40 Green's Function: Wave Equation

The Green's functions of wave equations are constructed directly or from those of Helmholtz equation. The radiation condition implies the specification of the time arrow.

Key words: retarded and advanced Green's function, propagator, afterglow effect, Helmholtz formula, causality, time arrow

Summary:
(1) If we use the retarded Green's function for the Helmholtz equation, we can obtain the retarded Green's function (→**40.1**).
(2) For wave equations the tiem arow is selected by the radiation condition.

40.1 Fundamental solution. A fundamental solution to the wave equation satisfies

$$\Box w(t, \boldsymbol{x}; t', \boldsymbol{x}') = \delta(t - t')\delta(\boldsymbol{x} - \boldsymbol{x}'), \tag{40.1}$$

where

$$\Box \equiv c^{-2}\partial_t^2 - \Delta \tag{40.2}$$

is called the *D'Alembertian.* Fourier-transforming this with respect to time, we obtain (→**39.1**, **27A.24**)

$$-(\Delta + \kappa^2)\hat{w}(\omega, \boldsymbol{x}; t', \boldsymbol{x}') = e^{-i\omega t'}\delta(\boldsymbol{x} - \boldsymbol{x}') \tag{40.3}$$

with $\kappa = \omega/c$. Thus basically this is the same as the problem of finding a fundmental solution for the Helmholtz equation in the whole space. If we use the retarded Green's function for the Helmholtz equation (→**39.6**), then inverse Fourier transformation gives

$$w(t, \boldsymbol{x}; t', \boldsymbol{x}') = w(t - t', \boldsymbol{x} - \boldsymbol{x}'; 0, 0) = \frac{1}{2\pi}\int d\omega \frac{e^{i\omega|\boldsymbol{x}-\boldsymbol{x}'|/c - i\omega(t-t')}}{4\pi|\boldsymbol{x} - \boldsymbol{x}'|} \tag{40.4}$$

This can easily be integrated to give (→**32C.8**)

$$w(t, \boldsymbol{x}; t', \boldsymbol{x}') = \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}'|}\delta(t - t' - |\boldsymbol{x} - \boldsymbol{x}'|/c). \tag{40.5}$$

Note that this is zero for any $t < t'$. This function is the Green function for 3-space, and is called the *retarded Green function.*

Discussion.
In terms of the retarded Green's function, the inhomogeneous wave equation

$$\Box u = q \tag{40.6}$$

can be solved as

$$u(t, \boldsymbol{x}) = \frac{1}{4\pi} \int_{|\boldsymbol{x}-\boldsymbol{y}|\le ct} \frac{q(t, \boldsymbol{y})}{|\boldsymbol{y} - \boldsymbol{x}|} d\boldsymbol{y}. \tag{40.7}$$

The formula is called the *Duhamel's formula.*

40.2 Advanced Green's function. We see above that the radiation condition (→**39.6**) imposes time reversal asymmetry (causality). Since the wave equation itself is time-reversal symmetric, the time reversed (40.5) should also be a solution to (40.1):

$$w_A(t, \boldsymbol{x}; t', \boldsymbol{x}') = \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}'|}\delta(t - t' + |\boldsymbol{x} - \boldsymbol{x}'|/c). \tag{40.8}$$

Note that this is zero for $t > t'$ everywhere. This is anti-causal, and is called the *advanced Green's function.*

40.3 Propagator. A fundamental solution $K(t, \boldsymbol{x}; t', \boldsymbol{x}')$ satisfying the boundary condition and symmetric in time is called the *propagator* of the problem. Its existence should be clear from the advanced and retarded Green's functions discussed above. The retarded Green's function is related to the propagator as

$$G(t, \boldsymbol{x}; t'\boldsymbol{x}') = \Theta(t - t')K(t, \boldsymbol{x}; t', \boldsymbol{x}'), \tag{40.9}$$

The fundamental solution satisfying the boundary condition and causality is called the retarded Green's function.

40.4 Symmetry of propagator.

$$K(t, \boldsymbol{x}|t', \boldsymbol{x}') = K(t - t', \boldsymbol{x}|0, \boldsymbol{x}'). \tag{40.10}$$

This time translation symmetry directly follows from **41.1**. This formula implies

$$K(t, \boldsymbol{x}|t', \boldsymbol{x}') = K(-t', \boldsymbol{x}| - t, \boldsymbol{x}'). \tag{40.11}$$

and consequently

$$\partial_t K(t, \boldsymbol{x}|t', \boldsymbol{x}') = -\partial_{t'} K(t, \boldsymbol{x}|t', \boldsymbol{x}'). \tag{40.12}$$

They imply that

$$K(t,\boldsymbol{x}|t',\boldsymbol{x}') = -K(t',\boldsymbol{x}|t,\boldsymbol{x}'). \tag{40.13}$$

Analogously

$$K(t,\boldsymbol{x}|t',\boldsymbol{x}') = K(t,\boldsymbol{x}'|t',\boldsymbol{x}), \tag{40.14}$$

so that we get

$$K(t,\boldsymbol{x}|t',\boldsymbol{x}') = -K(t',\boldsymbol{x}'|t,\boldsymbol{x}). \tag{40.15}$$

40.5 Eigenfunction expansion of propagator. Introducing the eigenfunction of the Laplacian with an appropriate homogeneous boundary condition (Dirichlet, Robin or Neumann condition) $\{|\lambda_n\rangle\}$ such that $-\Delta|\lambda_n\rangle = \lambda_n|\lambda_n\rangle$, we can separate the wave equation, to get

$$K(t,\boldsymbol{x}|t',\boldsymbol{x}') = \langle\boldsymbol{x}|\left\{\sum_{n=0}^{\infty}|\lambda_n\rangle\frac{c\sin[c^2\sqrt{\lambda_n}(t-t')]}{\sqrt{\lambda_n}}\langle\lambda_n|\right\}|\boldsymbol{x}'\rangle. \tag{40.16}$$

Here, if $\lambda_0 = 0$ (this happens only when the Neumann condition is imposed), the sine term is computed with the aid of l'Hospital's rule.

40.6 Propagator in infinite space. From (38.4) and the symmetry we can easily guess that[451]

$$K(t,\boldsymbol{x}|0,\mathbf{o}) = \frac{1}{4\pi x}[\delta(t-x/c)-\delta(t+x/c)]. \tag{40.17}$$

This is indeed the right answer as can be computed from the continuum version of (40.16):

$$K(t,\boldsymbol{x}|0,\mathbf{o}) = \frac{c}{(2\pi)^3}\int d^3k\frac{\sin(ckt)}{ck}e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = \frac{c}{(2\pi)^3}\frac{4\pi}{x}\int_0^\infty dk\sin(ckt)\sin(kx). \tag{40.18}$$

See **32C.8** Exercise.

40.7 Propagator in 2- and 1-spaces. For 2-space,

$$K^{(2)}(t,\boldsymbol{x}|0,\mathbf{o}) = sgn(t)\frac{1}{2\pi}\frac{\Theta(|t|-x/c)}{(t^2-x^2/c^2)^{1/2}}, \tag{40.19}$$

and for 1-space

$$K^{(1)}(t,\boldsymbol{x}|0,\mathbf{o}) = sgn(t)\frac{c}{2}\Theta(t^2-x^2/c^2). \tag{40.20}$$

[451] $|\boldsymbol{x}| = x$.

Of course, they can be obtained by integrating unnecessary coordinates out from the 3-space version (→**16C.3**).

40.8 Afterglow revisited. We can see explicitly from $G^{(2)}$ obtainable from $K^{(2)}$ that for $|\boldsymbol{x}| < tc$ $G^{(2)} > 0$, but this does not happen for 3-space. This is the afterglow in even dimensional spaces. (→**16C.4**, **32D.10**)

40.9 Helmholtz formula. The solution in 3-space to

$$\Box\psi(t, \boldsymbol{x}) = \varphi(t, \boldsymbol{x}) \tag{40.21}$$

can be written as

$$\begin{aligned}\psi(t, \boldsymbol{x}) &= \int_{T_1}^{t} dt' \int_{\Omega} d\boldsymbol{x} G(t, \boldsymbol{x}; t', \boldsymbol{x}')\varphi(t', \boldsymbol{x}') \\ &- \int_{T_1}^{t} dt' \int_{\partial\Omega} d\sigma(\boldsymbol{x}') \left[G(t, \boldsymbol{x}; t', \boldsymbol{x}')\frac{\partial\psi}{\partial n(\boldsymbol{x}')} - \psi\frac{\partial}{\partial n(\boldsymbol{x}')}G(t, \boldsymbol{x}; t', \boldsymbol{x}')\right] \\ &+ \frac{1}{c^2}\int_{\Omega} d\boldsymbol{x}'[G\partial_t\psi - \psi\partial_t G]_{t=T_1}.\end{aligned} \tag{40.22}$$

Just as in the case of the Helmholtz equation (→**39.8**), this is *not* the formula describing ψ in terms of the initial and boundary values.
[Demo] Just as a proof of Green's formula (→**16A.19**), we get

$$\int_{T_1}^{T_2} dt \int_{\Omega} d\boldsymbol{x}[(\Box f)g - f\Box g] = -\int_{T_1}^{T_2} dt \int_{\partial\Omega} d\boldsymbol{S}\cdot[f\nabla g - g\nabla f] + \int_{\Omega} d\boldsymbol{x}\frac{1}{c^2}[f\partial_t g + g\partial_t f]_{t=T_1}^{t=T_2}. \tag{40.23}$$

Take f to be the retarded Green's function (→**40.1**), and g to be the solution to (40.21), then this can be rewritten as the desired formula.

40.10 General causal solution. In (40.22) the surface integrals of the 4-volume $\Omega \times [T_1, t]$ describes the effects of the incoming waves into Ω from the past. Hence this can be rewritten as

$$\psi(t, \boldsymbol{x}) = \psi_{in}(t, \boldsymbol{x}) + \int_{T_1}^{t} dt' \int_{\Omega} d\boldsymbol{x}' G(t, \boldsymbol{x}; t', \boldsymbol{x}')\varphi(t', \boldsymbol{x}'). \tag{40.24}$$

Here ψ_{in} denotes the incoming wave. The Ausstrahlungsbedingung (→**39.6**) on ψ implies that $\psi_{in} \to 0$ when $\Omega \to \boldsymbol{R}^3$ and $T_1 \to -\infty$.