39 Green's Function: Helmholtz Equation

The Helmholtz equation results from diffusion and wave equations. Its Green's functions are constructed with the aid of generalized function theory. To single out physically meaningful solution, we need an extra condition (radiation condition).

Key words: Helmholtz equation, radiation condition, analogue of Green's formula

Summary:
(1) If the region is finite, then there is no special difficulty compared with the Laplace case (→**39.2**).
(2T) Juggling of generalized functions in **39.4-6** seems to be the simplest way to obtain physically meaningful Green's function. If the reader can follow the logic, that is enough. However,
(3) She must understand that a special condition is needed to guarantee the causality in the solution (Sommerfeld's radiation condition) (→**39.6**).

39.1 Helmholtz equation. The Helmholtz equation (→**27A.24**, **16C.4**)

$$-(\Delta + \kappa^2)\psi = 0 \tag{39.1}$$

appears when we Laplace transform (→**33**) the diffusion equation, or when we Fourier transform the wave equation (in this case $\kappa^2 = c^2/\omega^2$).
Convention. We will use the time Fourier transform with $e^{i\omega t}$. That is,

$$\psi(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega \psi(\omega) e^{-i\omega t}. \tag{39.2}$$

39.2 Green's function for Helmholtz equation on bounded domain. The formal formula for the Green's function is immediately obtained from the formal solution in, say, **35.2** or **37.7**. Let us solve

$$-(\Delta + \kappa^2)G = 1 \tag{39.3}$$

on a region D under the homogeneous boundary condition at the boundary ∂D. We know that the Laplacian has a set of eigenkets $\{|\lambda\rangle\}$ which makes an orthonormal basis ($\to$**37.1** $\sum|\lambda\rangle\langle\lambda| = 1$ $\to$**20.15**). Sandwiching (39.3) with an eigenket and bra

$$(\lambda - \kappa^2)\langle\lambda|G|\lambda'\rangle = \delta_{\lambda,\lambda'}, \tag{39.4}$$

so that we obtain

$$G = \sum|\lambda\rangle(\lambda - \kappa^2)^{-1}\langle\lambda|. \tag{39.5}$$

39.3 Example: Neumann condition on a rectangular region. The Green's function under a homogeneous Neumann condition (i.e., Neumann's function) for the Helmholtz equation in the rectangular domain $[0, a] \times [0, b]$ can be obtained as

$$N(x, y|x', y') = \frac{4}{ab}\sum_{n\geq 0, m\geq 0, nm\neq 0}\frac{\cos(n\pi x/a)\cos(n\pi x'/a)\cos(m\pi y/b)\cos(m\pi y'/b)}{(n\pi/a)^2 + (m\pi/b)^2 - \kappa^2} - \frac{1}{ab\kappa^2}. \tag{39.6}$$

39.4 Green's function for the whole space. We wish to solve

$$-(\Delta + \kappa^2)G(\boldsymbol{r}|\boldsymbol{r}_0) = \delta(\boldsymbol{r} - \boldsymbol{r}_0) \tag{39.7}$$

with the boundary condition $|u| \to 0$ as $|\boldsymbol{r}| \to \infty$. We interpret this equation in the generalized function sense ($\to$**14**). After Fourier transforming this ($\to$**32C.9**), we obtain

$$(k^2 - \kappa^2)\hat{u} = e^{ikr_0}. \tag{39.8}$$

Recalling **14.17(2)**, we can solve this equation as

$$\hat{u} = \hat{h}(k)e^{ikr_0}, \tag{39.9}$$

$$\hat{h}(k) = P\frac{1}{k^2 - \kappa^2} + C\delta(k^2 - \kappa^2), \tag{39.10}$$

where C is a constant, but may depend on κ. This can be rewritten as ($\to$**32C.13**, **8B.12**)

$$\hat{h}(k) = \frac{1}{2\kappa}\left\{\left[P\left(\frac{1}{k-\kappa}\right) + C\delta(k-\kappa)\right] - \left[P\left(\frac{1}{k+\kappa}\right) - C\delta(k+\kappa)\right]\right\}. \tag{39.11}$$

The Fourier inverse transform of u is given by the convolution of $h(\boldsymbol{r})$ and $\delta(\boldsymbol{r} - \boldsymbol{r}_0)$ ($\to$**32A.2**),

$$h(\boldsymbol{r}) = \frac{1}{4\pi^2 r}\frac{\partial}{\partial r}\int_{-\infty}^{+\infty} dk\frac{e^{ikr}}{2\kappa}\left\{\left[P\left(\frac{1}{k-\kappa}\right) + C\delta(k-\kappa)\right] - \left[P\left(\frac{1}{k+\kappa}\right) - C\delta(k+\kappa)\right]\right\}. \tag{39.12}$$

Here the angular integral has already been performed.

39.5 How to interpret the formal solution (39.11)? Using the Plemelj formula (→**32C.13**), we can rewrite

$$P\left(\frac{1}{k-\kappa}\right)+C\delta(k-\kappa)=\lim_{\epsilon\to+0}\frac{1}{k-\kappa\pm i\epsilon}+(C\pm i\pi)\delta(k-\kappa), \quad (39.13)$$

and

$$P\left(\frac{1}{k-\kappa}\right)-C\delta(k-\kappa)=\lim_{\epsilon\to+0}\frac{1}{k-\kappa\pm i\epsilon}-(C\mp i\pi)\delta(k-\kappa). \quad (39.14)$$

Thus, there are four combinations of + and − for (39.11). Consequently, we need an extra condition to select a solution.

39.6 Radiation condition (Ausstrahlungsbedingung). The extra condition to single out the physically meaningful solution from (39.11) is

$$\left|\frac{\partial}{\partial r}h-i\kappa h\right|\to 0. \quad (39.15)$$

for $r\to\infty$ This condition is called the Ausstrahlungsbedingung (out radiating condition due to Sommerfeld). This requires that − must be chosen in (39.13) and + in (39.14): the integrand in (39.11) now reads

$$\left\{\frac{1}{k-\kappa-i\epsilon}-\frac{1}{k+\kappa+i\epsilon}+(C-i\pi)[\delta(k-\kappa)+\delta(k+\kappa)]\right\}e^{ikr}. \quad (39.16)$$

Choosing $C=i\pi$, we can remove unwanted $e^{-i\kappa r}$. Thus, we can get

$$h(\boldsymbol{r})=-\frac{2\pi i}{8\pi^2 r\kappa}\frac{\partial}{\partial r}e^{i\kappa r}=\frac{e^{i\kappa}r}{4\pi r}. \quad (39.17)$$

That is,

$$G(\boldsymbol{r}|\boldsymbol{r}_0)=\frac{\exp(i\kappa|\boldsymbol{r}-\boldsymbol{r}_0|)}{4\pi|\boldsymbol{r}-\boldsymbol{r}_0|}, \quad (39.18)$$

which is called the *retarded Green's function* (cf. **40.1**, **16C.1**).

39.7 Green's functions for 2 and 1-spaces. With the aid of an analogous consideration, we can write down G in 2 and 1-space. For 2-space (→**27A.20** for $H_0^{(1)}$)

$$G(\boldsymbol{r}|\boldsymbol{r}_0)=\frac{i}{4}H_0^{(1)}(\kappa|\boldsymbol{r}-\boldsymbol{r}_0|). \quad (39.19)$$

For 1-space

$$G(\boldsymbol{r}|\boldsymbol{r}_0) = \frac{i}{2\kappa}\exp(i\kappa|\boldsymbol{r} - \boldsymbol{r}_0|). \tag{39.20}$$

The difference comes only from the angular integration.

39.8 Analogue of Green's formula. The equation corresponding to Green's formula **16A.19** is immediately obtained from Green's formula for the Laplace equation as

$$\int_D d\tau[u(\Delta + \kappa^2)v - v(\Delta + \kappa^2)u] = \int_{\partial D} d\sigma \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right). \tag{39.21}$$

How to use it should now also be obvious (→**16A.21**).