36 Green's Function: Laplace Equation

The Green's function method to solve the general boundary value problem for the Laplace equation is given. Neumann conditions need special care.

Key words: fundamental solution, Kirchhoff's formula, Neumann function

Summary:
(1) The reader must be able to explain the general idea of Green to her friend, and how to use Green's formula (→**36.6**).
(2) The Neumann function needs a special care, because homogeneous boundary conditions and the unit source are not compatible (→**36.7**, **37.9**).

36.1 Summary up to this point. Definition of Green's functions and fundamental solutions can be found in **14.2**. An intuitive idea was explained in **1.8**. Green's formula is in **16A.19** and some examples of Green's functions are in **16**.

36.2 Fundamental solution. The fundamental solution of the Laplace equation is a solution to

$$-\Delta\psi = \delta(x-y). \tag{36.1}$$

It is customary to put $-$ in front of the Laplacian, because $-\Delta$ is a positive definite operator (→**32A.3**). In d-space the following w is a fundamental solution. For $d \geq 3$ the function vanishes at infinity, so it is also a Green's function for free space $\boldsymbol{R}^d$ with the vanishing condition at infinity (→**16A.4**)

$$w(x|y) = \begin{cases} \frac{1}{S_{d-1}(d-2)|x-y|^{d-2}} & \text{for } d \geq 3, \\ -\frac{1}{2\pi}\ln|x-y| & \text{for } d = 2, \end{cases} \tag{36.2}$$

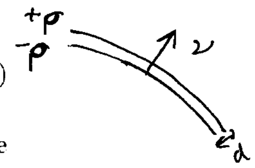
where S_{d-1} is the surface volume of the $(d-1)$-unit sphere.[434]

Notice that d-space function w can be obtained from the $(d+1)$-space counterpart through integrating along one coordinate direction (→**16A.5**).

[434] $S_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$.

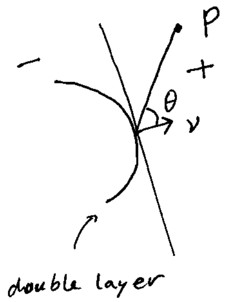
Discussion: Double layer.
(1) Consider two parallel surfaces with their spacing d. We assume that the surfaces are orientable[435] and let ν denote the outward normal direction. Let us assume that the outer surface has a uniformly distributed charge of area density $+\rho$, and the inner surface has the same distribution of the charge but of oppoisite sign. $p = \rho d$ is the area density of the dipole moment. We take the limit of $d \to 0$ while keeping p. The resultant double surface is called (electrical) *double layer.*
(1) Show that the electrical potential (assuming 0 potential at infinity) is given by (in 3-space) (ignore numerical coefficients)

$$V(\mathrm{P}) = \int_S d\sigma p \frac{\partial}{\partial \nu}\left(\frac{1}{r}\right), \tag{36.3}$$



where S is the surface, r is the distance between the point on the surface and the point P where we measure the potential.
(2) Let us introduce the angle θ between the outward normal and the line connecting the point on the surface and the point P. Then notice that

$$-\cos\theta = \frac{dr}{d\nu}, \tag{36.4}$$



so that (36.3) can be written as

$$V(\mathrm{P}) = \int_S \frac{p\cos\theta}{r^2} d\sigma. \tag{36.5}$$

(3) Notice that the solid angle of $d\sigma$ seen from P is given by

$$d\Omega = \pm\frac{d\sigma\cos\theta}{r^2}, \tag{36.6}$$

where the sign convention is $+$, if P is on the positive side of the double layer,[436] and $-$, if P is on the negative side. Hence, we have

$$V = \pm\int_S p d\Omega. \tag{36.7}$$

(4) This implies that, when $p = $ const., if P is outside a closed double layer S, then $V = 0$. If P is inside, then $V = -4\pi p$.

36.3 Theorem [Unique existence of Dirichlet problem Green's function]. For any well behaved[437] surface $\partial\Omega$ enclosing an open region Ω, there exists the unique Green's function $G_D(x|y)$ for $-\Delta$ which vanishes on $\partial\Omega$.□

[435] that is, there are two sides unlike the Möbius strip.
[436] This does not mean that P is located outside the layer even when the layer is closed. Simply, we draw a tangent plane on the shell and we ask on which side P exists.
[437] This vague statement will not be made precise here to avoid the technicality. Piecewise smooth surfaces are admissible. Cf. **1.19**(2) Discussion.

[Demo] The Green's function for the homogeneous Dirichlet problem is the solution to

$$- \Delta G_D(x|y) = \delta(x - y) \tag{36.8}$$

with $G_D = 0$ for $x \in \partial\Omega$. Here y is in Ω.[438] The problem can be rewritten as $G_D(x|y) = w(x|y) + u(x|y)$, where w is a fundamental solution in **36.2** and u satisfies

$$- \Delta u(x|y) = 0 \tag{36.9}$$

with the Dirichlet boundary condition $u(x|y) = -w(x|y)$ for $x \in \partial\Omega$. We have discussed that this problem has a unique solution at least informally (→**1.19**, **29.9**).

36.4 Symmetry of Dirichlet Green's function. In Green's formula (→**16A.19**) set $u(x) = G_D(x|y)$ and $v(x) = G_D(x|z)$. Then, we get

$$\int [G_D(x|y)\Delta G_D(x|z) - G_D(x|z)\Delta G_D(x|y)]dx = 0. \tag{36.10}$$

If we use (36.8), this immediately gives

$$G_D(y|z) = G_D(z|y), \tag{36.11}$$

the symmetry of the Green's function. We have already discussed this (formally in **35.2**, **16A.20**) (→**37.7**).

36.5 Free space Green's function is the largest. Let $G_D(x|y)$ be the Green's function for a region D. Then,

$$G_D(x|y) \le w(x|y). \tag{36.12}$$

Here w is the fundamental solution given in **36.2**, that is, the Coulomb potential.

This follows easily from the maximum principle **29.6**.

36.6 Solution to Dirichlet problem in terms of Green's function (16A.21 repeated). The solution to the following Dirichlet problem on an open region Ω

$$- \Delta u = \varphi, \quad u|_{\partial\Omega} = f, \tag{36.13}$$

where φ and f are integrable functions, is given by

$$u(x) = \int_\Omega G_D(x|y)\varphi(y)dy - \int_{\partial\Omega} f(y)\partial_{n(y)}G_D(x|y)d\sigma(y). \tag{36.14}$$

Here $\partial_{n(y)}$ is the outward normal derivative at y, τ is the volume element, and σ is the surface volume element.

[438] Inevitably, y is an internal point of Ω, since it is open.

Discussion

The Discussion in **36.2** allows us to understand (36.14) in terms of the charge distribution in Ω and the double layer on $\partial\Omega$. That is, Dirichle conditions can be understood as appropriate double layers.

36.7 Special feature of homogeneous Neumann condition. For a Neumann problem we do not know u but $\partial_n u$ on the boundary. We need the Green's function satisfying the homogeneous Neumann condition. However, we cannot impose a homogeneous boundary condition on a closed surface $\partial\Omega$ as seen below. Let G_N satisfy

$$-\Delta G_N(x|y) = \delta(x-y). \tag{36.15}$$

Then, Gauss' theorem (→**2C.13**) tells us that

$$\int_{\partial\Omega} \frac{\partial G_N}{\partial n} d\sigma = -1. \tag{36.16}$$

Therefore, the homogeneous Neumann condition cannot be imposed.[439] The simplest boundary condition compatible with (36.15) is

$$\frac{\partial G_N}{\partial n} = -1 \Big/ \int_{\partial\Omega} d\sigma = \frac{-1}{(\text{surface area of }\Omega)}. \tag{36.17}$$

36.8 Neumann function. The function satisfying (36.15) and (36.17) is called the *Neumann function*. In terms of the Neumann function, the solution to the following Neumann problem

$$-\Delta u = \varphi, \quad u|_{\partial\Omega} = h \tag{36.18}$$

reads

$$u(x) = \int_\Omega G_N(x|y)\varphi(y)dy + \int_{\partial\Omega} G_N(x|y)h(y)d\sigma(y). \tag{36.19}$$

Note that the solution to a Neumann problem is unique only up to an additive constant (→**1.19(3)**).

[Demo] In Green's formula let u be the solution and v be the Neumann function G_N. Then we have

$$\begin{aligned} u(x) &= \int_\Omega G_N(x|y)\varphi(y)dy + \int_{\partial\Omega}\left[G_N(x|y)h(y) + u(y)/\int_{\partial\Omega} d\sigma(y)\right]d\sigma(y), \\ &= \int_\Omega G_N(x|y)\varphi(y)dy + \int_{\partial\Omega} G_N(x|y)h(y)d\sigma(y) + const. \end{aligned} \tag{36.20}$$

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[439] If we wish to keep the homogeneous Neumann boundary condition, we must modify (36.15). This will be discussed in **37.9**.

The constant can be ignored, because we need the solution up to an additive constant.

36.9 Method of images. ($\rightarrow$**16A.7**, **16A.8**, **16A.14**) With the aid of the superposition principle and the conformal invariance (say, the reflection principle) ($\rightarrow$**16A.10**), we can construct Green's functions for special cases. For example, the half 3-space Green's function can be obtained by **16A.7**. Analogous half 2-space Green's function can be obtained. Notice that this Green's function vanishes at infinity in contrast to the free space counterpart.