# 37 Spectrum of Laplacian

The spectrum of Laplacian gives the energy level of quantum mechanical billiards. It is important to grasp its general feature to understand the general spectrum of a particle in a potential well. One of the most interesting questions was to determine the shape of the domain from the spectrum: Can you here the shape of the drum? Now, we know that this is impossible even in 2-space.

**Key words**: Fundamental theorem, nodes, eigenfunction expansion of Green's function

**Summary**:
(1) Understand the eigenfunction expansion of Green's functions (→**37.7**, **37.9**).
(2) Remember the general features of the spectrum and eigenfunctions of the Laplacian with the Dirichlet condition on a bounded domain (→**37.1**). (Theoreticians) This is an example of the spectrum of compact operators.
(3) We cannot hear the shape of the drum (→**37.6**).

**37.1 Theorem [Fundamental theorem].**[440] Let $\Omega$ be a bounded open region, and $\partial\Omega$ be smooth. Then, the following eigenvalue problem

$$-\Delta u = \lambda u, \quad u|_{\partial\Omega} = 0 \tag{37.1}$$

has the following properties:
(1) There are countably many eigenvalues $\{\lambda_n\}$ such that $0 \le \lambda_1 \le \lambda_2 \le \cdots$, and $\lim_{n\to\infty} \lambda_n = +\infty$.
(2) There is no finite accumulation point for $\{\lambda_n\}$.
(3) Let $\varphi_n$ be an eigenfunction belonging to $\lambda_n$. Then, $\{\varphi_n\}$ is an orthogonal basis of $L_2(\Omega)$.□

Physically, if we consider the eigenmodes of a drumhead, at least (1) and (2) are understandable. There should not be any upper limit in its frequency for an ideal continuum drumhead. For a finite frequency there cannot be infinitely many independent modes.

[Demo for 3-space] With the aid of the Green's function (→**36.3**), we can convert (37.1) into an integral equation problem:

$$u(x) = \lambda \int_\Omega G(x|y)u(y)dy \equiv \lambda(\mathcal{G}u)(x). \tag{37.2}$$

[440] Actually, much more general theorems are known, since the Laplacian can be defined on any Riemann manifold.

Since $G(x|y) - w(x|y)$ is everywhere finite on $\Omega$, if we can show

$$\int_{\Omega} |w(x|y)|^2 dx < +\infty \quad \text{for } \forall y \in \Omega, \tag{37.3}$$

the Hilbert-Schmidt theorem (→**34C.11**) tells us that $\mathcal{G}$ is a compact (self-adjoint) operator (→**34C.9**). Let $B_\epsilon$ be a ball of radius $\epsilon$ centered at $y$. On $\Omega \setminus B_\epsilon$ the integral is finite, so we have only to consider

$$\int_{B_\epsilon} |w(x|y)|^2 dx. \tag{37.4}$$

But this is finite as can be seen from the order $w^2 = O[|x-y|^{-2}]$. Hence, Theorem **34C.12** tells us (1)-(3) except nonnegativity of the eigenvalue. We know $-\Delta$ is non-negative, so eigenvalues cannot be negative.

**Discussion.**
According to the variational principle for the eigenvalues of self-adjoint operators, **34C.13**, we can say that the fundamental frequency of a drum goes up if the drum head is constrained; in contrast, if the drum head is torn, then its fundamental frequency goes down.

**37.2 Theorem [Monotonicity].** Let there be two open regions such that $\Omega \supset \Omega'$. Consider the eigenvalue problems $-\Delta u = \lambda u$ on $\Omega$ with the condition $u|_{\partial\Omega} = 0$, and that with $\Omega$ replaced by $\Omega'$. Let the $n$-th eigenvalue (arranged in the increasing order) for the problem with the region $\Omega$ be $\lambda_n$, and that for the region $\Omega'$ be $\lambda'_n$. Then, $\lambda_n \leq \lambda'_n$.□
[Demo] We use the variational principle for the eigenvalues of compact self-adjoint operators **34C.13**. Notice, however, the eigenvalue there is the reciprocal of the eigenvalues in our present context. That is, the variational principle gives us the eigenvalue with the *smallest* modulus. Due to the non-negativity of the eigenvalues, actually the variational principle gives us the smallest eigenvalue $\lambda_1$. More generally, the minimum of $\langle\varphi| - \Delta|\varphi\rangle$ under the condition $\langle\varphi|\varphi\rangle = 1$ is $\lambda_n$ in the orthogonal complement $V_n$ of the direct sum of the eigenspaces for $\lambda_1, \cdots, \lambda_{n-1}$. For any $n$ the minimum value of $\langle\varphi| - \Delta|\varphi\rangle$ on $V_n$ with the condition $\varphi|_{\partial\Omega} = \varphi|_{\partial\Omega'} = 0$ cannot be smaller than that with the condition $\varphi_{\partial\Omega} = 0$.

**37.3 Theorem.** Eigenvalues depend on $\Omega$ continuously.□[441]

**37.4 Theorem [Courant].** Let $u_n$ be the eigenfunction belonging to the $n$-th smallest eigenvalue of $-\Delta$ on $\Omega$ under the condition $u|_{\partial\Omega} = 0$. Then the *nodal set*:

$$\mathcal{N}(u_n) \equiv \{x : u_n(x) = 0, x \in \Omega\} \tag{37.5}$$

[441] See Courant-Hilbert, vol. I Chapter 6, Section 2 Theorem 10.

separates $\Omega$ into at most $n$ disjoint components.□[442]

**Discussion.**
Consider a Laplace eigenvalue problem in a bounded closed domain with a homogeneous Dirichlet boundary condition in 2-space. The curves on which the eigenfunction vanishes is called the *nodal curve*. Demonstrate that a nodal curve is perpendicular to the boundary curve, when the former touches the latter where the latter is smooth.

**37.5 Vibrating drumhead.** The eigenmodes of a 2-dimensional drumhead of shape $D$ obey

$$-\Delta u = \omega^2 u, \quad u|_{\partial D} = 0. \tag{37.6}$$

If $D$ is a disk of radius $a$, then the eigenfunctions (modes) are given by

$$u_{mn} = \begin{cases} J_m(r_n^{(m)} r/a)\cos m\varphi, \\ J_m(r_n^{(m)} r/a)\sin m\varphi, \end{cases} \tag{37.7}$$

where $\omega = r_n^{(m)}/a$ with $r_n^{(m)}$ being the $n$-th zero of $J_m$ (→**27A.2**). Illustration of low frequency modes can be found in Wyld p164-5.[443]

**37.6 Can one hear the shape of the drum?** Suppose the set of all the eigenvalues of $-\Delta$ on $\Omega_1$ and that on $\Omega_2$ are identical. Can we conclude that the shapes of the domains are congruent: $\Omega_1 \equiv \Omega_2$? If yes, we can hear the shape of a drum. Now, we know this is not true even for 2-d drums.[444] However, we can hear quite a lot. For example, we can here the area of the drumhead: Let $N(\lambda)$ be the number of eigenvalues less than $\lambda$. Then,

$$N(\lambda)/(\mu(\Omega)\lambda/4\pi) \to 1 \tag{37.8}$$

asymptotically for large $\lambda$, where $\mu(\Omega)$ is the volume of $\Omega$ (conjectured by Lorentz who gave a lecture on this at Göttingen. This was later proved by Weyl). We can also here the number of holes.

**37.7 Eigenfunction expansion of Green's function.** The formal

[442] See Courant-Hilbert, Chapter 6, Section 6 for a proof.

[443] Excellent pictures of modes of a kettledrum can be found in T. D. Rossing, "The Physics of Kettledrums," Sci. Am. **247** (5) (1982) [November 1982].

[444] A readable account can be found in M A Shubin (ed.) *Partial Differential Equations VII* (Springer, 1994) Section 16.7 (p165-). However, the counter examples are all on the domains with non-smooth boundaries. No smooth counterexample is known. This is still a major problem. Historically, the first negative answer to the question was given in 16-space by Smale.

theory in **35.2** can be justified exactly as in the regular Sturm-Liouville problem (→**35.3**) thanks to **37.1**. Hence we have:
**Theorem**. The Green's function for the Laplacian in a compact domain $\Omega$ can be written as

$$G(x|y) = \sum_{i=1}^{\infty} \lambda_i^{-1} u_i(x)\overline{u_i(y)}, \tag{37.9}$$

where $u_i$ is the normalized eigenvector belonging to the eigenvalue $\lambda_i$ of $-\Delta$.□
From this, the symmetry of Green's functions (→**36.4**) is obvious.

**37.8 Examples.**
(1) The Green's function for a rectangular domain $[0, a] \times [0, b]$. The eigenvalues and the corresponding normalized eigenfunctions are given by

$$u_{mn} = \frac{2}{\sqrt{ab}} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}, \quad \lambda_{mn} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \tag{37.10}$$

for positive integers $m$ and $n$. Hence, the Green's function for the present problem is, according to (37.9)

$$G(x, y|x', y') = \frac{4}{\pi^2 ab} \sum_{m,n>0} \frac{\sin\frac{m\pi x}{a} \sin\frac{m\pi x'}{a} \sin\frac{n\pi y}{b} \sin\frac{n\pi y'}{b}}{(m/a)^2 + (n/b)^2}. \tag{37.11}$$

(2) Cylindrically symmetric Green's function for 3-space. In this case it is sensible to define the $L_2$-space with weight $r$, because the volume element is $2\pi r dr dz$. Hence, the delta function with the same weight (→**20.25**) is convenient (that is, $\delta(r - r')\delta(z - z')/r$ →**20.26**). The Green's function is the solution to

$$-\Delta u = -\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) u = \delta(z - z')\frac{\delta(r - r')}{r} \tag{37.12}$$

with the vanishing condition at infinity. We first solve the eigenvalue problem

$$-\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) u = \kappa^2 u. \tag{37.13}$$

We get the eigenvalues and the corresponding normalized eigenfunctions as (→**27A.21**)

$$u_{\kappa,k} = \frac{1}{\sqrt{2\pi}} e^{i\kappa z} J_0(kr), \quad \lambda_{\kappa,k} = \kappa^2 + k^2. \tag{37.14}$$

Here, $\kappa \in \boldsymbol{R}$ and $k$ is any positive real. Thus **37.7** (or its natural extension) tells us that the Green's function for our problem is

$$G(r, z|r', z') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\kappa \int_0^{\infty} dk \frac{e^{i\kappa(z-z')}J_0(kr)J_0(kr')}{\kappa^2 + k^2}. \qquad (37.15)$$

**Exercise.**
Construct the Green's functions for the Laplace equation with the following boundary conditions:
(1) On $[0, \pi] \times [0, 2\pi]$ with a homogeneous Dirichlet boundary condition along $x = 0$, $x = \pi$ and $y = 2\pi$, and a homogeneous Neumann boundary condition on $y = 0$.
(2) On the same domain with a periodic boundary condition. in the y direction (the x direction is the same).

**37.9 Neumann function in terms of eigenfunctions**. Under the homogeneous Neumann boundary condition any constant is an eigenfunction belonging to the zero eigenvalue. Hence, as can clearly be seen in (37.9), we cannot construct the Green's function. However, still the following 'generalized Green's function' works:

$$\hat{G}_N(x|y) = \sum_i{}' \lambda_i u_i(x)\overline{u_i(y)}, \qquad (37.16)$$

where $'$ implies that zero eigenvalue is excluded from the summation, and $u_i$ is the normalized eigenfunction belonging to the eigenvalue $\lambda_i$. The solution to **37.8** can be written as

$$u(x) = \int_\Omega \hat{G}_N(x|y)\varphi(y)dy + \int_{\partial\Omega} \hat{G}_N(x|y)h(y)d\sigma(y). \qquad (37.17)$$

(This is essentially (36.19). The difference is a constant which we may ignore.)
[Demo] First we find the equation for $\hat{G}_N$

$$-\Delta\hat{G}_N(x|y) = \sum_i{}' u_i(x)\overline{u_i(y)} = \sum_i u_i(x)\overline{u_i(y)} - V^{-1}, \qquad (37.18)$$

where $V$ is the volume of $\Omega$. We have used that the normalized eigenfunction belonging to zero is $1/\sqrt{V}$. Since the eigenfunctions are with the homogeneous Neumann condition

$$\frac{\partial \hat{G}_N}{\partial n}|_{x\in\partial\Omega} = 0. \qquad (37.19)$$

This is compatible with the equation (37.18). Now put $v = \hat{G}_N$ in Green's formula **16A.19**, and we get (37.17), ignoring an additive constant.