35 Spectrum of Sturm-Liouville Problem

Eigenvalues for a regular Sturm-Liouville problem can be studied more conveniently through its Green's function which is a Hilbert-Schmidt kernel.

Key words: Sturm-Liouville eigenvalue problem, fundamental theorem

Summary:

(1) Remember that the inverse operator of the regular Sturm-Liouville operator is compact. All the fundamental properties of its spectrum follows from this fact (→**35.3**).
(2) Details of the Weyl-Stone-Titchmarsh-Kodaira theorem **35.5** need not be understood, but remember that there is a general way to expand a function in terms of functions in a fundamental system of solutions of a formally self-adjoint differential operator.

35.1 Rewriting of the eigenvalue problem as integral equation. The Sturm-Liouville eigenvalue problem is to find λ for

$$\mathcal{L}_{ST}u \equiv \left[\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\right] u = \lambda w(x)u \tag{35.1}$$

(with $p > 0$) under the following boundary condition:

$$B_a[u] \equiv Ap(a)u'(a) - Bu(a) = 0, \tag{35.2}$$
$$B_b[u] \equiv Cp(b)u'(b) - Du(b) = 0, \tag{35.3}$$

The problem can be rewritten with the aid of the Green's function (→**15.6**) as

$$u(x) = \lambda \int dy\, w(y)G(x|y)u(y) = \lambda(\mathcal{G}u)(x) \tag{35.4}$$

G is called the *kernel* of the integral operator $\mathcal{G}$.

35.2 Formal theory. [**20.28** repeated] (35.4) can be written as

$$|u\rangle = \lambda\mathcal{G}|u\rangle, \tag{35.5}$$

where bras and kets are defined with the weight function w (→**20.22**, **20.23**). Let $|i\rangle$ be an eigenket belonging to the eigenvalue λ_i:

$$\lambda_i\mathcal{G}|i\rangle = |i\rangle. \tag{35.6}$$

If $\{|i\rangle\}$ is an orthonormal basis of $L_2([a,b],w)$ ($\rightarrow$**20.19**), then from (35.5) we get

$$\mathcal{G} = \sum_i |i\rangle\lambda_i^{-1}\langle i|. \tag{35.7}$$

That is, the Green's function can be written as

$$G(x|y) = \langle x|\mathcal{G}|y\rangle = \sum_i \lambda_i^{-1}u_i(x)\overline{u_i(y)}. \tag{35.8}$$

We must justify this result.

35.3 Theorem [Fundamental theorem of Sturm-Liouville eigenvalue problem]. The eigenfunctions of a regular Sturm-Liouville problem ($\rightarrow$**15.4**) form an orthogonal basis of $L^2([a,b],w)$ ($\rightarrow$**20.19**), and the sequence of eigenvalues satisfies $|\lambda_n| \rightarrow \infty$ as $n \rightarrow \infty$.□

[Demo] We can explicitly construct the Green's function for this problem as in **15.6**, which is a continuous function of x and y, so that $\mathcal{G}$, whose kernel is given by **15.6**, is a compact operator ($\rightarrow$**34C.9**) thanks to Hilbert and Schmidt **34C.11**. Its self-adjointness is also easy to demonstrate. Hence, we can apply **34C.12**. Note that the eigenvalues here are the reciprocals of those in **34C.12**.

Discussion.

(A) **Classical approach due to Prüfer.**

Our demonstration heavily relied on functional analytic methods. The facts were known before functional analytic methods were widely available. Here a classical proof of the theorem due to Prüfer is given. The argument may seem more complicated and more artful, but more delicate results than those obtained by a high-tech functional analysis may be obtained.

(1) Suppose there is a solution $u \not\equiv 0$ to (35.1). Then, pu' and u do not vanish simultaneously. Hence, we can introduce a polar coordinate system such as

$$u(x) = \rho(x)\sin\theta(x), \tag{35.9}$$

$$p(x)u'(x) = \rho(x)\cos\theta(x). \tag{35.10}$$

(2) Our eigenvalue problem can be rewritten as follows:

$$\rho'(x) = (p(x)^{-1} + q(x) + \lambda w(x))\rho\sin\theta\cos\theta \tag{35.11}$$

$$\theta' = p(x)^{-1}\cos^2\theta + (-\lambda w(x) - q(x))\sin^2\theta. \tag{35.12}$$

The second equation does not contain ρ, so we can integrate this for $\theta(x)$ with an arbitrary initial condtion $\theta(0) = \alpha$.

(3) A necessary and sufficient condition for λ to be an eigenvalue of **35.1** is that $\theta(x)$ with the initial condtion $\theta(a) = \alpha$ satisfies $\theta(b) = \beta + n\pi$, where n is a positive integer. Here the angles α and β are determined as

$$\tan\alpha = A/B, \ \tan\beta = C/D \tag{35.13}$$

with $\alpha, \beta \in [0, \pi)$.

(4) **Prüfer's comparison theorem.** Let $\theta(x,\lambda)$ be the solution of (2) with the

initial condition $\theta(a) = \alpha$. Then, for $x \in (\alpha, \beta]$

$$\lambda_1 < \lambda_2 \Rightarrow \theta(x, \lambda_1) < \theta(x, \lambda_2). \tag{35.14}$$

This tells us that the eigenfunction corresponding to a larger eigenvalue oscillates faster. θ is monotonically increasing as a function of λ. In particular,
(5)

$$\lim_{\lambda \to -\infty} \theta(b, \lambda) \le 0, \tag{35.15}$$

$$\lim_{\lambda \to +\infty} \theta(b, \lambda) = +\infty. \tag{35.16}$$

This implies
(6) The Sturm-Liouville eigenvalue problem has a discrete set of eigenvalues such that

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n \to +\infty. \tag{35.17}$$

(7) Furthermore, the eigenfunction corresponding to the n-th largest eigenvalue has exactly $n-1$ simple zeros in (a, b). See **24A.13** (Discussion) for the simplicity of the zeros (non-degeneracy of eigenstates). For nodal sets, see **37.4**. Also note that this proves the statement about the positions of the zeros of orthogonal polynomials **21A.11** (2) (see **21A.7**).
(8) Completeness of the eigenfunctions: If a continuous function $h(x)$ satisfies

$$\int_a^b dx w(x) h(x) \phi_n(x) dx = 0, \tag{35.18}$$

for all $n \in \boldsymbol{N}$, then $h \equiv 0$, where ϕ_n is an eigenfunction belonging to λ_n.
Its proof depends on the fact that if (35.18) is true, then the solution to

$$\mathcal{L}_{ST} y = w(x) h(x) \tag{35.19}$$

with the homogeneous boundary condtion has a continuous solution for any real λ. However, this cannot be true if $h \equiv 0$.
(9) (8) gives us a generalized Fourier expansion: If

$$f(x) = \sum f_n \phi_n(x) \tag{35.20}$$

is uniformly and absolutely convergent, then the coefficient can be computed as a Fourier coefficien.
(10) Let f be piecewisely C^1. The formal series (35.20) is actually uniformly and absolutely convergent.
(B) Prüfer's technique allows us to prove the following theorem about the distribution of zeros of a Schödinger equation:

$$u'' + q(x) u = 0. \tag{35.21}$$

Suppose

$$m^2 \le q(x) \le M^2. \tag{35.22}$$

Then, for any solution $u \not\equiv 0$, the spacing of the zeros δ satisfies

$$\frac{\pi}{M} \le \delta \le \frac{\pi}{m}. \tag{35.23}$$

Exercise.
Suppose (35.21) is considered on $[a, b]$ with a Dirichlet condition. Demonstrate that the magnitude of the eigenvalue λ_n increases asymptotically as n^2.
Discussion.
(C) Find the eigenvalues and eigenfunctions of the operator $d^2/dx^2 + \lambda$ on $[-1, 1]$ with the following boundary conditions:
(1) $du/dx(-1) = du/dx(1) = 0$.
(2) $u - du/dx = 0$ at $x = \pm 1$.
(D) What happens if the regularity condition is dropped?[430]
Consider

$$\frac{d}{dt}\left(t^2 \frac{d}{dt} x\right) + \lambda x = 0, \tag{35.24}$$

with the following boundary conditions.
(1) $x(-1) + x'(-1) = x(1) + x'(1) = 0$ (no eigenvalue).
(2) $x(-1) + x'(-1) = 0$ and $x(1) - x'(1) = 0$ (-2 is the only eigenvalue. The corresponding eigenfunction is t.)
(E) Irrespective of the boundary conditions, the n-th eigenvalue of a Sturm-Liouville problem is a continuous function of the coefficients of the equation (Courant-Hilbert).

35.4 Justification of separation of variables. When the region of the problem is finite, very often the separated problems are regular Sturm-Liouville eigenvalue problem. Hence, **35.3** is the key (if the reader does not wish to use less elementary Friedrichs extension ($\rightarrow$**34B.5**)). However, notice that **35.3** is <u>not</u> enough to justify what we wish to do on unbounded regions. Friedrichs extensions work even in such cases. Here, however, a more constructive theory is posted.

35.5 Theorem [Weyl-Stone-Titchmarsh-Kodaira]. Let L be a second order linear differential operator which is formally self-adjoint:

$$L = -\frac{d}{dx} p(x) \frac{d}{dx} + q(x), \tag{35.25}$$

where p and q are C^∞ on (a, b).[431] For $\lambda \in \boldsymbol{R}$, consider

$$Lu = \lambda u. \tag{35.26}$$

Let $\{\psi_1(x;\lambda), \psi_2(x;\lambda)\}$ be a fundamental system of solutions ($\rightarrow$**24A.11**) of this equation. Then, there is a matrix measure ρ_{ij} $(i, j \in \{1, 2\})$[432]

[430] N
[431] a could be $-\infty$ and b ∞.
[432] That is, any component of the matrix $Matr\{\rho_{ij}(\lambda)\}$ is a measure.

such that we can make the following decomposition of unity

$$\delta(x-y) = \int_{-\infty}^{\infty} \sum_{i,j} \psi_i(x;\lambda) d\rho_{ij}(\lambda) \psi_j(y;\lambda). \qquad (35.27)$$

The equality here is in the L_2-sense.[433] Here the so-called density matrix ρ_{ij} can be constructed from the resolvent (→**34C.2**) of L.□
(35.27) implies the following:

$$f(x) = \int_{-\infty}^{\infty} \sum_{i,j} \psi_i(x;\lambda) d\rho_{ij}(\lambda) \hat{f}_j(\lambda), \qquad (35.28)$$

and

$$\hat{f}_j(\lambda) = \int_a^b dy \psi_j(y;\lambda) f(y). \qquad (35.29)$$

Thus $\hat{f}_i(\lambda)$ is a kind of generalized Fourier transform of f.

[433]That is, when it is applied to a ket, the difference of RHS and LHS measured in terms of the L_2-norm is zero.