# 12 Lecture 12. Lax pair and Toda lattice

The story in this lecture goes as follows chronologically.

Scott-Russel observed (1834) the first solitary wave propagation along the Union Canal, Scotland (and also he confirmed it experimentally) (see **12.6**). Then, Rayleigh and Bousinesque explained it theoretically (∼1870), and later Korteweg and his student de Vries derived the equation governing the solitary waves (the KdV equation; 1895).

Fermi had been interested in the foundation of statistical mechanics, and the availability of Maniac I in Los Alamos allowed him to pursue the equilibration process of nonlinear lattice systems (Fermi-Pasta-Ulam problem; ∼1955; see **12.5**). This study taught us that nonlinearity is not enough to establish thermal equilibrium due to persistent recurrence.

Zabusky and Kruskal studied a continuum approximation of the system (= the KdV equation) and discovered solitons (1965).[126] This led to the Gardener, Greene, Kruskal and Miura discovery of the relation between solitons and the Schrödinger equation.[127] This led to the general theory of the Lax pair (1968; see **12.1**).

The explanation of recurrence in the FPU problem in terms of solitons was wrong, but a lattice system with solitons was later discovered (the Toda lattice, see **12.3**), which can be reduced in a certain limit to the KdV equation.

## 12.1 Lax pair

Suppose

$$\dot{x} = f(x) \tag{12.1}$$

can be expressed as

$$\dot{L} = [A, L], \tag{12.2}$$

where $A$ and $L$ are square matrices whose components are functions of $x$.

**Proposition** [Lax] The eigenvalues of $L$ are first integrals of (12.1).
[Demo] Let us solve (12.2) for $L$: for sufficiently small $s$ (notice that $A$ may be time-dependent)

$$L(t+s) = e^{sA}L(t)e^{-sA} + o[s]. \tag{12.3}$$

[126] "Interaction of "solitons" in a collisionless plasma and the recurrence of initial states", PRL **15**, 240 (1965).
[127] "Method for solving the Korteweg-de Vries equation" PRL **19**, 1095 (1967).

Therefore,

$$\det[L(t+s)-\lambda] = \det[L(t)-\lambda] + o[s] \tag{12.4}$$

That is, the characteristic equation of $L(t)$ is time-independent.

**Remark** The $L$-$A$ pair has been found for almost all problems of classical mechanics previously integrated by other methods.

## 12.2 Euler's equation

Euler's equation reads

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\omega}. \tag{12.5}$$

The Lax pair representation is

$$L = \begin{pmatrix} 0 & M_1 & -M_2 \\ -M_1 & 0 & M_3 \\ M_2 & -M_3 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \omega_1 & -\omega_2 \\ -\omega_1 & 0 & \omega_3 \\ \omega_2 & -\omega_3 & 0 \end{pmatrix}. \tag{12.6}$$

Confirm the Lax pair representation.

## 12.3 Toda lattice

Consider a chain consisting of point masses with the nearest intereaction only given by the following potential

$$\phi(r) = \frac{a}{b} e^{-br} + ar. \tag{12.7}$$

If the position of the $n$th point mass is $x_n$, the equation of motion reads

$$m\frac{d^2 x_n}{dt^2} = a\left[e^{-b(x_n - x_{n-1})} - e^{-b(x_{n+1}-x_n)}\right]. \tag{12.8}$$

so $r_n = x_{n+1} - x_n$ obeys

$$m\frac{d^2 r_n}{dt^2} = 2a\left[e^{-br_n} - \frac{1}{2}\left(e^{-br_{n+1}} + e^{-br_{n-1}}\right)\right]. \tag{12.9}$$

A dimensionless form reads

$$\dot{Q}_n = P_n, \quad \dot{P}_n = e^{-(Q_n - Q_{n-1})} - e^{-(Q_{n+1}-Q_n)}. \tag{12.10}$$

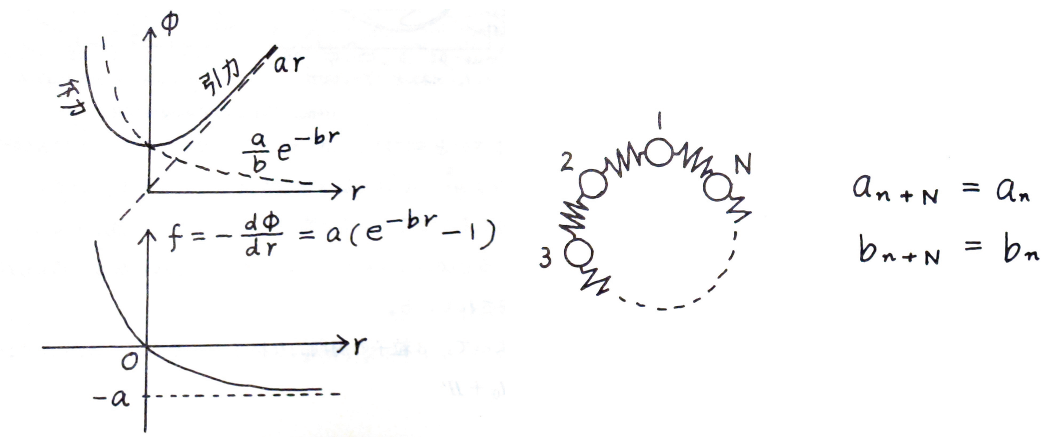


Figure 12.1: Toda lattice after M Toda

Now, introduce (these variables are not canonical variables)

$$a_n = \frac{1}{2} e^{-(Q_{n+1}-Q_n)/2}, \; b_n = \frac{1}{2} P_n. \tag{12.11}$$

Then the dimensionless form reads (with a cyclic boundary condition $a_{n+N} = a_n$, $b_{n+N} = b_n$.

$$\dot{a}_n = a_n(b_n - b_{n+1}), \tag{12.12}$$

$$\dot{b}_n = 2(a_{n-1}^2 - a_n^2). \tag{12.13}$$

This can be cast into the Lax pair form with

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & \cdots & 0 & a_N \\ a_1 & b_2 & a_2 & 0 & \cdots & \vdots & 0 \\ 0 & a_2 & b_3 & \ddots & \ddots & \vdots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & b_{N-2} & a_{N-2} & 0 \\ 0 & 0 & \cdots & 0 & a_{N-2} & b_{N-1} & a_{N-1} \\ a_N & 0 & \cdots & \cdots & 0 & a_{N-1} & b_N \end{pmatrix}. \tag{12.14}$$

and

$$A = \begin{pmatrix} 0 & -a_1 & 0 & \cdots & \cdots & 0 & a_N \\ a_1 & 0 & -a_2 & 0 & \cdots & \vdots & 0 \\ 0 & a_2 & 0 & \ddots & \ddots & \vdots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 & -a_{N-2} & 0 \\ 0 & 0 & \cdots & 0 & a_{N-2} & 0 & -a_{N-1} \\ -a_N & 0 & \cdots & \cdots & 0 & a_{N-1} & 0 \end{pmatrix}. \tag{12.15}$$

However, it is not straightforward to check that all the eigenvalues are independent invariants.

## 12.4 From Toda to KdV[128]

(12.16) may be rewritten by space-time scaling as

$$\frac{d^2 r_n}{dt^2} = 2e^{-r_n} - e^{-r_{n+1}} - e^{-r_{n-1}}. \tag{12.16}$$

In terms of interaction forces $f_n = e^{-r_n} - 1$ this equation reads

$$\frac{d^2}{dt^2}\log(1+f_n) = f_{n+1} + f_{n-1} - 2f_n. \tag{12.17}$$

Let us scale time and the function as ($h \in (0,1]$)

$$t = \tau/h^3, \quad f_n = h^2 u_n(\tau). \tag{12.18}$$

(12.17) reads

$$\frac{d^2}{d\tau^2}\log(1+h^2 u_n) = \frac{1}{h^4}(u_{n+1} + u_{n-1} - 2u_n). \tag{12.19}$$

Let us write $u_n(\tau) = u(hn,\tau)$. We introduce $x = nh$. Next, we observe this system from a moving coordinates:

$$x \to y = x - \left(\frac{1}{h^2} - h^2\right)\tau, \tag{12.20}$$

[128] Noriko Saito, "A transformation connecting the Toda lattice and the KdV equation," J Phys Soc Japan **49**, 409 (1980).

Notice that

$$\left.\frac{\partial}{\partial \tau}\right|_x u(x,\tau) = \left(\left.\frac{\partial}{\partial \tau}\right|_y + \frac{\partial y}{\partial \tau}\frac{\partial}{\partial y}\right) u(y,\tau) = \left(\left.\frac{\partial}{\partial \tau}\right|_y - \left(\frac{1}{h^2} - h^2\right)\frac{\partial}{\partial y}\right) u(y,\tau) \tag{12.21}$$

Therefore, the equation of motion reads[129]

$$\left(\frac{\partial}{\partial \tau} - \left(\frac{1}{h^2} - h^2\right)\frac{\partial}{\partial y}\right)^2 \log(1 + h^2 u(y,\tau)) = \frac{1}{h^4}(u(y+h,\tau) + u(y-h,\tau) - 2u(y,\tau)). \tag{12.22}$$

We consider $h \to 0$ limit. The LHS reads

$$\left(\frac{\partial^2}{\partial \tau^2} - 2\left(\frac{1}{h^2} - h^2\right)\frac{\partial^2}{\partial \tau \partial y} + \left(\frac{1}{h^2} - h^2\right)^2 \frac{\partial^2}{\partial y^2}\right)\left(h^2 u(y,\tau) - \frac{1}{2}h^4 u^2(y,\tau)\right) \tag{12.23}$$

$$\to -2\frac{\partial^2 u}{\partial \tau \partial y} + \frac{1}{h^2}\frac{\partial^2 u}{\partial y^2} - \frac{1}{2}\frac{\partial^2 u^2}{\partial y^2}. \tag{12.24}$$

The RHS reads

$$\frac{1}{h^4}(u(y+h,\tau) + u(y-h,\tau) - 2u(y,\tau)) \to \frac{1}{h^2}\frac{\partial^2 u}{\partial y^2} + \frac{1}{12}\frac{\partial^4 u}{\partial y^4}. \tag{12.25}$$

Thus, we have arrived at

$$\frac{\partial}{\partial y}\left[-2\frac{\partial u}{\partial \tau} - \frac{1}{2}\frac{\partial u^2}{\partial y} - \frac{1}{12}\frac{\partial^3 u}{\partial y^3}\right] = 0. \tag{12.26}$$

[ ] must be a function of $\tau$ only. Let us assume that the 'wave' is localized in space (i.e., if we chase it, it looks like a localize disturbance). Thus, [ ] must be zero. We have arrived at

$$\frac{\partial u}{\partial \tau} + \frac{1}{2}u\frac{\partial u}{\partial y} + \frac{1}{24}\frac{\partial^3 u}{\partial y^3} = 0. \tag{12.27}$$

This is the Korteweg-de Vries (KdV) equation.

Actual experiment of a solitary wave:
https://www.youtube.com/watch?v=w-oDnvbV8mY&frags=wn

[129] Saito starts from this equation; for $h = 1$ this is just the Toda lattice.

## 12.5 Fermi-Pasta-Ulam problem

Everybody knows that for a harmonic lattice no thermal equilibrium is realized. Thus, to study the foundation of thermal physics Fermi was interested in the effect of nonlinearity. Since analytic study is out of question, he utilized the Los Alamos computer Maniac I. The model is a chain of point masses interacting with the neighbor points with the potential

$$\phi(r) = \frac{\kappa}{2}r^2 + \frac{\kappa\alpha}{3}r^3. \tag{12.28}$$

The equation of motion is

$$m\frac{d^2y_n}{dt^2} = \phi'(y_n - y_{n-1}) - \phi'(y_{n+1} - y_n) \tag{12.29}$$

$$= \kappa(2y_n - y_{n+1} - y_{n-1}) + \kappa\alpha((y_n - y_{n-1})^2 - (y_n - y_{n+1})^2). \tag{12.30}$$

The numerical experiments exhibited almost complete recurrence of the intial condition.[130] Thus, it was clear that nonlinearity is not enough to thermalize the system.

FPU simulation (the quadratic case); you can observe recurrence:
https://www.youtube.com/watch?v=0WS548JX6D8

Zabusky and Kruskal studied its continuum limit. Assuming the lattice spacing is $h$, and we consider small $h$ limit.

$$y_{n\pm1} = y_n \pm h\frac{\partial y_n}{\partial x} + \frac{h^2}{2}\frac{\partial^2 y_n}{\partial x^2} \pm \frac{h^3}{3!}\frac{\partial^3 y_n}{\partial x^3} + \frac{h^4}{4!}\frac{\partial^4 y_n}{\partial x^4} \tag{12.31}$$

We see

$$2y_n - y_{n+1} - y_{n-1} = h^2\frac{\partial^2 y_n}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 y_n}{\partial x^4}, \tag{12.32}$$

and

$$(y_n - y_{n-1})^2 - (y_n - y_{n+1})^2 = (y_{n+1} - y_{n-1})(2y_n - y_{n+1} - y_{n-1}) \tag{12.33}$$

$$= 2h\frac{\partial y_n}{\partial x}\left(h^2\frac{\partial^2 y_n}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 y_n}{\partial x^4}\right). \tag{12.34}$$

Therefore, (12.30) becomes (with $\varepsilon = 2\alpha h$ and $c_0^2 = \kappa h/m$)

$$\frac{\partial^2 y}{\partial t^2} = c_0^2\left(1 + \varepsilon\frac{\partial y}{\partial x}\right)\frac{\partial^2 y}{\partial x^2} + \frac{c_0^2\alpha h^2}{12}\frac{\partial^4 y}{\partial x^4}. \tag{12.35}$$

[130] As to the experiment and the story surrounding it, see T. Dauxois, Fermi, Pasta, Ulam and a *mysterious* lady, arXiv:0801.1590.

Now, consider only the wave propagating to the right: new variables are $\xi = x - c_0 t$ and $\tau = t$. We have

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \tau}{\partial x}\frac{\partial}{\partial \tau} = \frac{\partial}{\partial \xi}, \;\; \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - c_0 \frac{\partial}{\partial \xi} \tag{12.36}$$

This gives

$$\frac{\partial^2}{\partial t^2} - c_0^2 \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial \tau} - c_0 \frac{\partial}{\partial \xi}\right)^2 - c_0^2 \frac{\partial^2}{\partial \xi^2} = \frac{\partial^2}{\partial \tau^2} - 2c_0 \frac{\partial}{\partial \tau}\frac{\partial}{\partial \xi}. \tag{12.37}$$

Thus (12.35) now reads

$$\frac{\partial^2 y}{\partial \tau^2} - 2c_0 \frac{\partial^2 y}{\partial \tau \partial \xi} = \varepsilon c_0^2 \frac{\partial y}{\partial \xi}\frac{\partial^2 y}{\partial \xi^2} + \frac{c_0^2 \alpha h^2}{12}\frac{\partial^4 y}{\partial \xi^4}. \tag{12.38}$$

We scale time as $s = \varepsilon \tau$. Then,

$$\varepsilon^2 \frac{\partial^2 y}{\partial s^2} - 2\varepsilon c_0 \frac{\partial^2 y}{\partial s \partial \xi} = \varepsilon c_0^2 \frac{\partial y}{\partial \xi}\frac{\partial^2 y}{\partial \xi^2} + \frac{c_0^2 \alpha h^2}{12}\frac{\partial^4 y}{\partial \xi^4}. \tag{12.39}$$

Ignoring the highest order in $\varepsilon$, and assuming $h^2$ and $\varepsilon$ are comparable, we get

$$\frac{\partial^2 y}{\partial s \partial \xi} = -\frac{1}{2} c_0 \frac{\partial y}{\partial \xi}\frac{\partial^2 y}{\partial \xi^2} - \frac{c_0 \alpha h^2/\varepsilon}{12}\frac{\partial^4 y}{\partial \xi^4}. \tag{12.40}$$

Define $u = \partial y / \partial \xi$. We get

$$\frac{\partial u}{\partial s} = -\frac{1}{2} c_0 u \frac{\partial u}{\partial \xi} - \frac{c_0 \alpha h^2/\varepsilon}{12}\frac{\partial^3 u}{\partial \xi^3}. \tag{12.41}$$

Notice that this can be rewritten as

$$\frac{\partial u}{\partial s} + B u \frac{\partial u}{\partial \xi} + \frac{\partial^3 u}{\partial \xi^3} = 0 \tag{12.42}$$

for any $B > 0$. First scale $u$ so that the ratio of the coefficients of the second and the third terms is $B$, then scale time. Thus, the KdV equation was obtained. Its numerical solution gave solitons.

## 12.6 What is KdV?

The Korteweg-de Vries equation has the following form

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{12.43}$$

This describes the propagation of waves in a shallow channel observed by an observer following the wave. It was motivated by the observation (and subsequent experiments) by John Scott Russell in 1834 along the Union Canal (Scotland) [John Scott Russell, Report on Waves, Report of the 14th Meeting of the British Association for the Advancement of Science, (1844), pp.311-390]

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

KdV can be written in the Lax form with

$$L = -\frac{d^2}{dx^2} + u(x,t), \quad A = -4\frac{d^3}{dx^3} + 3u\frac{d}{dx} + 3\frac{d}{dx}u. \tag{12.44}$$

Thus, eigenvalues of the Schrödinger operator are invariant.

The problem to find the potential (the inverse-scattering problem) and solving KdV are closely related (this leads to a very large research field).

3 soliton solution

https://www.youtube.com/watch?v=mqwi8UHYwJA https://www.youtube.com/watch?v=VFM48pSLwGc

Sine wave breaking into solitons

https://www.youtube.com/watch?v=agteGpbhEaE

### 12.7 Related equations

The equation with a constant $v$

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0 \tag{12.45}$$

has a general solution $u(t,x) = f(x - vt)$ where $f$ is an arbitrary differentiable function. If $v > 0$, this describes a translational motion to the right with speed $v$.

The following nonlinear equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{12.46}$$

may be locally (in space-time) understood, if we look at (12.45), as illustrated in Fig. 12.2. The position with height $u$ moves to the right with speed $u$:

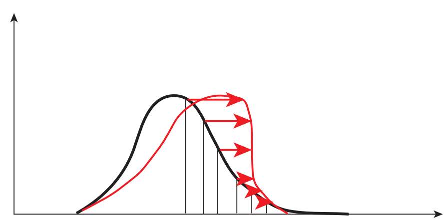

Figure 12.2: A short term evolution of $u$ governed by (12.46).

As is clear, the solution $u$ ceases to exist beyond some time as a differentiable function.

To prevent this singularity from being produced, there are two ways to modify (12.46). Both ways add a higher order spatial derivatives which adds dissipation (even orders) or dispersion (odd orders) to (12.46).

A famous example is Burgers' equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \tag{12.47}$$

which has a diffusion terms (viscosity effect) that kills sharp edges and prevent the emerging singularity. The KdV equation is the simplest dispersive case, where the speed of a plane wave depends on the wavelength (shorter wavelength waves travel relatively slower, so the 'wave front' is always dominated by longer wavelength waves; consequently the front is never sharp).

Incidentally, Burger's equation can be solved analytically with the aid of the Cole-Hopf transformation

$$u = -2\nu\frac{\partial}{\partial x}\log\phi = -2\nu\frac{1}{\phi}\frac{\partial \phi}{\partial x} \tag{12.48}$$

Then, we obtain a diffusion equation,

$$\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial x^2}, \tag{12.49}$$

which may be solved analytically.

This can be shown as follows: (12.47) may be rewritten as

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\nu \frac{\partial u}{\partial x} - \frac{1}{2}u^2\right). \tag{12.50}$$

Thus,

$$-2\nu \frac{\partial}{\partial x}\left(\frac{1}{\phi}\frac{\partial \phi}{\partial t}\right) = \frac{\partial}{\partial x}\left(-2\nu^2 \frac{\partial}{\partial x}\frac{1}{\phi}\frac{\partial \phi}{\partial x} - \frac{2\nu^2}{\phi^2}\left(\frac{\partial \phi}{\partial x}\right)^2\right). \tag{12.51}$$

That is

$$-2\nu \frac{\partial}{\partial x}\left(\frac{1}{\phi}\frac{\partial \phi}{\partial t}\right) = -2\nu^2 \frac{\partial}{\partial x}\left(-\frac{1}{\phi}\frac{\partial^2 \phi}{\partial x^2}\right). \tag{12.52}$$

This means

$$\frac{1}{\phi}\left(\frac{\partial \phi}{\partial t} - \nu \frac{\partial^2 \phi}{\partial x^2}\right) = f(t). \tag{12.53}$$

Assuming that $\phi$ is constant for large $|x|$ ($u$ vanishes there), we get (12.49).

## 12.8 Quick derivation of KdV from scratch[131]

If you read the original papers, or recent systematic expositions, the derivation via systematic expansion with respect to the smallness of the vertical displacements of water looks quite tedious and lengthy. Therefore, here we will 'cheat,' following the logic outlined in a recent textbook of classical physics.[132]

The strategy is to use the linearized Euler equation to obtain the nontrivial dispersion relation, and then the nonlinear transport ignored in the original Euler equation is recovered with the aid of the conservation of water.

[131] I am pretty sure that the KdV may be obtained as an RG equation just as almost all other 'named' equations. Publishable.

[132] The following is the logic in K. S. Thorne and R. D. Blandford, *Modern classical physics* (Princeton UP 2017) p852 made palatable.

### 12.9 Dispersion relation of shallow water waves

Consider the disturbance of surface height of a horizontal channel of depth $h_0$. Our starting point is always the Navier-Stokes equation

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\eta\Delta\boldsymbol{v} - \nabla P + \boldsymbol{F}, \tag{12.54}$$

with the incompressibility $\nabla\cdot\boldsymbol{v} = 0$, where $\rho$ is the density, $P$ the pressure and $\boldsymbol{F}$ the external force (we assume here gravity only, so it is a potential force). As long as the fluid speed is slow, we may drop the nonlinear term, and also we may ignore the viscosity term (i.e., we use the linearized Euler equation). In this case curl$\boldsymbol{v}$ is time independent, so if there is no vorticity initially, the flow is a potential flow, so we may introduce the velocity potential $\phi$:

$$\boldsymbol{v} = \nabla\phi. \tag{12.55}$$

The incompressibility means $\phi$ must be a harmonic function,

$$\Delta\phi = 0 \tag{12.56}$$

and the linearized Euler equation reads

$$\nabla\left(\frac{\partial\phi}{\partial t} + P/\rho + gz\right) = 0. \tag{12.57}$$

This means

$$\frac{\partial\phi}{\partial t} + P/\rho + gz = f(t). \tag{12.58}$$

If we are interested in a local disturbance, $f(t) = 0$.[133]

Let $\zeta(x,t)$ be the surface shape relative to the free surface at $z = 0$, where the pressure is constant $P_0$:

$$P_0 = -\rho\zeta g - \rho\frac{\partial\phi}{\partial t}. \tag{12.59}$$

This gives a boundary condition:

$$\frac{\partial\zeta}{\partial t}g + \frac{\partial^2\phi}{\partial t^2} = 0. \tag{12.60}$$

Since $\partial\zeta/\partial t = v_z \simeq \partial\phi/\partial z$; this is admissible since $\zeta$ is small. We have an equation for $\phi$

$$\left(\frac{\partial\phi}{\partial z} + \frac{1}{g}\frac{\partial^2\phi}{\partial t^2}\right)_{z=\zeta} = 0. \tag{12.61}$$

[133] This is what Landau did. We could instead redefine $\phi$ to absorb $f$ without spoiling $\boldsymbol{v} = \nabla\phi$.

However, $\zeta$ is small, so we may drop it. Thus, we have the following set of equation at the liquid surface:

$$\Delta\phi = 0, \tag{12.62}$$

$$\left(\frac{\partial\phi}{\partial z} + \frac{1}{g}\frac{\partial^2\phi}{\partial t^2}\right)_{z=0} = 0. \tag{12.63}$$

At the bottom $z = -h_0$ of the channel we must impose that condition that $v_z = 0$.

Let us solve the above boundary value problem for $\phi$ that is periodic:

$$\phi = f(z)\cos(kx - \omega t). \tag{12.64}$$

$\Delta\phi = 0$ reads

$$f''(z) - k^2 f = 0, \tag{12.65}$$

whose general solution is $f(z) = Ae^{kz} + Be^{-kz}$. Its derivative must vanish at the bottom $z = -h_0$:

$$kAe^{-kh_0} - kBe^{kh_0} = 0. \tag{12.66}$$

Therefore, $\phi$ must have the following form:

$$\phi = A\cosh k(z + h_0)\cos(kx - \omega t). \tag{12.67}$$

This must satisfy the boundary condition (12.63)

$$gk\sinh(kh_0)\cos(\omega t) - \omega^2\cosh(kh_0)\cos(\omega t) = 0. \tag{12.68}$$

Thus, we have obtained the dispersion relation:

$$\omega^2 = gk\tanh kh_0. \tag{12.69}$$

Let us assume that $kh_0$ is not large (shallow water wave[134]).

$$\omega^2 = gk\left(kh_0 - \frac{1}{3}(kh_0)^3\right) + \cdots = gh_0k^2\left(1 - \frac{1}{3}h_0^2k^2\right) + \cdots, \tag{12.70}$$

or

$$\omega = \sqrt{gh_0}k\left(1 - \frac{1}{6}h_0^2k^2\right). \tag{12.71}$$

[134]In the opposite limit we can compute the group velocity as $(1/2)\sqrt{g/k}$: long wavelength wave propagates fast.

This means that a propagating wave $\zeta$ obeys (cf $\omega \to i\partial_t$, $k \to -i\partial_x$)

$$\frac{\partial \zeta}{\partial t} + \sqrt{gh_0}\frac{\partial \zeta}{\partial x} + \frac{1}{6}h_0^2\frac{\partial^3 \zeta}{\partial x^3} = 0. \tag{12.72}$$

We cannot accurately reconstruct the nonlinear term (the original form of the second term on the LHS) from this.

## 12.10 Nonlinear transport term

To study the nonlinear term we restart from Euler's equation($/\rho$) and also the mass conservation equation:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + g\frac{\partial h}{\partial t} = 0, \tag{12.73}$$

$$\frac{\partial h}{\partial t} + \frac{\partial vh}{\partial x} = 0, \tag{12.74}$$

where $h = h_0 + \zeta$. That is, now we choose the bottom to be the origin of the height. Let us define $gh = \eta$. (12.74) read

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{\partial \eta}{\partial t} = 0. \tag{12.75}$$

$$\frac{\partial \eta}{\partial t} + v\frac{\partial \eta}{\partial x} + \eta\frac{\partial v}{\partial x} = 0, \tag{12.76}$$

If we wish to combine these two equations into one,[135] $v$ must be linearly combined with $\sqrt{\eta}$ (dimensional homogeneity requirement). Let us introduce $Z = \sqrt{\eta}$.

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + 2Z\frac{\partial Z}{\partial t} = 0. \tag{12.77}$$

$$2\frac{\partial Z}{\partial t} + 2v\frac{\partial Z}{\partial x} + Z\frac{\partial v}{\partial x} = 0, \tag{12.78}$$

Therefore,

$$\frac{\partial (v-2Z)}{\partial t} + v\frac{\partial (v-2Z)}{\partial x} - Z\frac{\partial (v-2Z)}{\partial t} = 0 \tag{12.79}$$

or

$$\frac{\partial (v-2Z)}{\partial t} + (v-Z)\frac{\partial (v-2Z)}{\partial x} = 0 \tag{12.80}$$

[135] The motivation for this is to find a conserved quantity that may be followed from a moving coordinate.

This implies that $v - 2\sqrt{gh}$ is constant, if we observe it from the moving coordinate with speed $v - \sqrt{gh}$.

Suppose the wave propagation starts from a quiet surface with height (depth) $h_0$ with $v = 0$. Initially, $v - 2\sqrt{gh} = -2\sqrt{gh_0}$. Since this must be constant, $v = 2\sqrt{gh} - 2\sqrt{gh_0}$. Putting this into (12.74), we obtain

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\sqrt{g}(h^{3/2} - hh_0^{1/2}) = \frac{\partial h}{\partial t} + (3\sqrt{gh} - 2\sqrt{gh_0})\frac{\partial h}{\partial x} = 0. \tag{12.81}$$

Since

$$3\sqrt{g(h_0+\zeta)} - 2\sqrt{gh_0} = \sqrt{gh_0} + \frac{3}{2}\sqrt{\frac{g}{h_0}}\zeta, \tag{12.82}$$

the equation for $\zeta$ reads

$$\frac{\partial \zeta}{\partial t} + \left(\sqrt{gh_0} + \frac{3}{2}\sqrt{\frac{g}{h_0}}\zeta\right)\frac{\partial \zeta}{\partial x} = 0 \tag{12.83}$$

Now, observe this from a moving coordinate with speed $\sqrt{gh_0}$. We arrive at

$$\frac{\partial \zeta}{\partial t} + \frac{3}{2}\sqrt{\frac{g}{h_0}}\zeta\frac{\partial \zeta}{\partial x} = 0. \tag{12.84}$$

Augmenting this with the higher order dispersion relation (or combining (12.85) with (12.72)) we finally obtain

$$\frac{\partial \zeta}{\partial t} + \frac{3}{2}\sqrt{\frac{g}{h_0}}\zeta\frac{\partial \zeta}{\partial x} + \frac{1}{6}h_0^2\frac{\partial^3 \zeta}{\partial x^3} = 0, \tag{12.85}$$

the Korteweg-de Vries equation.