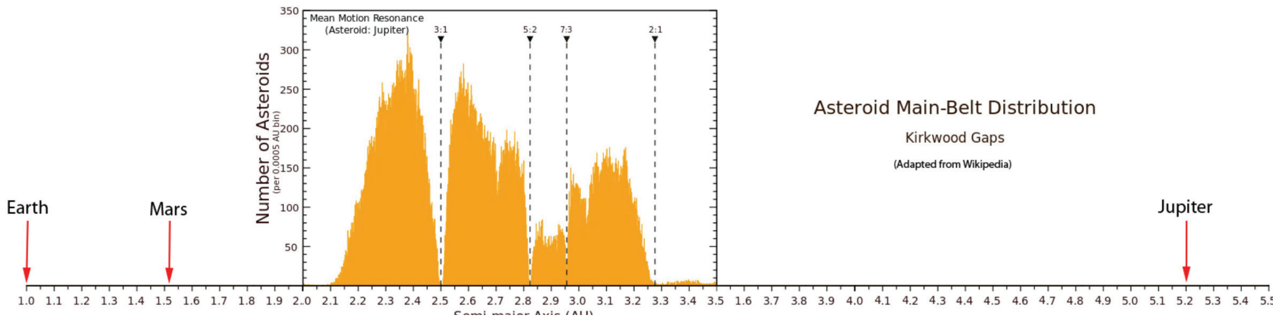

Figure 14.3: Asteroid belt and Kirkwood gaps

14.14 Special solutions of restricted three-body problem

Needless to say, many people tried to find special solutions (orbits) for the restricted three-body problem. There are 5 equilibrium points in the moving coordinate system (x, y). Three of them are along the SJ axis (Euler's linear solutions), but they are not stable. The remaining two are Lagrange's regular triangle solutions: the points are at the apexes of the regular triangles on the rotational plane (the plane spanned by (x, y)) whose one edge is the SJ segment.

Are the Lagrange points stable? For this to be stable it is not very hard to show that $\mu(1-\mu) < 1/27$ (or $\mu < 0.0385\cdots$). To show the stability actually, we must show the existence of small invariant tori surrounding these orbits. Thus, it is related to KAM and highly nontrivial, but except for three values of $\mu < 0.0385\cdots$ their stability was shown by Arnold.

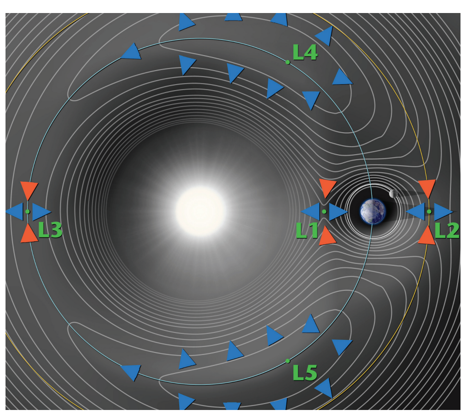

Figure 14.4: A gravitational potential contour plot showing Earth's Lagrangian points; L4 and L5 are above and below the planet, respectively. [Wikipedia: Jupiter Trojan]

The equilibrium points may not be so hard to find from the gravitational potential plot (Fig. 14.4, although this is for the earth). Lagrange's points are potential max points. Then, how can a particle stay near the hill tops? Physically speaking, this is due to Coriolis' force.

We actually observe asteroids around Lagrange's equilibrium points along Jupiters orbit (the so-called Trojan and Greek asteroid groups; the Trojan 'camp' trails Jupiter). The Hilda group asteroids are strong resonance with Jupiter. A good illustration of Trojan (green), Greek (red), and Hilda (white) families of asteroids is:
https://people.duke.edu/~ng46/borland/hilda%20family.gif.[152]
Here the Hilda is separated:
https://www.youtube.com/watch?v=yt1qPCiOq-8

14.15 Did asteroids cause mass extinctions?

The biggest mass extinction (the end Permian MS) is certainly not due to asteroids. I am not so convinced by the physicists' asteroid theory of the KPg mass extinction (65.5 MaBP; Cretaceous-Paleogene mass extinction 'due to' an impact at Chicxulub). It is sure that an asteroid hit had a severe effect. However, this does not mean the asteroid hit was the main cause; it could well be the last straw. Remember that big scale mass extinctions are always with drastic sea level changes as seen in Fig. 14.5. How can asteroid hit be predicted by the sea level changes? It should be fair to claim that an asteroid can cause havoc only when the biosphere is strained severely already.

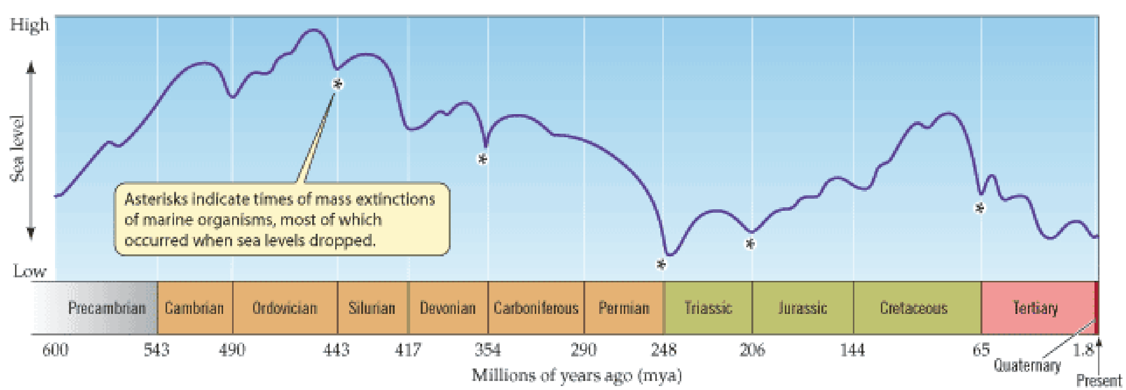

Figure 14.5: Mass extinctions and see level changes

[152]from DrBill's Astronomy Web Site http://hildaandtrojanasteroids.net

14.16 Rings

Saturn detail:

https://www.youtube.com/watch?v=ENwQ7-qLlrA