11 Lecture 11. Determination of motion

Here, we discuss (analytically) solvable classical mechanics problems. First, we understand the completely integrable cases geometrically (Liouville-Arnold theorem), and then describe such systems in terms of the action-angle variables.

Then, in Lecture 12, a recent integration technique (the Lax pair approach) is introduced. It will be illustrated with the Toda lattice. To discuss the method historically, we go back to Korteweg-de Vries (KdV) equation. [This naturally leads us to solitons, but I do not go into this vast topic.]

11.1 Determination of motion

Here the word 'determine' implies that we can construct a map that allows us to transform the dynamical flow in the phase space into a simple flow ('laminar flow' on a torus just like the rectifiability theorem **3.19**): (Q, P) (P is constant and $Q = Pt+c$) as we will see below. That is, to rectify the flow while preserving the structure of Hamiltonian dynamical systems (i.e., by a canonical transformation). If this can be done, the system is said to be integrable.

Thus, we need at least a rudimentary familiarity to the canonical transformation.

Needless to say, not all the systems allow such transformations (only integrable systems). Poincaré's recognition that the three (celestial) body problem is not integrable in this sense was the birth of modern classical mechanics/study of dynamical systems.

11.2 Canonical transformation

The transformation $T : (q, p) \to (Q, P)$ that preserves the form of the canonical equation of motion (10.14) is called a *canonical transformation*.[117] Thus, the transformation T to be canonical implies that the equation of motion in terms of the new variables reads

$$\frac{dQ_i}{dt} = \frac{\partial K}{\partial P_i}, \ \frac{dP_i}{dt} = -\frac{\partial K}{\partial Q_i}, \tag{11.1}$$

where $K(Q, P) = H(q(Q, P), p(Q, P))$ is the Hamiltonian in terms of the new variables.

[117] Corresponding to unitary transformations in quantum mechanics.

11.3 Generator of canonical transformation

For (11.1) to hold, we must have the action principle in terms of the new variables (Q, P).

Let (q, p) be the canonical coordinates and (Q, P) the new coordinates (they are phase functions). Then, the Hamilton's principle reads

$$\delta \int (P\dot{Q} - K)dt = 0. \tag{11.2}$$

Thus, we demand (see the remark below)

$$\delta \int \Big[(p\dot{q} - H) - (P\dot{Q} - K)\Big]\, dt = 0. \tag{11.3}$$

This means that the difference of these two integrals can be constant, so

$$(pdq - Hdt) - (PdQ - Kdt) = dF, \tag{11.4}$$

or

$$dF = pdq - PdQ + (K - H)dt \tag{11.5}$$

must be path-independent. That is, dF must be exact.[118] Thus, there is a function $F = F(q, Q, t)$ such that

$$dF = \sum_i p_i dq_i - \sum_i P_i dQ_i + (K - H)dt, . \tag{11.6}$$

F is called the *generator* of the canonical transformation $T : (q, p) \to (Q, P)$. (11.6) gives

$$\frac{\partial F}{\partial q} = p,\ \frac{\partial F}{\partial Q} = -P,\ \frac{\partial F}{\partial t} = K - H. \tag{11.7}$$

Solving these equations we can construct the canonical transformation T. In particular, if F is time-independent, then $K = H$ is obtained by replacing q and p in H in terms of Q and P.

Remark: For the system described by q and p to satisfy the canonical equation of motion, much more general transformations are allowed, but the form with the generator discussed above is the most convenient, especially because the Hamiltonian is virtually preserved if the generator is time-independent. Thus, when we say

[118] If form θ can be written as an (external) derivative of another form ω as $\theta = d\omega$, θ is called an exact form. A form θ that satisfies $d\theta = 0$ is called a closed form. Obviously, an exact form is a closed form.

'canonical transformation' we restrict ourselves to this type of transformations.

Applying a sort of Legendre transformation to generators, we can construct different (perhaps more convenient) transformations:

$$d(F + PQ) = pdq + QdP + (K - H)dt. \tag{11.8}$$

Thus, replacing F with $G = F + \sum_i P_i Q_i$, we obtain

$$dG = \sum_i p_i dq_i + \sum_i Q_i dP_i + (K - H)dt. \tag{11.9}$$

11.4 Complete integrability

Let $F_1, \cdots, F_n$ ($F_1 = H$) be pairwise involutive (i.e., $[F_i, F_j]_{PB} = 0$) smooth functions on $2n$-mfd M. If F_i's are functionally independent (or ∇F_i's are linearly independent on a dense set of M), then,
(1) $M_f = \{x \mid F_i(x) = f_i\}$ is diffeomorphic to $T^k \times \mathbb{R}^{n-k}$.
(2) The Hamiltonian flow on M_f takes the following form with the coordinate system on $T^k \times \mathbb{R}^{n-k}$: $\varphi_1, \cdots, \varphi_k$ mod 2π, $y_1, \cdots, y_{n-k}$

$$\dot{\varphi}_i = \omega_i, \;\; \dot{y}_j = c_j, \tag{11.10}$$

where ω and c are constants.

11.5 Slightly general form of integrability[119]

For a n-dimensional Hamiltonian system suppose there are n first integrals $F_1, \cdots, F_n$ (i.e., $[H, F_i] = 0$) such that

$$[F_i, F_j] = \sum_k c_{ij}^k F_k, \tag{11.11}$$

where c_{ij}^k are constants. If
(1) on $M_f = \{x \mid F_i(x) = f_i\}$ F_i are functionally independent
(2) $\sum_k c_{ij}^k f_k = 0$.
(3) The Lie algebra $\mathcal{A}$ of linear combination $\sum \lambda_i F_i$ is solvable.[120]

[119] Theorem 1 Chapter 4 of V. I. Arnold, V. V. Kozlov and A. I. Neishtadt, Mathematical aspects of classical and celestial mechanics in *Dynamical systems III* ed. V. I. Arnold (Springer, 1988).

[120] ⟪**Solvable Lie algebra**⟫ A Lie algebra $\mathfrak{g}$ is solvable, if

$$\mathfrak{g} \supset [\mathfrak{g}, \mathfrak{g}] \supset [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] \to \{0\}$$

.

Then, the Hamiltonian dynamical system is solvable by quadrature. p107

11.6 Liouville-Arnold's theorem

This is a special case of **11.4** when the motion is bounded (confined in a finite space; M is compact). In this case **11.4** reads:
Let $F_1, \cdots, F_n$ ($F_1 = H$) be pairwise involutive (i.e., $[F_i, F_j]_{PB} = 0$) smooth functions on $2n$-mfd M. If F_i's are functionally independent (or ∇F_i's are linearly independent on a dense set of M), then
(1) $M_f = \{x \mid F_i(x) = f_i\}$ is diffeomorphic to T^n.
(2) The Hamiltonian flow on M_f takes the following form with the coordinate system on T^n: $\varphi_1, \cdots, \varphi_n$ mod 2π

$$\dot{\varphi}_i = \omega_i, \tag{11.12}$$

where ω are constants.

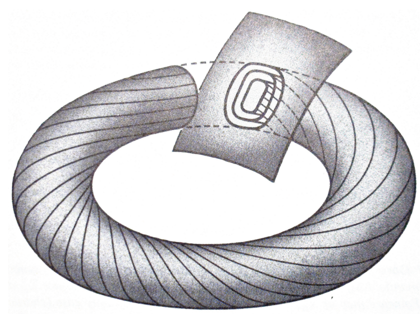

Figure 11.1: Arnold-Liouville foliated torus [Fig. 8.3.2 of Abraham & Marsden]

[Demo] Consider n Hamiltonian dynamical systems on M each of which is governed by F_j ($j \in \{1, \cdots, n\}$) as its Hamiltonian. We introduce a canonical coordinate system (q, p) for the phase space. The canonical equation of motion (s is the time variable) governed by the Hamiltonian F_j reads

$$\frac{dq}{ds} = \frac{\partial F_j}{\partial p}, \; \frac{dp}{ds} = -\frac{\partial F_j}{\partial q}. \tag{11.13}$$

Since F_j are involutive, the tangent vectors parallel to the flows for these n systems commute. Let us check this:[121]

$$[F_{1,p}\partial_q - F_{1,q}\partial_p, F_{2,p}\partial_q - F_{2,q}\partial_p] = [F_{1,p}\partial_q, F_{2,p}\partial_q] + [F_{1,q}\partial_p, F_{2,q}\partial_p] - [F_{1,p}\partial_q, F_{2,q}\partial_p] - [F_{1,q}\partial_p, F_{2,p}\partial_q] \tag{11.14}$$

$$\begin{aligned}
[F_{1,p}\partial_q, F_{2,p}\partial_q] &= (F_{1,p}F_{2,pq} - F_{2,p}F_{1,pq})\partial_q. && (11.15)\\
[F_{1,q}\partial_p, F_{2,q}\partial_p] &= (F_{1,q}F_{2,qp} - F_{2,q}F_{1,qp})\partial_p. && (11.16)\\
[F_{1,p}\partial_q, F_{2,q}\partial_p] &= F_{1,p}F_{2,qq}\partial_p - F_{2,q}F_{1,pp}\partial_q && (11.17)\\
[F_{1,q}\partial_p, F_{2,p}\partial_q] &= F_{1,q}F_{2,pp}\partial_q - F_{2,p}F_{1,qq}\partial_p && (11.18)\\
& && (11.19)
\end{aligned}$$

Collecting the terms containing ∂_q, we get

$$F_{1,p}F_{2,pq} - F_{2,p}F_{1,pq} + F_{2,q}F_{1,pp} - F_{1,q}F_{2,pp} = -\partial_p(F_{1,q}F_{2,p} - F_{1,p}F_{2,q}) = \partial_p[F_2, F_1]_{PB} = 0. \tag{11.20}$$

Similarly, collecting the terms containing ∂_p, we get

$$F_{1,q}F_{2,qp} - F_{2,q}F_{1,qp} - F_{1,p}F_{2,qq} + F_{2,p}F_{1,qq} = \partial_q(F_{1,q}F_{2,p} - F_{2,q}F_{1,p}) = \partial_q[F_1, F_2]_{PB} = 0. \tag{11.21}$$

Thus, the Lie bracket (11.14) vanishes.[122] This implies that the time evolution groups due to $\{F_i\}$ commute. Thus there are n independent directions.

The compact manifold M_f must be invariant under the n-dimensional Abelian group defined by the time evolutions due to $\{F_i\}$. This implies that M_f is diffeomorphic to T^n.

11.7 Action-angle variables[123]

If M is compact for a completely integrable system, then M_f is diffeomorphic to T^n.
(1) The small nbh of M_f in the ambient symplectic mfd M is diffeomorphic to $D \times T^n$, where D is a small domain in $\mathbb{R}^n$.
(2) In $D \times T^n$ there exists canonical coordinates φ mod $2\pi$and I ($I \in D$, $\phi \in T^n$ in which F_k's depend only on I.

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[121] We are checking $[X, Y]f = 0$ for any differentiable f. That is, Two paths going from $(a, b) \to (A, B)$, that is, $(a, b) \to (A, b) \to (A, B)$ and $(a, b) \to (a, B) \to (A, B)$ give the same result. Notation: $\partial_x f = f_{,x}$ is used.

[122] We have shown that $[F, G]_{PB} = 0$ implies that the Hamiltonian flows defined by F and by G determine independent (commutative) time evolutions.

[123] Nekhoroshev proved a version when the involution relation holds for a subset of $\{F_j\}$ (Arnold DS III p117).

I are called action variables and φ angle variables. The completely integrable Hamiltonian H has the form $H = H(I)$. Consequently, the equation of motion reads

$$\dot{I} = 0, \quad \dot{\varphi} = \omega(I) = \frac{\partial H}{\partial I}. \tag{11.22}$$

Thus I labels the invariant tori. If the Hessian of $H(I)$ is nondegenerate, the systems is said to be non-degenerate.

11.8 Demonstration of 11.7[124]

Thanks to the Liouville-Arnold theorem **11.6** in the nbh of the torus $M_f \equiv T^n$, we can take as coordinates the functions $I_i = F_i$ and the angles φ_i. Since the differentials dF_i are linearly independent, we can use I and ϕ as a coordinate system for $D \times T^n$. We have to show that I, φ system is a canonical coordinate system. We know $[I_i, I_j]_{PB} = 0$ (involutive invariants). $[\varphi_j, I_k]_{PB}$ is a constant on M_f according to **11.6**, so it is a function of I only. The Jacobi identity applied to I_k, φ_i and φ_j implies (here $[\ ,\ ]$ implies $[\ ,\ ]_{PB}$)

$$[I_k, [\varphi_i, \varphi_j]] + [\varphi_i, [\varphi_j, I_k]] + [\varphi_j, [I_k, \varphi_i]] = 0, \tag{11.23}$$

so $[I_k, [\varphi_i, \varphi_j]]$ is a function of I only. Since I and φ describe the system dynamics, the Poisson bracket determinant $\det[I_k, \varphi_i]$ must not be zero. Therefore, we can solve the following equation

$$[I_k, [\varphi_i, \varphi_j]] = \sum_s [I_k, \varphi_s] \frac{\partial}{\partial \varphi_s} [\varphi_i, \varphi_j] \tag{11.24}$$

to conclude that $\partial[\varphi_i, \varphi_j]/\partial \varphi_s$ is a function of I only. Therefore, we may write

$$[\varphi_i, \varphi_j] = \sum_s f_{ij}^s(I) \varphi_s + g_{ij}(I). \tag{11.25}$$

Since the derivatives of φ must be single-valued, when φ_i and φ_j go around the torus, the Poisson bracket should also return to the original value. Therefore, φ_s dependence should not exist: $f_{ij}^s(I) = 0$.

We change $I \to J$ to make $[J_i, \varphi_k] = \delta_{jk}$. Let us check the possibility.

$$[I_i, \varphi_j] = \sum_k \frac{\partial I_i}{\partial J_k} [J_k, \varphi_j] = \frac{\partial I_i}{\partial J_j}. \tag{11.26}$$

[124] The proof here paraphrases with details the proof on p115 of Arnold, DS III. A general version for the situation **11.5** also holds (p118 Th10).

The consistency condition reads

$$\frac{\partial}{\partial J_s}[I_i,\varphi_j]=\frac{\partial}{\partial J_j}[I_i,\varphi_s] \iff \sum_k [I_k,\varphi_s]\frac{\partial}{\partial I_k}[I_i,\varphi_j]=\sum_k [I_k,\varphi_j]\frac{\partial}{\partial I_k}[I_i,\varphi_s], \tag{11.27}$$

but this follows from the Jacobi identity for I_i, φ_j nd φ_k:

$$[I_i,[\varphi_j,\varphi_k]]+[\varphi_j,[\varphi_k,I_i]]+[\varphi_j,[I_i,\varphi_k]]=0 \Rightarrow [\varphi_j,[\varphi_k,I_i]]+[\varphi_j,[I_i,\varphi_k]]=0. \tag{11.28}$$

Thus, we can obtain J such that $[J_i,\varphi_j]=\delta_{ij}$.

The remaining task[125] is to guarantee $[\varphi_i,\varphi_j]=0$. If this is not the case, set $\psi_i=\varphi_i+f_i(J)$. Then, we must solve the following equation for f:

$$[\psi_i,\psi_j]=[\varphi_i,\varphi_j]-\frac{\partial f_i}{\partial J_j}+\frac{\partial f_j}{\partial J_i}=0. \tag{11.29}$$

Consider formally

$$q=\sum_{i<j}[\varphi_i,\varphi_j]dJ_i\wedge dJ_j=\sum_{i<k}\left(\frac{\partial f_i}{\partial J_k}-\frac{\partial f_k}{\partial J_i}\right)dJ_i\wedge dJ_k=d\sum_i f_i dJ_i, \tag{11.30}$$

so $dq=0$ is the consistency (solvability) condition. This follows from the closedness of the symplectic 2-form $d\varphi\wedge dJ$ (which is a shorthand of $\sum_i d\varphi_i\wedge dJ_i$; that is, the invariance of (13.26) in **13.11**): $d(d\varphi\wedge dJ)=0$; we can compute this 2-form as

$$\sum_i d\varphi_i\wedge dJ_i=\sum_{i,j}\frac{\partial\varphi_i}{\partial J_j}dJ_j\wedge dJ_i=\sum_{i<j}\left(\frac{\partial\varphi_i}{\partial J_j}-\frac{\partial\varphi_j}{\partial J_i}\right)dJ_j\wedge dJ_i, \tag{11.31}$$

but note that

$$\frac{\partial\varphi_i}{\partial I_j}-\frac{\partial\varphi_j}{\partial I_i}=[\varphi_i,\varphi_j]. \tag{11.32}$$

Thus, q is a symplectic 2-form, so $dq=0$.

11.9 Calculation of action variables

Let $q.p$ be the symplectic coordinates in $\mathbb{R}^{2n}$, and $\gamma_1,\cdots,\gamma_n$ are the fundamental cycles of $M_f\equiv T^n$. Since $pdq-Id\varphi$ is closed, the difference

$$\oint_{\gamma_s}pdq-\oint_{\gamma_s}Id\varphi=\oint_{\gamma_s}pdq-2\pi I_s \tag{11.33}$$

[125] $[J_i,J_k]=0$, because $J=J(I)$ and I does not depend on φ.

is a constant. Since the action variables can be determined up to additive constants, this means

$$I_s = \frac{1}{2\pi} \oint_{\gamma_s} p dq. \tag{11.34}$$

11.10 Action for harmonic oscillator

Let the Hamiltonian be

$$H = \frac{1}{2}(a^2p^2 + b^2q^2). \tag{11.35}$$

Then, M_E specified by $H = E$ is an ellipse

$$\frac{p^2}{2E/a^2} + \frac{q^2}{2E/b^2} = 1. \tag{11.36}$$

Its area is $2\pi E/ab$. The angular frequency is $\omega = ab$ as can be seen from $\dot{p} = -b^2 q$, $\dot{q} = a^2 p \Rightarrow \ddot{q} = -(ab)^2 q$. Thus,

$$I(E) = \frac{1}{2\pi} \oint p dq = E/ab = E/\omega. \tag{11.37}$$

11.11 Adiabatic invariants

Let $G(q, I)$ be the generating function of the canonical transformation $(q, p) \to (\omega, I)$ (see (13.4)) for a completely integrable system:

$$\frac{\partial G}{\partial q} = p, \frac{\partial G}{\partial I} = \omega. \tag{11.38}$$

Assume that the system is subjected to a small perturbation $\lambda(t)$. Then, the above canonical transformation now depends on time as well (through $\lambda(t)$), so the new Hamiltonian reads to order $O[\lambda]$

$$K = H(I) + \frac{\partial G}{\partial t} = H(I) + \Lambda \dot{\lambda}(t), \tag{11.39}$$

where $\Lambda = \partial G/\partial\lambda$. Therefore, the perturbed Hamilton's equation reads

$$\dot{I} = -\frac{\partial K}{\partial \omega} = \frac{\partial \Lambda}{\partial \omega}\dot{\lambda}(t). \tag{11.40}$$

Lam is a bounded function, so it must be a multiple periodic function. Therefore its derivative must have the vanishing time average. Therefore, it λ change very slowly, $\dot{I}$ must be zero. That is, I is invariant to order $\dot{\lambda}$. Such a quantity is called an adiabatic invariant.