17 Fourier Expansion

Fourier expansion and its salient features are summarized. We should pay due attention to the relation between the decay rate of the Fourier coefficients and the smoothness of the function. Impacts of Fourier's idea on Modern Mathematics is also briefly outlined.

Key words: Fourier expansion, periodic extension, Gibbs phenomenon, Riemann-Lebesgue lemma, countable.

Summary:
(1) Three basic facts (**17.5**) for piecewise smooth functions are worth memorizing as well as the formal expansion formulas in **17.1**.
(2) Fourier coefficients decay faster if the function is smoother. This is due to the Riemann-Lebesgue lemma (**17.11-13**).
(3) To use Fourier expansion to solve a boundary problem, a problem-adapted form should be looked for (**17.15-17**).
(4) Attempts to rationalize Fourier series almost dictated modern mathematics (**17.18**).

17.1 Fourier expansion of function with period 2ℓ: A formal statement. If f is a periodic function with period 2ℓ, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}\right] \text{ for } x \in [-\ell, \ell]. \tag{17.1}$$

where

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx, \quad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx. \tag{17.2}$$

Or, we may write

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{in\pi x/\ell}, \tag{17.3}$$

where

$$c_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)e^{-in\pi x/\ell}. \tag{17.4}$$

This is what Fourier asserted, but he could not convince mathematicians he admired ($\to$**1.6**, **1.7**, **17.18**). The formal series (17.1) or (17.3)

are called *Fourier series.* If we may freely exchange the order of summation and integration, then it is easy to check Fourier's claim (see also **20.14**).
The real value condition of f in terms of its Fourier coefficients is $c_n = \overline{c_{-n}}$.
The best introductory book of Fourier analysis is E. Körner's *Fourier Analysis* (Cambridge, 1988). For solved problems, Schaum's outline series is useful as usual.

Discussion.
(A) **Fourier expansion as least square approximation.**
Let

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell}\right]. \tag{17.5}$$

where a_n and b_n are given by (17.3). The Fourier coefficients minimizes the following integral:

$$\int_{-\ell}^{\ell} |f(x) - g(x)|^2 dx. \tag{17.6}$$

We say g is the closest to f in the L_2-norm (→**20.5**). That is, the Fourier expansion is understood as the least square approximation of a function in terms of trigonometric functions up to a given wavelength (→**20.13**).
(B) **Acceleration of convergence.** The convergence of Fourier series can be accelerated. Consider in $(-\pi, \pi)$

$$f(x) = \sum_{n=2}^{\infty} (-1)^n \frac{n^3}{n^4 - 1} \sin nx \tag{17.7}$$

We know

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sin nx}{n} = \frac{x}{2}. \tag{17.8}$$

Let us subtract this from f:

$$f(x) - \frac{x}{2} = \sin x + \sum_{n=2}^{\infty} (-1)^n \frac{\sin nx}{n^5 - n}. \tag{17.9}$$

The series should be faster convergent than the original one, so this subtraction trick is useful in numerical calculation.

Try a similar trick to

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n + a}, \tag{17.10}$$

where $a > 0$.
(C) **Crystal periodicity**. Let $f(\boldsymbol{r})$ be a function defined on $\boldsymbol{R}^3$ with the following 'lattice structure':

$$f(\boldsymbol{r} + \boldsymbol{a}_i) = f(\boldsymbol{r}) \tag{17.11}$$

for three (linearly independent) vectors $\boldsymbol{a}_i$ ($i = 1, 2, 3$) called the *crystal lattice vectors*. The parallelepiped spanned by these vectors is called the *unit cell.* Such a function can be expanded as

$$f(\boldsymbol{r}) = \sum_{\boldsymbol{h}} A_{\boldsymbol{h}} \exp(2\pi i \boldsymbol{h} \cdot \boldsymbol{r}), \tag{17.12}$$

where the summation is over all the vectors $\boldsymbol{h}$ such that

$$\boldsymbol{h} = \sum_{i=1}^{3} h_i \boldsymbol{b}_i \tag{17.13}$$

for any integer h_i. The vectors $\boldsymbol{b}_i$ ($i = 1, 2, 3$) called *reciprocal lattice vectors* are give by

$$\boldsymbol{b}_1 = \frac{\boldsymbol{a}_2 \times \boldsymbol{a}_3}{V_c}. \tag{17.14}$$

and cyclical permutations of the suffices, where V_c is the volume of the 'uint cell':

$$V_c = \boldsymbol{a}_1 \cdot \boldsymbol{a}_2 \times \boldsymbol{a}_3. \tag{17.15}$$

The expansion coefficient can be obtained by

$$A_{\boldsymbol{h}} = \frac{1}{V_c} \int_{\mathrm{cell}} d\boldsymbol{r}\, f(\boldsymbol{r}) \exp(-2\pi i \boldsymbol{h} \cdot \boldsymbol{r}). \tag{17.16}$$

Exercise.
(1) Fourier-expand the following functions of x (here a is a real such that $a \in (-1, 1)$.

$$\frac{1 - a^2}{1 - 2a\cos x + a^2}. \tag{17.17}$$

and

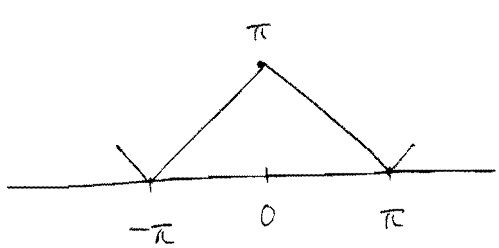
$$\frac{x \sin x}{1 - 2a\cos x + a^2}. \tag{17.18}$$

See **16D.10**.
(2) Fourier expand

$$f(x) = |\cos ax|. \tag{17.19}$$

(3) Find the Fourier expansions of the following graphically given periodic functions. (period 2π)

(a)



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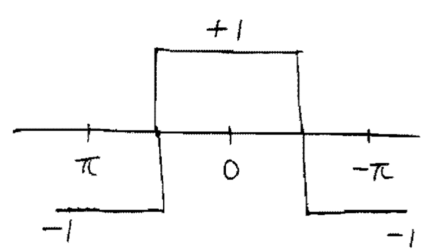
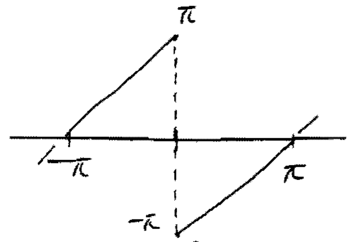


(C)



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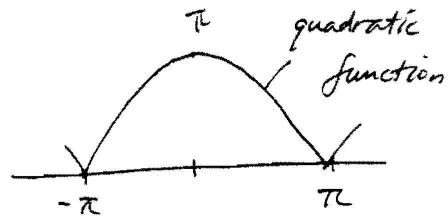


(4) Let $f(x) = Ax^2 + Bx + C$ in $(-\pi, \pi)$, where A, B, C are constants. Find its Fourier expansion, or show

$$Ax^2 + Bx + C = \frac{A\pi^2}{3} + C + 4A\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2} - 2B\sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n}. \quad (17.20)$$

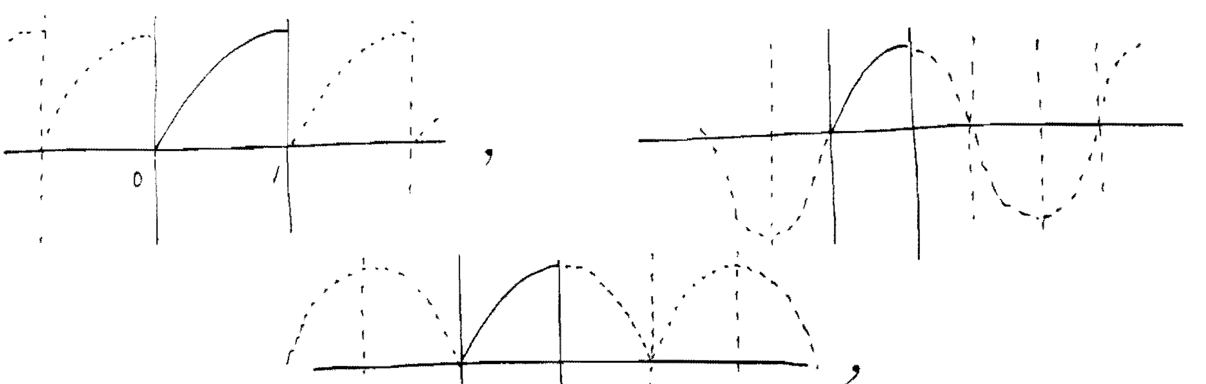
If the range is $(0, 2\pi)$, then

$$Ax^2 + Bx + C = \frac{4A\pi^2}{3} + B\pi + C + 4A\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} - 2B\sum_{n=1}^{\infty}\frac{\sin nx}{n}. \quad (17.21)$$

With the aid of these expansions, we can compute the following series

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n}, \quad \sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}. \quad (17.22)$$

17.2 Periodic extension of function. If f is defined only on $[-\ell, \ell]$, or one is interested in f on this interval, f can be extended to a periodic function F defined on the whole $\boldsymbol{R}$, and we may use **17.1**. There are many ways to define a function which is a periodic extension of f



As we will discuss in detail later ($\rightarrow$**17.15**-) we should, in practice, make the extended function F to be as smooth as possible.

17.3 Theorem [Weierstrass]. Any continuous function on (a, b) can be approximated in the sup-norm sense[233] by a polynomial. More pre-

[233] The sup-norm $\| \ \|_{sup}$ is defined by $\|f(x)\|_{sup} = \sup_{x\in(a,b)}|f(x)|$. That is, we measure the distance between two functions f and g by the widest possible separation of their graphs. See **20.3** for 'norm.'

cisely. for a given continuous function f defined on (a, b). and for any specified positive ϵ there is a polynomial P such that $\|f - P\|_{sup} < \epsilon$. □

We say that the set of polynomials is *complete* in the set of continuous functions.

It is straightforward to generalize the theorem for multivariable functions.

Discussion: Theorem(Hausdorff) on the moment problem.
Let $[a, b]$ be a finite interval and let f, g be continuous functions. Then, if

$$\int_a^b x^n f(x)dx = \int_a^b x^n g(x)dx \tag{17.23}$$

for all $n \in \boldsymbol{N}$. $f = g$ on $[a.b]$. □
The condition is equivalent to

$$\int_a^b P(x)(f - g)dx = 0 \tag{17.24}$$

for any polynomial P. Weierstrass (→**17.3b**) tells us that there is a sequence of polynomials P_n uniformly converging to $f - g$ on $[a, b]$. Hence. the condition implies $\int (f - g)^2 dx = 0$. Hence. $f = g$ follows.

If the domain is not bounded. then Hausdorff's theorem does not hold. That is. the knowledge about all the moments do not uniquely specify a distribution function.

17.3a Bernstein polynomial. A constructive demonstration of Weierstrass' theorem is the following in terms of the *Bernstein polynomial.* We study a continuous function f defined on $[0, 1]$. Let

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} f\left(\frac{k}{n}\right). \tag{17.25}$$

This uniformly converges to f as $n \to \infty$.

This tells us that any continuous function is approximated as a linear combination of monomials $1, x, x^2, \cdots$. The set of monomials is *complete* in the space of continuous functions (= any continuous function is in the closure of the totality of the linear combination of monomials). [234]

Exercise.
(1) Demonstrate that for $f(t) = 1, t$. and t^2 $B_n(t)$ converges to the respective target

[234] The good function principle (→**19.16**) can be restated as follows: If a relation among integrals on a finite closed interval is correct for polynomials. then it is correct for any integrable functions.

functions uniformly.[235]
(2) $\{1, \cos nt\}$ is complete on $[0, \pi]$ but not so onn $[-\pi, \pi]$. The same is true for $\{\sin nt\}$.

Discussion. Here. a theoretical physicists' formal demonstration of the convergence of (17.25) is given. We first note that the Taylor expansion can be written as[236]

$$f(x+y) = \exp\left(y\frac{d}{dx}\right) f(x). \tag{17.26}$$

(17.25) can be rewritten as

$$\begin{aligned} B_n(x) &= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \exp\left(\frac{k}{n}\frac{d}{dt}\right) f(t)\bigg|_{t=0}, & (17.27)\\ &= \left(1 - x + x\exp\left(\frac{1}{n}\frac{d}{dt}\right)\right)^n f(t)\bigg|_{t=0}. & (17.28)\end{aligned}$$

Here. we have used the binomial theorem. Now n is extremely large. so the exponent can be expanded to obtain

$$\begin{aligned} B_n(x) &= \left(1 - x + x\left(1 + \frac{1}{n}\frac{d}{dt} + \cdots\right)\right)^n f(t)\bigg|_{t=0}, & (17.29)\\ &= \left(1 + \frac{x}{n}\frac{d}{dt} + \cdots\right)^n f(t)\bigg|_{t=0}. & (17.30)\\ &\to \exp\left(x\frac{d}{dt}\right) f(t)\bigg|_{t=0}. & (17.31)\\ &= f(x). & (17.32)\end{aligned}$$

Mathematically. this is not a proof (note that this works only for analytic functions (→**7.1**)): this is just the Euler style 'algebraic formalism' (→**4.4** Discussion). but it is not empty.

17.3b Who was Weierstrass? Karl Theodor Wilhelm Weierstrass was born on October 31. 1815 at Ostenfelde in Münsterland. His family was rather poor. but a very cultivated one. He was a student *cum laude* every year. good at German. Greek. Latin. and Math. His father wished him to be a politician. so he studied law and economics at the

[235] This demonstrates that B_n converges uniformly to any continuous target function thanks to the Bohman-Korovin theorem:
Let L_n be a linear operator on $C[a, b]$ (continuous functions on $[a, b]$) which is monotonic (i.e.. if $f \leq g$. then $L_n f \leq L_n g$). The following two conditions are equivalent:
(A) $L_n f \to f$ is uniform for any $f \in C[a, b]$.
(B) $L_n f \to f$ is uniform for $f = 1, x$ and x^2. □
The theorem is the shortest route to the Weierstrass approximation theorem.

[236] If the reader knows that the momentum operator is the generator of translation in quantum mechanics. the formula should be obvious.

University of Bonn (from Fall of 1834). However. he soon realized that these were not true scholarly disciplines but only for bread, and began to feel them as a waste of his life. He studied Laplace' *Mechanique celeste*. Jacobi's *Fundamenta nova*, etc., but he could never patiently attend mathematics classes except the one by Plücker's geometry.[237] He studied only mathematics for four years without taking any exams in any subject. He also loved taverns and became an expert in fencing with his great physical strength and agility. Hence, when he returned home after four years, naturally he was treated very coldly.

Since he knew he could not go to a good university to learn mathematics. he decided to be a teacher. and enrolled in the Theological and Philosophical Academy at Münster on May 22, 1839, where Gudermann[238] was teaching mathematics. Weierstrass quit the Academy in the same Fall. and prepared for the exam to be a teacher. In the exam in 1840. he gave a new result on elliptic functions.

He became a teacher of Münster gymnasium in 1841. He wished to complete his work submitted as a part of the exam. but thought that he should first clarify the foundation of the theory of general functions. He completed a paper proving Cauchy's theorem without using double integrals (note that Gauss' letter (→**8A.4** Discussion (A)) was not known until 1880). The paper also contained Laurent's theorem (**8A.8**). In 1842. he completed a fundamental paper on analytic functions. There he introduced the idea of function elements (→**7.7**). analytic continuation (→**7.8**). singularities (→**8A**). natural boundary (→**7.11**). etc.[239] In the Fall of 1842. he moved to the Royal Catholic Gymnasium at Deutsch-Krone (West Prussia). He had to teach calligraphy. geography. and gymnastics. He published his first paper on the gamma function (→**9**. the infinite product formula) in the proceedings of the gymnasium. Of course. no one read it who could understand it. For the next six years. he stayed in the position without becoming desperate.

In 1848 fall. he was promoted to a teacher of an Obere-Gymnasium (high school) at Braunsberg on the Baltic. Fortunately. the principal.

[237] Plücker was a professor of physics concentrating on analytical geometry. However. it was said that a non-physicist occupying a physics position was inappropriate, so after 1846 for about twenty years he concentrated on physics. However. he could not stop studying mathematics. so that he decided to return to math. and started to construct a new geometry (introducing the Plücker coordinate), but could not finish it.

[238] C. Gudermann (1798-1851). He stressed the importance of power series. and this gave a profound influence on Weierstrass. who always appreciated Gudermann in every possible opportunity (e.g.. on his (=KW) seventieth and eightieth birthdays). Weierstrass never attended any class but Gudermann's. There were 13 students in the first class. but in the second class Weierstrass was the only student.

[239] The paper was not published for a few tens of years.

Mr F. Schurz, understood him. His research was at the second stage of completing the theory of elliptic functions. His work on Abel functions appeared in Crelle's journal[240] in 1854. It was a sensational paper, making him famous instantly. University of Königsberg decided to give him *Doctor honoris causa*, and they (including math Professor Richelot who proposed this) went to Braunsberg to hand the doctorate.[241]

In 1856, he became a professor of Geverbe-Institute in Berlin from July 1, and then from the Fall, thanks to the recommendation of Kummer, he was also an associate professor of the University of Berlin. Kronecker was also there. However, he had to give lectures 12 hours a week at the Institute, and also had to give a lecture on Gauss' theory on the dispersion of light, so he did not have enough time to do research. He and Kummer founded the first seminar in Germany devoted exclusively to pure mathematics in 1861. After 1862 he could lecture only while seated in a chair because of brain spasms and the onset of recurrent attacks of bronchitis and phlebitis. During his classes an advanced student assisted him by writing on the blackboard.

He became a full professor in 1864, and was already very well known all over the world (even in the US).[242] By the 1870s as many as 250 students attended his classes each year. This enrollment was exceptionally high for advanced mathematics courses in his time. He removed the requirement that doctoral dissertations be in Latin.

He tried hard to eliminate use of intuition as much as possible. He analyzed intuitive concepts and wished to reconstruct everything on the concept of integers. Also he made effort to find the shortest path from the very basic.

Sonia Kowalevskaya[243] became his private student from 1870 Fall, because she had an excellent recommendation from Königsberger, his former student. He became the Provost of University of Berlin in 1873, but he continued to teach her. She received her PhD in 1874 with the now famous work about PDE (Cauchy-Kowalevskaya theorem).

Kowalevskaya died in 1889; Kronecker died in the same year.[244]

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[240] This is still the top-ranking mathematics journal.

[241] According to Mittag-Leffler, at his 80th birthday, Weierstrass recollected with tears in his eyes that it was his most delightful event that Professor Richelot came over in person to hand him the degree. 'However, I still regret that the day came too late for me.' He spent 15 years teaching boys.

[242] When Mittag-Leffler went to Paris to learn with Hermite, Hermite said, "You have made an error. You should have attended courses of Weierstrass in Berlin. He is our master of all." (in 1873 just after the Franco-Prussian war).

[243] January 15, 1850 Moscow - February 10, 1889 Stockholm.

[244] Kronecker did not like Weierstrass' theory of irrational numbers. He aimed at arithmetization of mathematics, saying "Die ganze Zahl schuf der liebe Gott, alles Übriges is Menschenwerk." Needless to say, Kronecker hated Cantor (→**17.19**), but Weierstrass defended him.

He retired in 1892. He chose as his successor Frobenius (→**24B**). He died on February 19. 1897 of aggravated influenza.

17.4 Set of trigonometric functions is complete.[245] Let $f(\theta)$ be a 2π-periodic function. Introduce

$$\varphi(x, y) = rf(\theta), \tag{17.33}$$

where $x = r\cos\theta$ and $y = r\sin\theta$. This agrees with $f(\theta)$ on the unit circle $x^2 + y^2 = 1$. φ can be uniformly approximated by a polynomial of x and y on $[0.1] \times [0.1]$. Setting $r = 1$ the resultant formula becomes a polynomial of $\cos\theta$ and $\sin\theta$. However. with the aid of the formulas of trigonometric functions. this can always be reduced to the form of the partial sum of Fourier series.

Discussion [Münz' theorem].
The set of powers $\{x^{\alpha_i}\}$ with $\alpha_i \to \infty$ is complete (w.r.t. the ordinary sup norm) on $[0.1]$. if and only if $\sum_i(\alpha_i)^{-1}$ diverges. □[246]
From this we realize that $\{e^{-\alpha_i t}\}$ is a complete set on a finite interval $[0.s]$ for any $s > 0$ under the same condition. Thus we can approximate correlation functions $c(t)$ (→**32.13**) on any large time interval $[0.T]$ with the linear combination of exponentially decaying functions.[247]

17.5 Three basic facts for piecewise smooth functions.[248] Let f_N be the partial sum of (17.1) up to the $n = N$ terms. We assume f to be piecewisely smooth. Then. there are three basic facts:
(1) $\lim_{N\to\infty} f_N(x) = [f(x+0) + f(x-0)]/2$.
(2) On any closed interval $[a.b]$ which is in an open region where f is smooth.[249] the convergence is uniform:
$\lim_{N\to\infty} \max_{x\in[a.b]} |f(x) - f_N(x)| = 0$.
(3) At an isolated jump discontinuity at x_0. *Gibbs' phenomenon*[250] occurs: for sufficiently small $\delta > 0$:
$\lim_{N\to\infty}[\max_{|x-x_0|<\delta} f_N(x) - \min_{|x-x_0|<\delta} f_N(x)] = C|f(x_0+0) - f(x_0 - 0)|$. where C is a universal constant given by $C = \frac{2}{\pi}\int_0^\pi \frac{\sin x}{x}dx \simeq 1.17897974\cdots$ (i.e.. there is about 18% overshooting).

[245] This smart proof is found in Courant-Hilbert (Chapter 2. Section 5.4).

[246] H. Münz. Festschrift H A Schwarz. p303 (1914): The lecturer has not read the original. This is quoted in Courant-Hilbert Chapter 2, Section 10.6. See P Borwein and T Erdelyi. "Polynomials and Polynomial Inequalities" (Springer, 1995) for detailed information about the theorem and the related topics.

[247] This implies that we can approximate any Gaussian process with a linear combination of Gauss-Markov processes.

[248] A function which is continuously differentiable except finitely many points (at most countably many points) is called a ***piecewise smooth function.***

[249] that is. f is continuous and piecewise C^1 (the so-called strong Dini condition).

[250] Read Körner. Section 17.

Discussion.
(A) Intuitive understanding of the fundamental theorem of Fourier expansion.
Let f be a periodic function with period 2π. We have

$$f(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi} ds f(s) e^{in(t-s)} \tag{17.34}$$

Truncate the sum as

$$f_N(t) = \frac{1}{2\pi}\sum_{n=-N}^{N}\int_{-\pi}^{\pi} ds f(s) e^{in(t-s)}. \tag{17.35}$$

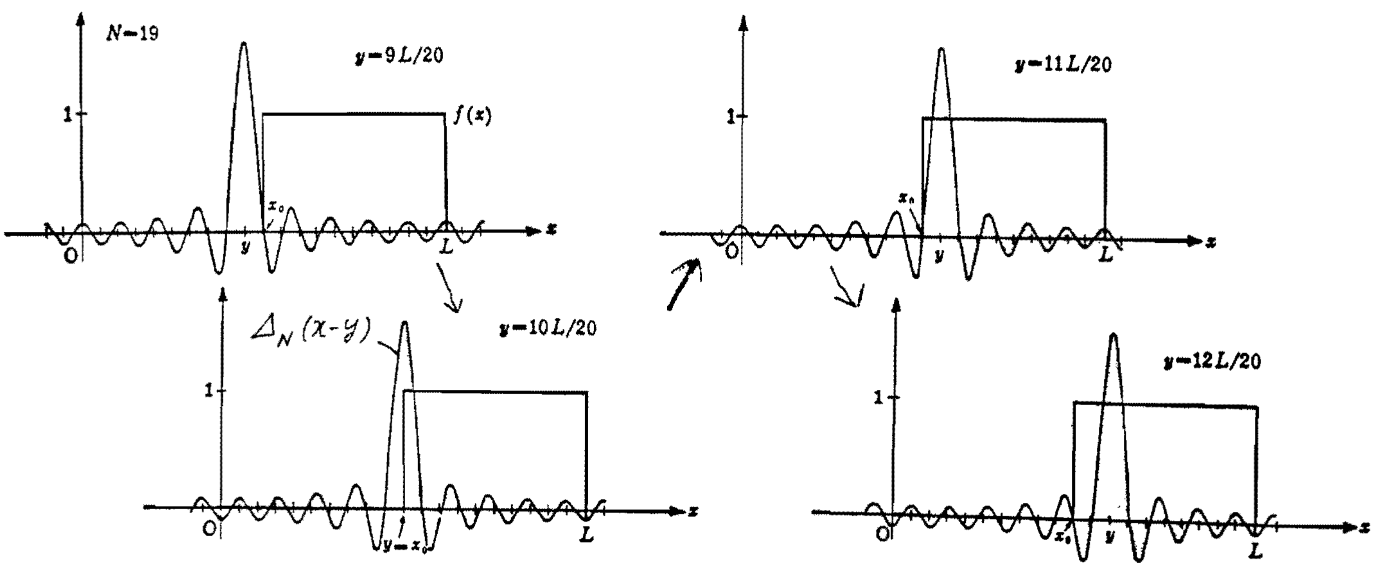
This can be rewritten as

$$f_N(t) = \int_{-\pi}^{\pi} ds f(s) \Delta_N(t-s). \tag{17.36}$$

where

$$\Delta_N(z) = \frac{1}{\pi}\frac{\sin Nz}{\sin z}. \tag{17.37}$$

This is called the Dirichlet kernel (→**14.13** Discussion (D)). Its graph looks like the ones in the figure below. The Gibbs phenomenon and the average value property can be seen from the following figures.[251]



(B) Dirichlet integral: Let $a < 0 < b$ and f be piecewise monotonic in $[a, b]$. Then,

$$\lim_{\lambda\to\infty}\int_a^b f(x)\frac{\sin\lambda x}{x}dx = \frac{\pi}{2}(f(+0)+f(-0)). \tag{17.38}$$

[251] Ezawa

Discuss the relation of this to (2) above.

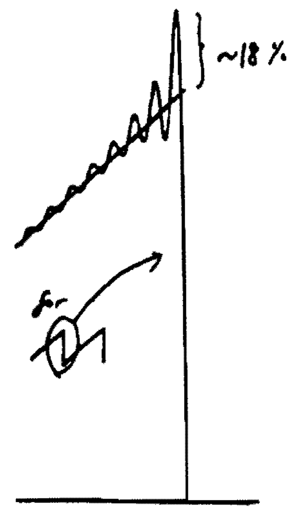
17.6 Gibbs phenomenon. The pathology called Gibbs' phenomenon occurs near the jump. Any jump could be used to check the assertion in **15.5**(3).[252] We may use the following sawtooth function

$$f(x) = x \text{ for } x \in (-\pi, \pi) \text{ and } f(\pm\pi) = 0. \tag{17.39}$$

Let S_n be the partial sum up to the n-th term of its sine Fourier expansion formula ($\rightarrow$**17.13**). Then, it is not hard to see

$$S_n(\pi - \pi/n) \rightarrow C\pi, \tag{17.40}$$

where C is given in **17.5**(3).



Gibbs' phemenon

17.7 Dirichlet's sufficient condition for expandability: practical condition. The basic facts in **17.5** are for piecewise smooth functions. Much wilder functions can be written as Fourier series. A sufficient condition for **17.5**(1) is:
f is periodic with at most finite number of extremal points and discontinuities.
This is sufficiently general for practitioners, but an ultimate version is:

17.8 Theorem [Riemann-Lebesgue].[253] A necessary and sufficient condition for (17.1) to converge to f at x is that

$$\lim_{\lambda\to\infty} \int_0^\delta \frac{1}{2}[f(x+t) + f(x-t) - 2f(x)]\frac{\sin \lambda t}{t}dt = 0 \tag{17.41}$$

for some $\delta \in (0, \pi)$. If this holds uniformly in $[a, b]$, then (17.1) uniformly converges to $f(x)$ on $[a, b]$.
Corollary [Dini]. If $|f(x+t) + f(x-t) - 2f(x)|/t$ is integrable as a function of t on $(0, \delta)$ for some $\delta \in (0, \pi)$, then (17.1) converges to $f(x)$ at x. In other words, if for any $\delta > 0$

$$\int_{-\delta}^{\delta} \frac{f(x+t) - f(x-t)}{t}dt \tag{17.42}$$

exists, then (17.1) converges to f at x.

17.9 Advanced theorems.
Theorem[Dini]. If $f(x)$ is L_1 (Lebesgue integrable, $\rightarrow$**19.8**), and is

[252] However, this localization of pathology is only in 1-space.
[253] See Y. Katznelson, *An Introduction to Harmonic Analysis* (Dover, 1968), Section 5, II-2, p51-55.

Hölder continuous.[254] then its Fourier series converges to $f(x)$ at x.
Theorem[Carlson] (1966). For any L_2-function (square Lebesgue integrable. →**19.8**). there is a convergent subsequence of its Fourier finite series such that it converges pointwisely to f for almost all (→**19.5**) points.

17.10 Remark.
(1) **Warning**. There exists continuous functions whose Fourier expansions do not converge at some point [duBois-Reymond].[255] Hence. continuity is not enough to ensure the convergence, although we know the Fourier series of a continuous function contains all the information needed to recover the original continuous function if summed according to Cesaro:[256]
Fejer'stheorem.[257] Let S_n be the partial sum of the Fourier series (17.1) up to the n-th (both sine and cosine) term. Define

$$\sigma_n \equiv \frac{1}{n+1}\sum_{k=0}^{n} S_k. \tag{17.43}$$

If f is 2ℓ-periodic continuous function. then σ_n uniformly converges to f. □
Fejer's theorem can be written as (→**14.13** Discussion (D))

$$f(x) = \lim_{n\to\infty}\frac{2}{\pi}\int_{-\pi}^{\pi} f(y)\left(\frac{\sin\frac{n(y-x)}{2}}{2\sin\frac{y-x}{2}}\right)^2 dy. \tag{17.44}$$

Note that the kernel (→**14.13** Discussion (D)) of the integral does no change its sign in contrast to the Dirichlet kernel in Discussion (A) of **17.5**.
(2) There exists L_1-functions (i.e.. Lebesgue integrable functions →**19.8**) whose Fourier series diverges everywhere.
(3) For any L_2-function (i.e.. square Lebesgue integrable functions →**19.18**). the set on which its Fourier series diverges is measure zero (→**19.3**). This explains partially why Lebesgue integral is the most natural framework to treat Fourier analysis (→**19**). See also **17.9**.

[254] That is. there are positive numbers α and C such that for any $\epsilon > 0$ there is $\delta > 0$ such that

$$|x-y| < \delta \Rightarrow |f(x)-f(y)| < C|x-y|^\alpha.$$

[255] Paul David Gustave du Bois Reymond, 1831-1889.
[256] As seen here. even a divergent series can sometimes be used to reconstruct the original function. We will come to another example later in asymptotic expansions (→**25.18-20**).
[257] Lipót Fejér (1880-1959) proved this sensational theorem when he was 19.

17.11 Theorem [Riemann-Lebesgue lemma]. Let f be integrable on $[a.b]$. Then.

$$\lim_{m\to\infty}\int_a^b f(x)e^{imx}dx = 0. \tag{17.45}$$

Here m need not be an integer. $\square$[258]

Of course. this implies that sine and cosine Fourier coefficients also vanish in the $m \to \infty$ limit.

Physically. the essence of the lemma is that if the total energy carried by the wave is finite, then the energy carried by every high frequency modes must be sufficiently small to avoid any 'ultraviolet catastrophe.' because the total energy ought to be the sum of the energy carried by each mode.

17.12 Smoothness and decay rate. If f is a k-times differentiable periodic function and $f^{(k)}$ is integrable. then

$$\int_{-\pi}^{\pi} f(x)e^{inx}dx = o[n^{-k}] \text{ as } |n| \to \infty. \tag{17.46}$$

This follows easily from the Riemann-Lebesgue lemma through integration by parts (cf. **25.11**).

(1) This supports our intuition that smoother functions have less high-frequency components.

(2) If $f \in C^\infty$. then its Fourier coefficients must decay in the $n \to \infty$ limit faster than any negative power of n.[259]

A precise statement is as follows:[260]

Theorem. Let $k \in \boldsymbol{Z}$. If $\sum_{n=-\infty}^{\infty}|n^k g(n)| < \infty$. then $f(x) = \sum_{n=-\infty}^{\infty} g(n)e^{inx}$ is a C^k-function.

(3) **Theorem** [Paley-Wiener]. A necessary and sufficient condition for a real analytic periodic function $f(x)$ to be analytic on a strip $|Im\, z| < \sigma$ is that for any $a \in (0.\sigma)$ there is a positive constant C (which may depend on a) such that $|\hat{f}(n)| \le Ce^{-a|n|}$, where $\hat{f}(n)$ is the Fourier coefficient.

Discussion.

Around a nonsmooth point the convergence is slow as shown in the figure.[261]

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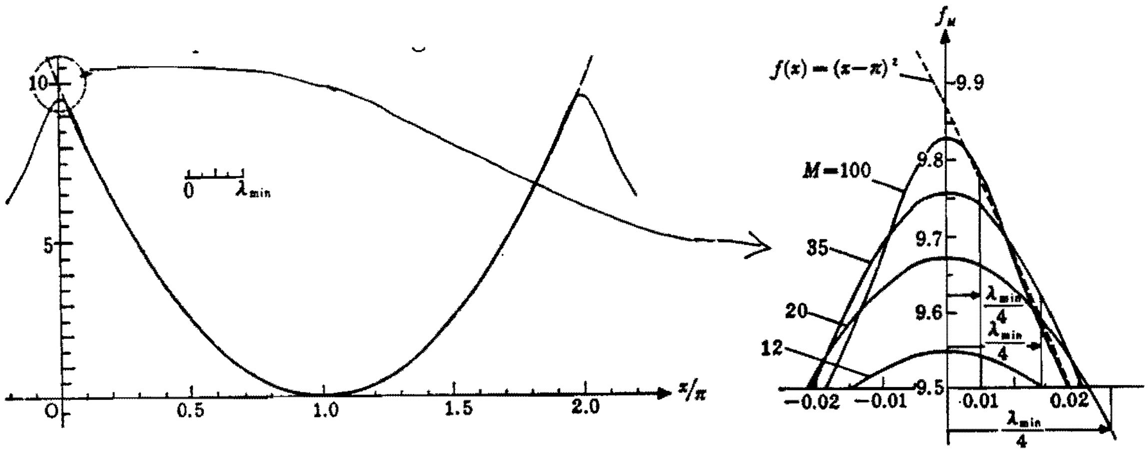
[258] See Katznelson p13.

[259] This property turns out to be crucial for the definition of the Fourier transforms of generalized functions (→**32C.6**).

[260] K. Tanishima. *Buturisugaku nyumon* (Univ. Tokyo Press. 1994).

[261] Ezawa



Exercise.
(1) Compute the Fourier expansion of $x|x|$ on $[-\pi, \pi]$. Then, discuss the relation of your result and the smoothness of the function.
(2) For $f(x) = x^{2n+1}e^{-a|x|}$, where n is a positive integer and a is a positive constant, we estimate its k-th Fourier coefficient $\sim k^{-(2n+3)}$.

17.13 Smoothness examples. If a function f is such that $f^{(k)}$ is continuous, but that $f^{(k+1)}$ is not, then $a_n \sim n^{-(k+2)}$: on $(-\pi, \pi)$:

$$x = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx. \tag{17.47}$$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum\frac{1}{(2n-1)^2}\cos(2n-1)x, \tag{17.48}$$

$$x(\pi^2 - x^2) = 12\sum\frac{(-1)^{n+1}}{n^3}\sin nx, \tag{17.49}$$

$$x^2(2\pi^2 - x^2) = \frac{7\pi^4}{15} + 48\sum\frac{(-1)^n)}{n^4}\cos nx. \tag{17.50}$$

17.14 Nontrivial numerical series obtained via Fourier expansion. Fourier expansions could be used to get the following series results. From $|x|$ we get

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots, \tag{17.51}$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots, \tag{17.52}$$

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots. \tag{17.53}$$

From x^3 we get. for example.

$$\frac{\pi^3}{12} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots. \tag{17.54}$$

17.15 Importance of smoothness. We wish to use Fourier expansions to solve PDE. Therefore, the convergence property of the series is very important. We should be able to differentiate the series termwisely. For this to be allowed a sufficient condition is the uniform convergence of the termwisely differentiated series. Hence, we wish to have the Fourier coefficients to decay as fast as possible (see the next entry). The previous entry explains why we must pay careful attention to the smoothness of periodic extension (→**17.2**) of a function defined on an interval.

17.16 Sine and cosine Fourier expansion. If f is defined only on $[0. \ell]$. then f is extended periodically as a function of period ℓ to use Fourier expansion formulas (→**17.2**). It is often convenient to extend f as an even or odd function of period 2ℓ (or longer →**17.17**). When we extend the function. it is advantageous to make the extended function as smooth as possible to ensure the good converging property of the series as discussed above.

(1) If $f(0) = 0$. then f defined on $[0. \ell]$ should be *sine-Fourier* expanded as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell} \text{ for } x \in [0. \ell]. \tag{17.55}$$

where

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx. \tag{17.56}$$

(2) If $f \neq 0$. then f defined on $[0. \ell]$ can be *cosine-Fourier* expanded as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} \text{ for } x \in [0. \ell], \tag{17.57}$$

where

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\frac{n\pi x}{\ell}dx. \tag{17.58}$$

[In the latter case we can subtract $f(0)$ from f to apply the sine-Fourier expansion. too.]

Exercise.

Expand the following functions on $[0. \pi]$ in Fourier cosine series:

(1)

$$f(x) = \cos ax \tag{17.59}$$

(2)

$$f(x) = \Theta(b - x). \tag{17.60}$$

where $b \in (0, \pi)$.

17.17 More sophisticated extension. To pursue the smoothness of the function to be expanded. for example, we can use the following trick to extend the original function on $[0, \ell]$ into a periodic function of period 4ℓ:
The following set could be used to expand any function on $(0, \ell)$

$$\left\{ \sin \frac{(2n-1)\pi x}{2\ell} \right\}. \tag{17.61}$$

The formulas are

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{(2n-1)\pi x}{2\ell}, \tag{17.62}$$

with

$$a_n = \frac{2}{\ell} \int_0^{\ell} dx\, f(x) \sin \frac{(2n-1)\pi x}{2\ell}. \tag{17.63}$$

This expansion is particularly useful when $f(0) = 0$ and $f'(\ell) = 0$
Analogously. we could use the cosine counterpart.

17.18 Impact of Fourier. The impact of Fourier's general assertion was not confined within applied mathematics. As we see below, it almost dictated Modern Mathematics.
(1) Function concept had to be clarified. Fourier claimed that any function can be expanded into Fourier series (→**1.6**. **1.7**). In those days the idea of function was not very clear. For example, there was a dispute between d'Alembert (→**2B.7**) and Euler (→**4.4**): Euler thought every hand-drawable function is a respectable function, but d'Alembert thought only analytically expressible functions are respectable. Therefore. to make sense out of Fourier's claim, the concept of function had to be clarified. Eventually, the modern concept of function as a map culminated through the work of Cauchy (→**6.11**) and Dirichlet: if a value $f(x)$ is uniquely specified for a given value of the independent variable x. then f is a respectable function. Then, inevitably, many strange functions began to be found (see. e.g., **2A.1** Discussion (B)). Now, we know many examples such as fractal curves.[262] Nowhere continuous functions were also found. A famous example is the *Dirichlet function*: $D(x) = 1$ if x is rational. and 0 otherwise (→**19.1**).

[262] B. B. Mandelbrot. *The Fractal Geometry of Nature* (Freeman. 1985).

(2) Convergence condition. The convergence condition of infinite series had to be considered. This spurred Cauchy to construct his theory of convergence (→**6.11**).
(3) Concept of integration had to be sharpened. Fourier proposed an integral formula for the Fourier coefficients as summarized in **17.1**. However, if a function f is not continuous, then it was not clear how to interpret the integral. To clarify this point. Riemann invented the concept of Riemann integration with clear integrability condition (in 1853 →**7.15**).
(4) Set theory became necessary. Cantor (→**17.19**) found that even if the values of the function at infinitely many points were unknown. still the Fourier series was determined uniquely. He studied very carefully how large 'sets' of points could be removed without affecting the Fourier coefficients. Soon he had to characterize these collections of points. The first surprise he found was that infinity of the totality of real numbers and that of rational numbers are distinct.[263] To organize his theory of infinity. Cantor attempted to introduce the concept of 'set.' However. many antinomies ('paradoxes') were found.[264]
(5) Securing foundation required axiomatic set theory. Eventually. to secure the foundation of set theory a set of axioms[265] were introduced by Zermelo.[266] Hence. the currently most popular axiomatic system of mathematics is under almost the direct impact of Fourier's idea.
(6) Further sharpening of integration concept was required. According to Cantor the area of $D(x)$ for $x \in [0, 1]$ must be zero (→**19.2**). but we cannot make any sense out of the Riemann integral of the Dirichlet function D (→**19.1**). A more powerful integral was needed. which was eventually provided by Lebesgue as the Lebesgue integration (→**19.8**).

[263] Cantor's first important result (December. 1873). →**19.3**.

[264] Perhaps the most famous antinomy is the Russel paradox (1902). The Russel paradox is as follows. 'Sets' can be classified into two classes: 'sets' which contain themselves as their elements ($x \in x$) and 'sets' which do not contain themselves ($x \notin x$). Make the 'set' Z of all the 'sets' x such that $x \notin x$: $Z \equiv \{x : x \notin x\}$. Is Z in Z or not? If $Z \notin Z$. then $Z \in Z$. but if $Z \in Z$. then $Z \notin Z$. a paradox.

[265] Y N. Moschovakis. *Notes on Set Theory* (Springer. 1994) and J. Winfried and M Weese. *Discovering Modern Set Theory I. the basics* (AMS. 1996) are recommended. P. Maddy. *Realism in Mathematics* (Oxford, 1990) may be used to understand the background of axiomatic set theories.

[266] Ernst Friedrichs Ferdinand Zermelo. 1871-1953. For physicists. Zermelo is famous for his discussion against Boltzmann: the 'Rückkehreinwand.' He was an assistant of Planck in those days and was against atomism (as his boss was). See G. H. Moore. *Zermelo's Axiom of Choice. its origins. development. and influence* (Springer. 1982). Perhaps this is more entertaining than many novels.

Discussion.
The reader must know and be able to explain to her lay friend the argument showing that $\boldsymbol{Q}$ is countable, but $[0,1]$ is not. Also she must be able to explain why $[0,1]^n$ for any $n \in \boldsymbol{N}$ has the same density as $[0,1]$ (i.e., there is a one-to-one correspondence between any dimensional cube and the interval $[0,1]$.

17.19 Who was Cantor? Georg Ferdinand Cantor was born in 1845 into a cosmopolitan merchant family in St. Petersburg. He was an artistically inclined child (a dessin is reproduced in his biography by Dauben[267]). He got his university education at Berlin (from 1863) from Weierstrass ($\rightarrow$**17.3b**), Kummer, Kronecker and others. His thesis solved a problem left unsettled by Gauss.

After briefly teaching at a Berlin's girls' school, he got his permanent job at Halle in 1869, where he became a junior colleague of Heine,[268] who urged Cantor to study the question about the uniqueness of the Fourier coefficients. Cantor quickly found what is outlined in (**4**) of **17.18**. In 1891, Cantor invented an entirely different proof of uncountability of reals, the so-called *diagonal method* (or method of diagonalization). This allowed him to make an ascending hierarchy of transfinite (=infinite) numbers. Cantor accepted the concept of actual infinity through his study of Plato, Aquinas, Spinoza and Leibniz. This put him at odds with a tradition stretching from Aristotle to Gauss that accepted only potential infinity. Most of his first papers were published in Acta Mathematica published by Mittag-Leffler.

To organize his theory of transfinite numbers, Cantor attempted to introduce the concept of 'set.' By a "set" he meant any collection M of definite, distinct objects m (called elements of M) which we can perceive or think. Cantor published his famous Beiträge[269] part I in 1895. This was the birth of set theory.

Cantor wished to move to a position more prestigious than Halle, but Kronecker hated his theory of transfinite numbers and opposed his appointment at Berlin. Mental illness afflicted his final decades of his life. Beginning in 1884 he suffered sporadically from depression. Although his studies in other fields than mathematics may look strange (to try to prove that Bacon wrote Shakespeare's plays, Freemasonry, etc), he continued to work actively in mathematics. In 1890

[267] J. W. Dauben, *Georg Cantor, His Mathematics and Philosophy of the Infinite* (Princeton UP, 1979) is a very informative and enjoyable book.

[268] Heinrich Eduard Heine, 1821-1881, well known for his covering theorem (Heine-Borel).

[269] which means 'contributions.' "Beiträge zur Begrundung der transfiniten Mengenlehre" Math. Ann. **46**, 481 (1895). A Dover translation is available: *Contribution to the founding of the theory of transfinite numbers* (Dover, 1955; the original translation by P E B Jourdain in 1915).

he founded the Association of German Mathematicians. He advocated international congress of mathematicians and made arrangements for the first of these held in Zurich in 1897. He died in January 1918 at the University of Halle mental hospital.

Cantor showed a unique ability in the art of asking questions that opened vast new areas of mathematical inquiry, an ability that he considered more valuable than solving questions.