33 Laplace Transformation

Laplace transformation is a disguised Fourier transformation for causal functions (the functions that are zero in the past), and is a very useful tool to study transient phenomena. The inverse transformation is often not easy, but clever numerical tricks may be used to invert the transforms. Appendix **a33** discusses a disguised Laplace transformation, Mellin transformation, which is useful when we wish to solve problems on fan shaped domains.

Key words: Laplace transform, fundamental theorem, convolution, time-delay, fast inverse Laplace transform.

Summary:
(1) Laplace transformation **33.2** allows one to solve many ODE algebraically with the aid of tables (**33.14**).
(2) Basic formulas like the convolution theorem, delay theorem, etc should be known to this end (**33.7**-**10**).

33.1 Motivation. Due to causality, we often encounter functions of time t that are zero for $t < 0$ (or often so for $t \leq 0$ due to continuity). Then, the so-called one-sided Fourier transform

$$F[\omega] = \int_0^\infty f(t)e^{i\omega t}dt \tag{33.1}$$

appears naturally. However, if $f(t)$ grows as e^{at} ($a > 0$), then this does not make sense even in the sense of generalized functions (→**14.4**). Even in this case, if we choose sufficiently large $c > 0$, the one-sided Fourier transform of $e^{-ct}f(t)$ exists in the ordinary sense. If $f(t)e^{-ct}\Theta(t)$ ($\Theta(t)$ is the Heaviside step function →**14.15(3)**) is absolutely integrable, and f' is piecewise continuous for $t > 0$, then from the Fourier transform of this function, $f(t)$ for $t > 0$ can be recovered.

33.2 Definition of Laplace transform. The following transformation $\mathcal{L}_s$ is called the *Laplace transformation*:

$$\mathcal{L}_s[u(t)] = \int_0^\infty e^{-st}u(t)dt, \tag{33.2}$$

where $s = c - i\omega$ and c is chosen sufficiently large so that the integral exists. $\mathcal{L}_s[u]$ is called the *Laplace transform* of u.[411]

Discussion.

(A) A discrete counterpart is the so-called *z-transformation*: The z-transform $A(z)$ of $\{a_n\}$ is defined by

$$A(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{33.3}$$

This is also called the *generating function* of the sequence $\{a_n\}$. The inverse transform is given by

$$a_n = \frac{1}{2\pi i}\int_{\partial D} dz \frac{A(z)}{z^{n+1}}, \tag{33.4}$$

where D is a disc containing the origin but excluding all the singularities of $A(z)$.

(B) z-transform is a convenient way to solve linear difference equation:

$$a_0 x_{n+r} + a_1 x_{n+r-1} + \cdots + a_{r-1}x_{n+1} + a_r x_n = 0. \tag{33.5}$$

For example, let us solve

$$x_{n+2} - 2x_{n+1} + x_n = 0 \tag{33.6}$$

with the 'initial conditions' $x_0 = 1$, and $x_1 = 0$. The z-transform $X(z)$ obeys

$$X(z) - 1 + 2z(X(z) - 1) + z^2 X(z) = 0. \tag{33.7}$$

From this we can solve $X(z)$. The inverse transform gives $x_n = 1 - n$.

(C) An inhomogeneous linear difference equation is given by

$$a_0 x_{n+r} + a_1 x_{n+r-1} + \cdots + a_{r-1}x_{n+1} + a_r x_n = f_n \tag{33.8}$$

The general solution to this equation is given by the sum of the general solution of (33.5) and a special solution to (33.8) just as the linear differential equation. If we can compute the z-transform of $\{f_n\}$, then at least $X(z)$ can be obtained. However, to obtain x_n from X may not be very easy.

33.3 Who was Laplace (1749-1827) ? The 'Newton of France' was born into a cultivated provincial bourgeois family in Normandy (Beaumont-en-Auge) in 1749. After his secondary school education he attended University of Caen n 1766 to study the liberal arts, but two of his professors (Gadbled and LeCanu) urged this gifted student to pursue mathematics. With LeCanu's letter to d'Alembert (→**2B.5**) he left for Paris at age 18 in 1768. He impressed d'Alembert, who secured a position for him at the Ecole Militaire. In 1773 he demonstrated that the acceleration observed in Jupiter and Saturn was not cumulative but periodic. This was the principal advance in dynamical astronomy since

[411] For a history, see M. F. Gardner and J. L. Barnes, *Transients in Linear Systems* vol.I (Wiley, 1942) Appendix C.

Newton toward establishing the stability of the solar system. This work won him election to the Paris Academy in 1773.

Between 1778 and 1789 he was at his scientific prime. Laplace introduced his transformation in 1779, which was related to Euler's work. In 1780 he worked together with Lavoisier to make a calorimeter to establish that respiration is a form of combustion. Although he played a decisive role to design the metric system in 1790, he wisely avoided Paris when the Jacobins dominated until 1794. In the late 1790s with three well received books (one of which, *Systéme du Monde*, was not only a fine science popularizer but also a model of French prose), he became a European celebrity.

Laplace advanced applied mathematics and theory of probability substantially. He based his theory on generating functions, and extended Jakobi Bernoulli's work on the law of large numbers. He was amply honored by Napoleon and by Louis XVIII. During his final years he lived at his country home in Arceuil, next to his friend chemist Berthollet, surrounded by the adopted children of his thoughts, Arago, Poisson, Biot, Gay-Lussac, von Humboldt and others.

33.4 Fundamental theorem of Laplace transform.
(1) The Laplace transform of f (33.2) exists for s such that $e^{-(Re\, s)t}f(t) \in L_1([0,\infty))$.
(2) There is a one-to-one correspondence between $f(t)$ and its Laplace transform $\mathcal{L}_s[f]$. More explicitly, we have

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\mathcal{L}_s[f]ds, \tag{33.9}$$

where c is a real number larger than the convergence coordinate c^* such that all the singularities of $\mathcal{L}_s[f]$ lie on the left side of $z = c^*$ in $\boldsymbol{C}$.[412]
[Demo] (1) is obvious. At least formally, (2) follows from the motivation **33.1**. Fourier inverse transform of $\mathcal{L}_s[f]$ gives

$$f(t) = e^{ct}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\mathcal{L}_{c-i\omega}[u(t)]d\omega. \tag{33.10}$$

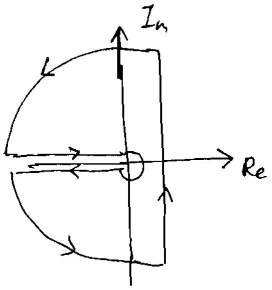
Since $d\omega = ids$, (33.10) becomes

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{L}_s[f(t)]e^{st}ds. \tag{33.11}$$

For this integral to be meaningful, we need the following theorem:

[412] This was formally shown by Riemann by 1859. Mellin proved this in Acta Soc. Sci. Fenn. **21**, 115 (1896). Hence, there is absolutely no justification to call this integral the 'Bromwitch integral.' History must not be distorted due to national interests.

Discussion.
(1) $f(t) = \exp(t^\sigma)$ with $\sigma > 1$ does not have Laplace transforms.
(2) The minimum real number r making $f(t)e^{-rt} \in L_2([0, +\infty))$ is called the *convergence coordinate.*
Exercise.
Although practically, there is almost no need (→**33.14**) of calculating the integral (33.9), still it is a good exercise of complex integration. Demonstrate the following inverse transform relations with the aid of the residue theorem (→**8B**).
(1)
$$\mathcal{L}_s^{-1} \frac{1}{(s+\alpha)^n} = \frac{t^{n-1}}{(n-1)!} e^{-\alpha t}, \tag{33.12}$$
where $\alpha > 0$ and n is a positive integer.



(2) How can we do a similar thing, if n is not an integer? In this case, $s = 0$ is a branch point (→**8A.2-4**). If $n \in (0, 1)$, then a straightforward contour integration along the contour in the figure works. The contribution from the small circle vanishes in the small radius limit, and the contribution from the large circle is zero thanks to the Jordan lemma **8B.7**. We need **9.5** to streamline the formula. If n is larger, then probaly the cleverest way is to use **33.7**(5) and reduce the problem to the case of $n \in (0, 1)$.

33.5 Theorem. $\mathcal{L}_s[f]$ is holomorphic (→**5.4**) where $\mathcal{L}_s[f]$ exists. □[413] Therefore, if $\mathcal{L}_s[f]$ exists for $c > c^*$, then $\mathcal{L}_s[f]$ has no singularity on the half plane $Re\, z \geq c$.
This implies that
(1) $\mathcal{L}_s[f]$ is differentiable with respect to s,
(2) $\mathcal{L}_s[f]$ is determined by its behavior on the portion of the real axis $x > c^*$ through analytic continuation (→**7.8**).

33.6 Theorem. If s goes to s_0 along a curve lying inside the convergence domain, then
$$\lim_{s \to s_0} \mathcal{L}_s[f] = \mathcal{L}_{s_0}[f]. \tag{33.13}$$
Especially,
$$\lim_{s \to \infty} \mathcal{L}_s[f] = 0. \tag{33.14}$$
[Demo] (33.14) follows from (33.13), which follows trivially from an elementary property of the Lebesgue integral.

[413] To prove this we need the following elementary theorem about Lebesgue integration
Theorem. Suppose
(1) $f(x, s)$ is integrable (→**19.8**) for each s as a function of x,
(2) $f(x, s)$ is holomorphic for almost all x as a function of s,
(3) There is an integrable function Φ such that $|f(x, s)| \leq \Phi(x)$.
Then, $\int dx f(x, s)$ is holomorphic as a function of s. □

33.7 Some properties of Laplace transform.
(1) $a\mathcal{L}_s[f(at)] = \mathcal{L}_{s/a}[f(t)]$, where a is a positive constant. This can be shown by a straightforward calculation.
(2) $\mathcal{L}_s[e^{-bt}f(t)] = \mathcal{L}_{s+b}[f(t)]$. This is straightforward, too.
(3) $\mathcal{L}_s[t^n f(t)] = (-1)^n(d/ds)^n\mathcal{L}_s[f(t)]$. In particular, $\mathcal{L}_s[tf(t)] = -d/ds\mathcal{L}_s[f(t)]$.
(4) $\mathcal{L}_s[f^{(n)}(t)] = s^n\mathcal{L}_s[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - s^{n-k}f^{(k-1)}(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$. In particular,

$$\mathcal{L}_s[f'(t)] = s\mathcal{L}_s[f(t)] - f(0). \tag{33.15}$$

This is due to integration by parts.
(5) $\mathcal{L}_s\left[\int_0^t f(t')dt'\right] = s^{-1}\mathcal{L}_s[f(t)]$.
(6) $\mathcal{L}_s[t^{-1}f(t)] = \int_s^\infty ds\mathcal{L}_s[f(t)]$.
(3) - (6) imply that calculus becomes algebra through the Laplace transformation. This is the most important and useful property facilitating the solution of linear ODE.

Discussion
The following equation is called the *Airy equation* (→**27A.23** Exercise (3))

$$\frac{d^2y}{dt^2} - ty = 0. \tag{33.16}$$

Since the coefficient is only a linear function of t, Laplace transformation is advantageous. Let z be a function of s that is the Laplace transform of y with respect to t. Then,

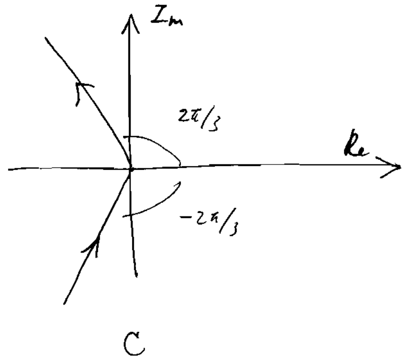
$$\frac{dz}{ds} - s^2 z = 0, \tag{33.17}$$

which can be solved easily as

$$z = e^{s^3/3}. \tag{33.18}$$

Hence, a solution can be written as

$$Ai(t) = \frac{1}{2\pi i}\int_C \exp\left(st - \frac{1}{3}s^3\right) ds. \tag{33.19}$$



Here C can be a path as shown in the figure. The integral is called the *Airy integral*
.
Show that

$$Ai(0) = 3^{-1/6}\Gamma(1/3)/2\pi. \tag{33.20}$$

33.8 Convolution. If we adapt the ordinary definition of convolution **14.22** to functions that are zero for $t < 0$, we get

$$(f_1 * f_2)(t) = \int_0^t f_1(t-u)f_2(u)du. \tag{33.21}$$

A straightforward calculation gives

$$\mathcal{L}_s[f_1 * f_2] = \mathcal{L}_s[f_1]\mathcal{L}_s[f_2]. \tag{33.22}$$

Exercise.

$$\int_0^x \sin(x-y)u(y)dy + u(x) = \cos x. \tag{33.23}$$

33.9 Time-delay.

$$\mathcal{L}_s[f(at-b)\Theta(at-b)] = \frac{1}{a}e^{-bs/a}\mathcal{L}_{s/a}[f(t)] \tag{33.24}$$

This is also demonstrated by a simple calculation. $e^{-\tau s}$ is often called a delay factor.

33.10 Periodic functions. If f is a function with period T, then

$$\mathcal{L}_s[f(t)] = (1-e^{-sT})^{-1}\int_0^T e^{-st}f(t)dt. \tag{33.25}$$

[Demo] Thanks to the periodicity, we get

$$\int_0^\infty e^{-st}f(t)dt = \int_0^\infty e^{-st}f(t+T)dt = \int_T^\infty e^{-s\tau}f(\tau)d\tau e^{sT}, \tag{33.26}$$

where $t = \tau - T$. This implies that

$$\mathcal{L}_s[f(t)] = \left\{\mathcal{L}_s[f(t)] - \int_0^T e^{-st}f(t)dt\right\}e^{sT}. \tag{33.27}$$

Solving this equation for $\mathcal{L}_s[f]$, we get the desired formula.

33.11 Examples.
(1) $\mathcal{L}_s[1] = 1/s$ is obvious by definition.
(2) This with (2) of **33.7** implies $\mathcal{L}_s[e^{-bt}] = 1/(s+b)$.
(3) Linearity of the Laplace transformation and (2) give, for example,

$$\mathcal{L}_s[\cos\omega t] = \frac{1}{2}(\mathcal{L}_s[e^{i\omega t}] + \mathcal{L}_s[e^{-i\omega t}]) = \frac{s}{s^2+\omega^2}. \tag{33.28}$$

Analogously, we get $\mathcal{L}_s[\cosh at] = s/(s^2-a^2)$, $\mathcal{L}_s[\sin\omega t] = \omega/(s^2+\omega^2)$, etc.
(4) (3) with (2) of **33.7** gives for example

$$\mathcal{L}_s[e^{-bt}\cos\omega t] = \frac{s+b}{(s+b)^2+\omega^2}. \tag{33.29}$$

(5) (1) and (3) of **33.7** imply

$$\mathcal{L}_s\left[\frac{t^n}{n!}\right] = \frac{1}{s^{n+1}}. \tag{33.30}$$

More generally, for $\nu > -1$

$$\mathcal{L}_s\left[\frac{t^\nu}{\Gamma(\nu+1)}\right] = \frac{1}{s^{\nu+1}}. \tag{33.31}$$

This can be shown immediately by the definition of the Gamma function (→**9**).
(6) Combining (33.30) and (2) of **33.7** gives

$$\mathcal{L}_s[e^{-bt}t^n] = \frac{n!}{(s+b)^{n+1}}. \tag{33.32}$$

(7) An application of **33.10** is

$$\mathcal{L}_s[|\sin t|] = \frac{1}{s^2+1}\coth\frac{\pi s}{2}. \tag{33.33}$$

(8) Applying the convolution theorem **33.8** we can demonstrate

$$\int_0^t J_0(\tau)J_0(t-\tau)d\tau = \sin t \tag{33.34}$$

This follows from (→**27A.15**)

$$\mathcal{L}_s[J_0(t)] = \frac{1}{\sqrt{s^2+1}}. \tag{33.35}$$

Exercise.
(A) Show

$$\mathcal{L}_s\frac{1}{\sqrt{t}} = \frac{\sqrt{\pi}}{\sqrt{s}}. \tag{33.36}$$

(B) Find
(1) $\mathcal{L}_s \cos^2 \omega t$.
(2) For $\tau > 0$ and $a > 0$ $\mathcal{L}_s(t-t_1)E^{-a(t-t_2}\Theta(t-\tau)$.

33.12 Laplace transform of delta function. We can define Laplace transforms of generalized functions. We will not discuss this, since the relation between Fourier and Laplace transformations **33.1** explains virtually everything we need practically. A subtlety may remain in the definition of the Laplace transformation of $\delta(x)$, since the definition **33.2** requires an integration from 0. That is, we must consider the product of $\delta(x)$ and $\Theta(x)$, which is meaningless (→**14.6**) as generalized functions. Without any ambiguity for $a > 0$

$$\mathcal{L}_s[\delta(t-a)] = e^{-as}. \tag{33.37}$$

This means the Laplace transform of the weak limit $\lim_{\epsilon\to 0+}\delta(t-\epsilon)$ is 1. Hence, as a generalized function it is sensible to define (→**14.18**)

$$\mathcal{L}_s[\delta(t)] = 1. \tag{33.38}$$

From this (33.37) is obtained with the aid of the time delay formula **33.9**.

33.13 Short time limit.

$$\lim_{t\to 0+} f(t) = \lim_{s\to\infty} s\mathcal{L}_s[f(t)]. \tag{33.39}$$

[Demo] **33.7**(4) with $n = 1$ reads $\mathcal{L}_s[f'(t)] = s\mathcal{L}_s[f(t)] - f(0)$. Apply **33.6** to f', and we get $\lim_{s\to\infty}\mathcal{L}_s[f'(t)] = 0$.

33.14 Practical calculation of Laplace inverse transformation: Use of tables. Although the fundamental theorem **33.4**(2) gives a method to compute the inverse transforms, practically, an easier method is to use a table of Laplace transforms of representative functions. The uniqueness of the transforms (→**33.4**(2)) guarantees that once we can find an inverse transform, that is the inverse transform of a given function of s. Also numerical fast Laplace inverse transform is available.

Exercise.

(1) Solve the following differential equation with the aid of Laplace transformation

$$\frac{d^2y}{dt^2} + 2a\frac{dy}{dt} + (a^2+b^2)y = e^{-at}\sin bt.$$

Here a and b are positive constants, and the initial condition is $y(0) = y'(0) = 0$.

(2) Using Laplace transformation, solve the following integrodifferential equation

$$y(t) = y'(t) + t + 2\int_0^t (t-u)y(u)du$$

with the initial condition $y(0) = 0$.

33.15 Heaviside's expansion formula.[414] Let $F(s)$ be a rational

[414] Heaviside (1850-1925) introduced an algebraic method to solve ODEs, which can be understood as the Laplace transform method explained below. The method,

function[415] $F(s) = P(s)/Q(s)$, where P and Q are mutually prime polynomials, and the order of Q is higher than that of P. If $Q(s) = A(s-a_1)\cdots(s-a_n)$ and $a_1, \cdots, a_n$ are all distinct, then

$$\frac{P(s)}{Q(s)} = \sum_{s=1}^{n} \frac{c_k}{s-a_k} \tag{33.40}$$

with $c_k = P(a_k)/Q'(a_k)$. □
This is obvious, and implies that

$$\mathcal{L}_s^{-1}[P(s)/Q(s)] = \sum_{k=1}^{n} P(a_k)e^{a_k t}/Q'(a_k). \tag{33.41}$$

33.16 Examples.

$$\mathcal{L}_s^{-1}\left[\frac{s^2+s+1}{(s^2+1)^3}\right] = \frac{1}{8}(4+t)\sin t - \frac{1}{8}(4t+t^2)\cos t. \tag{33.42}$$

$$\mathcal{L}_s^{-1}\left[\frac{2s+3}{2s^3+3s^2-2s}\right] = -\frac{3}{2} - \frac{1}{10}e^{-2t} + \frac{8}{5}e^{t/2}. \tag{33.43}$$

$$\mathcal{L}_s^{-1}\left[\frac{s^2+1}{2(s^4+s^2+1)}\right] = 1 - \frac{\sqrt{3}}{3}\left[e^{t/2}\cos\left(\frac{\sqrt{3}}{2}t + \frac{\pi}{6}\right) + e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right)\right]. \tag{33.44}$$

Exercise.
(1) Find the inverse transform of

$$g(s) = \frac{s^2 - \omega s + \omega^2}{s(s^2+\omega^2)}. \tag{33.45}$$

(Answer: $\Theta(t) - \sin \omega t$).

$$g(s) = \frac{1+e^{\pi s}}{s(s^2+1)}. \tag{33.46}$$

33.17 Fast inverse Laplace transform. T. Hosono, "Numerical inversion of Laplace transform and some applications to wave optics," Radio Science **16**, 1015 (1981); *Fast Laplace transform in Basic*, (Kyoritsu Publ., 1984)

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which requires generalized functions like the Heaviside step function, and even the delta function, was never accepted by mathematicians of his day. According to an anecdote, he said that we could eat even though we did not know the mechanism of digestion. This story is often told as a story of a triumph of a self-educated genius. **However**, the method was actually invented by Cauchy long ago. Therefore the story must be quoted as a failure of premature ossification of mathematics due to mediocre mathematicians.

[415] A rational function is a ratio of two polynomials.

Table

$$g(p) = \mathcal{L}_p[f(t)] = \int_0^\infty e^{-pt} f(t)\,dt$$

$f(t)$		$g(p)$
t^ν	$[\Re\nu>-1]$	$\Gamma(\nu+1)/p^{\nu+1}$
$(t^2+2at)^\nu$	$[a>0,\ \Re\nu>-1]$	$\dfrac{\Gamma(\nu+1)}{\sqrt{\pi}}\left(\dfrac{2a}{p}\right)^{\nu+(1/2)} e^{ap}K_{\nu+\frac{1}{2}}(ap)$
$(t^2+it)^\nu$	$[\Re\nu>-1]$	$-\dfrac{i\sqrt{\pi}\,\Gamma(\nu+1)e^{ip/2}}{2p^{\nu+(1/2)}}H^{(2)}_{\nu+\frac{1}{2}}\left(\dfrac{p}{2}\right)$
$(t^2-it)^\nu$	$[\Re\nu>-1]$	$\dfrac{i\sqrt{\pi}\,\Gamma(\nu+1)}{2p^{\nu+(1/2)}e^{ip/2}}H^{(1)}_{\nu+\frac{1}{2}}\left(\dfrac{p}{2}\right)$
$\log t$		$-\dfrac{1}{p}(\log p+\gamma)$
$t^{\nu-1}\log t$	$[\Re\nu>0]$	$\dfrac{\Gamma(\nu)}{p^\nu}[\psi(\nu)-\log p]$
$\log(t^2+1)$		$\dfrac{2}{p}[\mathrm{Ci}(p)\cos p-\mathrm{si}(p)\sin p]$
$\operatorname{arcsinh} t=\log(t+\sqrt{t^2+1})$		$\dfrac{\pi}{2p}[\mathbf{H}_0(p)-N_0(p)]$
$\operatorname{arccosh} t=\log(t+\sqrt{t^2-1})$	$[t\geqq1]$	$\dfrac{1}{p}K_0(p)$
0	$[t<1]$	
$e^{\alpha t}$		$1/(p-\alpha)$
$e^{\alpha t}t^{\nu-1}$	$[\Re\nu>0]$	$\Gamma(\nu)/(p-\alpha)^\nu$
$e^{\alpha t}/\sqrt{t}$		$\sqrt{\pi}/\sqrt{p-\alpha}$
$(1-e^{-at})/t$	$[a>0]$	$\log[1+(a/p)]$
$t^n/(1-e^{-t/a})$	$[a>0]$	$(-a)^{n+1}\psi^{(n)}(ap)$ (注 2)
$(1-e^{-t/a})^{\nu-1}$	$[\Re\nu,\ a>0]$	$aB(\nu,ap)=a\dfrac{\Gamma(\nu)\Gamma(ap)}{\Gamma(\nu+ap)}$
$\dfrac{1-e^{at}}{1-e^{-t}}$		$\psi(p-a)-\psi(p)$ (注 2)
$\dfrac{1}{t}\left(\dfrac{1}{1-e^{-t}}-\dfrac{1}{t}-\dfrac{1}{2}\right)$		$\log\dfrac{\Gamma(p)}{\sqrt{2\pi}}-\left(p-\dfrac{1}{2}\right)\log p+p$
$e^{-t^2/(4a)}$	$[a>0]$	$2\sqrt{a}\,e^{ap^2}\,\mathrm{Erfc}(\sqrt{a}\,p)$
$\sinh\alpha t$		$\alpha/(p^2-\alpha^2)$
$\cosh\alpha t$		$p/(p^2-\alpha^2)$
$\sinh^2\alpha t$		$2\alpha^2/(p^3-4\alpha^2 p)$
$\cosh^2\alpha t$		$(p^2-2\alpha^2)/(p^3-4\alpha^2 p)$
$t^{\nu-1}\sinh\alpha t$	$[\Re\nu>-1,\ \nu\neq0]$	$\dfrac{\Gamma(\nu)}{2}\left[\dfrac{1}{(p-\alpha)^\nu}-\dfrac{1}{(p+\alpha)^\nu}\right]$
$t^{\nu-1}\cosh\alpha t$	$[\Re\nu>0]$	$\dfrac{\Gamma(\nu)}{2}\left[\dfrac{1}{(p-\alpha)^\nu}+\dfrac{1}{(p+\alpha)^\nu}\right]$
$\dfrac{\sinh\alpha t}{t}$		$\dfrac{1}{2}\log\dfrac{p+\alpha}{p-\alpha}$
$\dfrac{\sinh^2\alpha t}{t}$		$-\dfrac{1}{4}\log\left(1-\dfrac{4\alpha^2}{p^2}\right)$

$f(t)$		$g(p)$
$\sin\alpha t$		$\alpha/(p^2+\alpha^2)$
$\cos\alpha t$		$p/(p^2+\alpha^2)$
$\lvert\sin at\rvert$	$[a>0]$	$\dfrac{a}{p^2+a^2}\coth\dfrac{\pi p}{2a}$
$\lvert\cos at\rvert$	$[a>0]$	$\dfrac{1}{p^2+a^2}\left[p+a\,\mathrm{cosech}\dfrac{\pi p}{2a}\right]$
$t^{\nu-1}\sin\alpha t$	$[\Re\nu>-1]$	$\dfrac{i\Gamma(\nu)}{2}\left[\dfrac{1}{(p+i\alpha)^\nu}-\dfrac{1}{(p-i\alpha)^\nu}\right]=\dfrac{\Gamma(\nu)}{(p^2+\alpha^2)^{\nu/2}}\sin\left(\nu\arctan\dfrac{\alpha}{p}\right)$
$t^{\nu-1}\cos\alpha t$	$[\Re\nu>0]$	$\dfrac{\Gamma(\nu)}{2}\left[\dfrac{1}{(p+i\alpha)^\nu}+\dfrac{1}{(p-i\alpha)^\nu}\right]=\dfrac{\Gamma(\nu)}{(p^2+\alpha^2)^{\nu/2}}\cos\left(\nu\arctan\dfrac{\alpha}{p}\right)$
$(\sin\alpha t)/t$		$\arctan(\alpha/p)$
$\dfrac{\sin\alpha t}{\sqrt{t}}$		$\sqrt{\dfrac{\pi}{2}}\,\dfrac{\sqrt{\sqrt{p^2+\alpha^2}-p}}{p^2+\alpha^2}$
$\dfrac{\cos\alpha t}{\sqrt{t}}$		$\sqrt{\dfrac{\pi}{2}}\,\dfrac{\sqrt{\sqrt{p^2+\alpha^2}+p}}{p^2+\alpha^2}$
$\dfrac{1-\cos\alpha t}{t}$		$\dfrac{1}{2}\log\left(1+\dfrac{\alpha^2}{p^2}\right)$
$e^{\beta t}\sin(\alpha t+\theta)$		$\dfrac{(p-\beta)\sin\theta+\alpha\cos\theta}{(p-\beta)^2+\alpha^2}$
$e^{\beta t}\cos(\alpha t+\theta)$		$\dfrac{(p-\beta)\cos\theta-\alpha\sin\theta}{(p-\beta)^2+\alpha^2}$

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Appendix a33 Mellin Transformation

a33.1 Mellin transformation. The *Mellin transform* $\check{f}$ of $f(r)$ is defined as

$$\check{f}(p) = \int_0^\infty f(r) r^{p-1} dr. \tag{33.47}$$

This is well-defined for p satisfying $\sigma_1 < Re\, p < \sigma_2$, where

$$\int_0^1 r^{\sigma_1 - 1}|f(r)|dr < +\infty, \quad \int_1^\infty r^{\sigma_2 - 1}|f(r)|dr < +\infty. \tag{33.48}$$

a33.2 Theorem [Fundamental theorem of Mellin transformation].
(1)

$$\check{f}(p) = \int_0^\infty f(r) r^{p-1} dr \tag{33.49}$$

is analytic in the strip $\sigma_1 < Re\, p < \sigma_2$.
(2) Inverse transformation:

$$f(r) = \frac{1}{2\pi i} \int_\Gamma \check{f}(p) r^{-p} dp, \tag{33.50}$$

where Γ is a straight line in the above strip.□
[Demo] (1) is shown just as the counterpart for the Laplace transformation (→). (2) is also a disguised version of the inversion formula for the Laplace transformation (→**33.2**). Introduce t as $r = e^{-t}$. Then (33.47) reads

$$\check{f}(p) = \int_0^\infty e^{-pt} f(e^{-t}) dt \tag{33.51}$$

This is the Laplace transformation (→**33.3**). Therefore, we can apply the inverse transformation formula to obtain

$$f(e^{-t}) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \check{f}(p) e^{pt} dp. \tag{33.52}$$

In terms of r, this is just what we wanted.

a33.3 Applications to PDE. If the region of the problem is fan-shaped, then the Mellin transformation is particularly useful. 2-Laplace problem in the cylindrical coordinates is

$$r^2 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \right) u + \frac{\partial^2}{\partial \varphi^2} u = 0. \tag{33.53}$$

Melling transforming this, we get

$$p^2 \check{u} + \frac{d^2}{d\varphi^2} \check{u} = 0, \tag{33.54}$$

which can be solved easily. The rest is to compute the inverse transform. To calculate it as the Laplace transform (33.52) may be advantageous, since there is the so-called fast Laplace transform algorithm (→**33.17**).