32 Fourier Transformation

Basics of Fourier transform including the principle of FFT, major qualitative features like the uncertainty principle, sampling theorem, Wiener-Khinchine theorem are discussed in the first two subsections. Then, Fourier analysis of generalized functions and related topics such as Poisson's sum formula, the Plemelj formula are treated in the third subsection. As a related topic, Radon transform is discussed in the last subsection, which underlies many tomographic techniques.

32.A Basics

Fourier analysis is reviewed. The relation between smoothness of the function and the decay rate of its Fourier transform is important. As theoretical applications, uncertainty principle, sampling theorem and the Wiener-Khinchin theorem about spectral analysis are discussed.

Key words: Fourier transform, deconvolution, inverse Fourier transform, sine (cosine) transform, bra-ket notation, Plancherel's theorem, Riemann-Lebesgue lemma

Summary:
(1) Fix your convention of Fourier transform (**32A.1**, **32A.7**). Deconvolution is often the place where Fourier transformation is effective (**32A.2**). Linear differential operators become multiplicative operators (**32A.3**).
(2) The decay rate of the Fourier transform and the smoothness of its original function are closely related just as in the Fourier expansion cases (**32A.11**).

32A.1 Fourier transform. Let f be an integrable function (→**19.8**) on $\boldsymbol{R}$. If the following integral exists

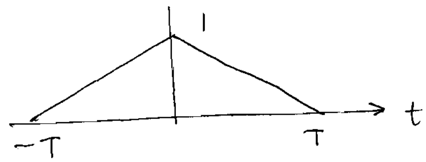
$$\hat{f}(k) = \mathcal{F}(f)(k) \equiv \int_{-\infty}^{\infty} dx f(x) e^{-ikx}, \tag{32.1}$$

it is called the *Fourier transform* of f. Multidimensional cases can be treated similarly.

Exercise.
(A) Consider the Fourier transform of a wave train of finite duration. Or, more concretely, compute the Fourier transform of

$$f(t) = [\Theta(t+T) - \Theta(t-T)]\cos at, \tag{32.2}$$

Sketch the Fourier transform.
(B)
(1) Demonstrate the Fourier transform of the following triangular function



is given by

$$X(\omega) = \frac{4\sin^2(\omega T/2)}{T\omega^2}. \tag{32.3}$$

(2) Demonstrate

$$\int_{-\infty}^{\infty} \frac{\sin^2 ax}{\pi a x^2} dx = 1. \tag{32.4}$$

for any $a \neq 0$ with the aid of (1).

32A.2 Deconvolution. As can be demonstrated with the aid of Fubini's theorem (→**19.14**).

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g), \tag{32.5}$$

This is a very useful relation.

Exercise.
In the following a and b are positive real numbers.
(i) Fourier transform

$$\chi(x) = \Theta(b - |x|). \tag{32.6}$$

(ii) Fourier transform $e^{-a|x|}$.
(iii) Fourier transform

$$f(x) = e^{-a|x|}\frac{\sin bx}{x}. \tag{32.7}$$

32A.3 Differentiation becomes multiplication. We have an important relation

$$\hat{f'} = +ik\hat{f}. \tag{32.8}$$

The sign in front of the formula depends on our choice of the definition **32A.1**. We have the following formulas (→**2C.7**, **2C.9**, **2C.11**):

$$\mathcal{F}(div\, \boldsymbol{v}) = +i\boldsymbol{k}\cdot \boldsymbol{v}_{\boldsymbol{k}} \tag{32.9}$$

$$\mathcal{F}(curl\, \boldsymbol{v}) = +i\boldsymbol{k}\times \boldsymbol{v}_{\boldsymbol{k}} \tag{32.10}$$

$$\mathcal{F}(-\Delta f) = k^2 f_{\boldsymbol{k}}. \tag{32.11}$$

The last formula explains why $-\Delta$ is a natural combination – it is a positive definite operator.

32A.4 Theorem. If $f : \boldsymbol{R} \to \boldsymbol{C}$ is continuous (and bounded), and both f and $\hat{f}$ are absolutely integrable, then the inversion formula holds

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{+ikx}dk \equiv \mathcal{F}^{-1}(\hat{f}). \tag{32.12}$$

□

The formula could be guessed from the Fourier expansion formula **17.1**; actually Fourier reached this result in this way. (32.12) appears so often that we have fairly a standard abbreviation

$$\int_k \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty}, \quad \int_{\boldsymbol{k}} \equiv \left(\frac{1}{2\pi}\right)^d \int d\boldsymbol{k}. \tag{32.13}$$

32A.5 Theorem [Inversion formula for piecewise C^1-function]. Let f be piecewise C^1-function on $\boldsymbol{R}$. Then (cf. **17.7**)

$$\frac{1}{2}[f(x_0-0)+f(x_0+0)] = \frac{1}{2\pi}P\int_{-\infty}^{\infty} dk e^{ikx_0}\hat{f}(k). \tag{32.14}$$

P denotes the Cauchy principal value (→**8B.10**, **14.17**). □

We can write the formula as

$$\frac{1}{2}[f(x_0-0)+f(x_0+0)] = \lim_{\lambda\to\infty}\int_{-\infty}^{\infty} d\xi \frac{\sin[\lambda(x_0-\xi)]}{x_0-\xi}\hat{f}(\xi). \tag{32.15}$$

□

32A.6 More general convergence conditions. As can easily be imagined from **17.8** for a pointwise convergence of the Fourier transform, we need some conditions. For example, if f is of bounded variation[402]

[402] If a function can be written as a difference of two monotonically increasing functions, we say the function is of *bounded variation*.

near x, then (32.12) holds with $f(x)$ being replaced by $[f(x+0)+f(x-0)]/2$. If f is continuous and of bounded variation in (a,b), then (32.12) holds uniformly there.

32A.7 Remark

(1) Mathematicians often multiply $1/\sqrt{2\pi}$ to the definition of Fourier transform as

$$\tilde{f} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx f(x) e^{-ikx}, \tag{32.16}$$

to symmetrize the formulas (as we will see in **32A.9** or **32B.1** sometimes this is very convenient), because

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk. \tag{32.17}$$

However, this makes the convolution formula (32.5) awkward. For physicists and practitioners, the definition in **32A.1** (the sign choice may be different) is the most convenient, because we wish to compute actual numbers.

(2) The integral over k may be interpreted as a sum over n such that $k = 2\pi n/L$, where L is the size of the space. The following approximation is very useful in solid-state physics

$$\frac{1}{V} \sum_{\boldsymbol{k}} f_{\boldsymbol{k}} \simeq \frac{1}{2\pi^d} \int f_{\boldsymbol{k}} d\boldsymbol{k} \equiv \int_{\boldsymbol{k}} f_{\boldsymbol{k}}. \tag{32.18}$$

32A.8 Sine and cosine transforms. If the space is limited to $x \geq 0$, then *Fourier sine* and *Fourier cosine transformations* may be useful (cf. **17.16**). If $f(0) = f(0+)$, then

$$g(k) = \int_0^{\infty} f(x) \cos kx dx, \quad f(x) = \frac{2}{\pi} \int_0^{\infty} g(k) \cos kx dk. \tag{32.19}$$

If $f(0) = 0$, then

$$g(k) = \int_0^{\infty} f(x) \sin kx dx, \quad f(x) = \frac{2}{\pi} \int_0^{\infty} g(k) \sin kx dk. \tag{32.20}$$

These can also be written concisely as

$$\frac{2}{\pi} \int_0^{\infty} \cos kx \cos k'x dx = \delta(k-k'), \tag{32.21}$$

$$\frac{2}{\pi} \int_0^{\infty} \sin kx \sin k'x dx = \delta(k-k'). \tag{32.22}$$

They can be shown easily with the aid of the Fourier transform of 1 (→**32C.8**); Put $\cos kx = (e^{ikx} + e^{-ikx})/2$, etc. into (32.21) or (32.22).

Exercise.
There is an infinite medium whose thermal diffusivity is D. Its initial temperature distribution is given by $T|_{t=0} = T_0(\boldsymbol{x})$. Find the physically meaningful solution (→**1.18**(5) Warning). There are many ways to solve this. For example, we can use the free space Green's function (→**16B.1** and the initial condition trick **16B.5**. We can also use the Fourier transformation as follows.
(1) Show that for any[403] function g on $\boldsymbol{R}^3$

$$g(x,y,z) = \frac{1}{\pi^3}\int_0^\infty\int_0^\infty\int_0^\infty d\alpha d\beta d\gamma \int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty dadbdc$$
$$g(a,b,c)\cos\alpha(x-a)\cos\beta(y-\beta)\cos\gamma(z-c). \qquad (32.23)$$

(2) The integrands are linearly independent (no mode coupling, or super position principle), so that each term must satisfy the diffusion equation. Introducing $A(t)\cos\alpha(x-a)\cos\beta(y-\beta 0\cos\gamma(z-c)$ into the diffusion equation, show that

$$A(t) = f(a,b,c)e^{-D(\alpha^2+\beta^2+\gamma^2)t}. \qquad (32.24)$$

(3) Combining (1) and (2), obtain the following formula, which can be obtained directly with the use of the free space Greeen's function.

$$T(x,y,x,t) = \pi^{-3/2}\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty d\eta d\xi d\zeta e^{-(\eta^2+\xi^2+\zeta^2)} f(x+2\sqrt{DT}\eta, y+2\sqrt{DT}\xi, z+2\sqrt{DT}\zeta). \qquad (32.25)$$

[Perform the integration over Greek letters.]

32A.9 Bra-ket notation of Fourier transform or momentum (wave-vector) kets. **32A.7** has the following symbolic representation (→**20.21-23** for notations).

$$f(x) = \langle x|f\rangle = \int_{-\infty}^\infty \langle x|k\rangle dk\langle k|f\rangle, \qquad (32.26)$$

$$\langle x|k\rangle = \frac{1}{\sqrt{2\pi}}e^{-ikx}, \qquad (32.27)$$

$$\tilde{f}(k) = \langle k|f\rangle = \int_{-\infty}^\infty \langle x|k\rangle dk\langle k|f\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{ikx}f(x).(32.28)$$

$\langle k|f\rangle$ is the Fourier transform of f in this bra-ket symmetrized version (**32A.7**), and the normalization is <u>different</u> from that given in **32A.1**. Notice that

$$\langle x|y\rangle = \delta(x-y) = \int\langle x|k\rangle dk\langle k|y\rangle = \frac{1}{2\pi}\int_\infty^\infty e^{ik(x-y)}dk. \qquad (32.29)$$

[403] If you wish to be within the ordinary calculus, it must be integrable, but we may proceed formally.

To rationalize this, we need the theory of Fourier transform of generalized functions (→**32C.8**).

32A.10 Plancherel's theorem.

$$\langle f|f\rangle = \int \langle f|k\rangle dk \langle k|f\rangle \tag{32.30}$$

is called *Plancherel's formula.* In our normalization (for physicists) in **32A.1** this reads

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk. \tag{32.31}$$

The theorem tells us that if f is square integrable (that is, the total energy of the wave is finite), then the total energy is equal to the energy carried by individual harmonic modes. This is of course the counterpart of Parseval's equality (→**20.12**).

32A.11 Theorem [Riemann-Lebesgue Lemma]. For an integrable function f

$$\lim_{|\boldsymbol{k}|\to\infty} \hat{f}(\boldsymbol{k}) = 0. \tag{32.32}$$

If all the n-th derivatives are integrable, then $\hat{f}(\boldsymbol{k}) = o[|\boldsymbol{k}|^{-n}]$.□
There is an analogue of **17.11**. There we have already discussed its physical meaning.[404]

32.B Applications of Fourier Transform

Fundamental applications of Fourier transformation important in practice are summarized: uncertainty principle, sampling theorem, the Wiener-Khinchine theorem (the relation between power spectrum and correlation function). Also the principle of FFT is outlined.

Key words: uncertainty principle, coherent state, band-limited function, sampling theorem, sampling function, aliasing, time-correlation function, power spectrum, Wiener-Khinchine theorem, fast Fourier transform

[404] see Katznelson p123.

Summary:
(1) The uncertainty principle is a basic property of Fourier transformation. Its essence is the elementary Cauchy-Schwarz inequality (**32B.1**).
(2) If the band width of a signal (function) is finite, then discrete sampling with sufficiently frequent sampling points perfectly captures the signal. This is the essence of the sampling theorem (**32B.5**).
(3) Spectral analysis is a fundamental tool of experimental physics. Its theoretical basis is the Wiener-Khinchine theorem – Fourier transform of the time-correlation function is the power spectrum (**32B.10**).
(4) Spectral analysis becomes practical after the popularization of fast Fourier transform (FFT) (**32B.11-13**).

32B.1 Theorem [Uncertainty principle]. Let f be in $L_2(\boldsymbol{R})$ (→**20.19**). Define the following averages

$$\langle x \rangle \equiv \int x|f(x)|^2 dx \Big/ \int |f(x)|^2 dx, \tag{32.33}$$

$$\langle k \rangle \equiv \int k|\hat{f}(k)|^2 dk \Big/ \int |\hat{f}(k)|^2 dk, \tag{32.34}$$

$$\Delta x^2 \equiv \int (x - \langle x \rangle)^2 |f(x)|^2 dx \Big/ \int |f(x)|^2 dx, \tag{32.35}$$

$$\Delta k^2 \equiv \int (k - \langle k \rangle)^2 |\hat{f}(k)|^2 dk \Big/ \int |\hat{f}(k)|^2 dk. \tag{32.36}$$

Then,

$$\Delta x \Delta k \geq 1/2. \tag{32.37}$$

[Demo] Without loss of generality, we may assume $\langle x \rangle = 0$, and also assume that f is already normalized. Define

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int dx e^{ikx} f(x). \tag{32.38}$$

Using Plancherel's theorem (→**32A.10**), we get (cf. **32A.3**)

$$\int dx |f'(x)|^2 = \int |k\tilde{f}(k)|^2 dk, \quad \int dx |f(x)|^2 = \int |\tilde{f}(k)|^2 dk, \tag{32.39}$$

so that

$$\Delta k^2 = \int |f'(x) - \langle k \rangle f(x)|^2 dx. \tag{32.40}$$

The Cauchy-Schwarz inequality (→**20.7**) implies

$$\Delta k^2 \Delta x^2 = \int |f'(x) - \langle k \rangle f(x)|^2 dx \int x^2 |f(x)|^2 dx \geq \left| \int [f'(x) - \langle k \rangle f(x)] x \overline{f(x)} dx \right|^2, \tag{32.41}$$

but since $\langle x \rangle = 0$, the last formula reads

$$|f'(x)x\overline{f(x)}dx|^2 \geq |Re \int f'(x)x\overline{f(x)}dx|^2 = 1/4. \qquad (32.42)$$

The last number comes from the following integration by parts

$$\int f'(x)x\overline{f(x)}dx = -\int \overline{f'(x)}xf(x)dx - \int |f(x)|^2 dx. \qquad (32.43)$$

32B.2 Remark. As can be seen from the proof of **32B.1**, the uncertainty principle is a disguised Cauchy-Schwarz inequality (→**20.7**) which says that the modulus of cosine cannot be larger than 1. Note that obvious mathematical theorems can have profound implication in real life.

32B.3 Coherent state. The equality in the uncertainty principle is realized if the wave function f is Gaussian

$$f(x) = \frac{1}{\pi^{1/4}\sigma^{1/2}} e^{-x^2/2\sigma^2}. \qquad (32.44)$$

Check indeed $\Delta x \Delta k = 1/2$. A state with this equality is called a *coherent state.*

32B.4 Band-limited function. If a function has a Fourier transform which has a compact support (i.e., $\hat{f}(k)$ is zero if $|k| > k_0$ for some $k_0 > 0$), then f is called a *band-limited function.*

32B.5 Theorem [Sampling theorem]. Let f be a band-limited function such that $\hat{f}(k)$ be zero if $|k| > k_0 > 0$. Then,

$$f(x) = \sum_{n=-\infty}^{\infty} f(n\pi/k_0) \frac{\sin(k_0 x - n\pi)}{k_0 x - n\pi}. \qquad (32.45)$$

That is, f can be reconstructed from the discrete sample values $\{f(n\pi/k_0)\}_{n\in \boldsymbol{Z}}$.□
The sampling theorem is extremely important in communication (multichannel communication, bandwidth compression, etc.), and information storage (digitization as in CD).

[Demo] Since $\hat{f}(k)$ is non-zero only on $[-k_0, k_0]$, we can Fourier expand this as a function of period $2k_0$ (→**17.2**)

$$\hat{f}(k) = \sum_{n\in \boldsymbol{Z}} c_n e^{ikn\pi/k_0} \qquad (32.46)$$

with

$$\frac{1}{2k_0}\int_{-k_0}^{k_0} \hat{f}(k) e^{-in\pi k/k_0} dk = c_n. \qquad (32.47)$$

On the other hand due to the band-limitedness

$$f(x) = \frac{1}{2\pi}\int_{-k_0}^{k_0} \hat{f}(k)e^{-ikx}dk. \tag{32.48}$$

Comparing (32.47) and (32.48), we get

$$c_n = \frac{\pi}{k_0} f(n\pi/k_0). \tag{32.49}$$

(32.46), (32.48) and (32.49) give the desired result.

Exercise.
Determine the minimum sampling rate (or frequency) for the signal $10\cos\omega t + 2\cos 3\omega t$. This is a trivial question, so do not think too much.

32B.6 Sampling function. The function

$$\varphi_n(x) = \frac{\sin(k_0 x - n\pi)}{k_0 x - n\pi} \tag{32.50}$$

appearing in (32.45) is called the *sampling function.* $\{\varphi_n\}_{n\in \boldsymbol{Z}}$ is an orthogonal system. There is an orthogonality relation:

$$\int_{-\infty}^{\infty} \varphi_n(x)\varphi_m(x)dx = \frac{\pi}{k_0}\delta_{nm}. \tag{32.51}$$

Exercise.
Demonstrate that the sampling functions $\{\varphi_n\}$ make an orthogonal system. That is, show (32.51).

32B.7 Band-limited periodic function. The sampling theorem would naturally tell us the following. A band-limited periodic function with no harmonics of order higher than N can be uniquely specified by its values sampled at appropriate $2N + 1$ points in a single period.

32B.8 Aliasing. If the function we sample is strictly band-limited, then the above theorem of course works perfectly. However, often the function has higher frequency components beyond the sample frequency. Then, just as we watch fast rotating wheel in the movie, what we sample is the actual frequency modulo the sample frequency (that is, the beat between these frequencies). This phenomenon is called *aliasing.* To avoid unwanted aliasing, often we filter the original signal (through a low-pass filter) and remove excessively high frequency components.

32B.9 Time-correlation function. Let $x(t)$ be a stochastic process or time-dependent data which is statistically stationary. Here 'stochastic' means that we have an ensemble of such signals (more precisely, we have a set of signals $\{x(t;\omega)\}$, where ω is the probability parameter specifying each sample signal. That is, if the reader wishes to start an observation, one ω is given (by God) and she will observe $x(t;\omega)$. The word 'stationary' implies that the ensemble average of $x(t,\omega)$ does not depend on t.[405] Let us denote the ensemble average by $\langle\ \rangle_\omega$. The *time correlation function* is defined by

$$C(t) = \langle x(t)x(0)\rangle_\omega \tag{32.52}$$

and is a fundamental observable in many practical cases.
The ensemble average of

$$\sigma(\nu) = \langle |x_\nu|^2\rangle_\omega \tag{32.53}$$

is called the *power spectrum* of the signal $x(t)$, where x_ν is the Fourier transform of $x(t)$. Thanks to the advent of FFT (→**32B.12**), it is easy to obtain the power spectrum experimentally (easier than the correlation function).

32B.10 Theorem [Wiener-Khinchin]. The Fourier transform of the power spectrum of a stationary stochastic process is its power spectrum. That is,[406]

$$C(t) \propto \int_{-\infty}^{\infty} e^{-i\nu t}\sigma(\nu)d\nu. \tag{32.54}$$

Its demonstration is a straightforward calculation. We compute (→**32C.8**)

$$\begin{aligned}\langle x_\nu x_{-\mu}\rangle &= \left\langle \int_{-\infty}^{\infty} dt x(t)e^{i\nu t}\int_{-\infty}^{\infty} ds x(s)e^{-i\mu s}\right\rangle \\ &= \int_{-\infty}^{\infty} dt\int_{-\infty}^{\infty} ds e^{i\nu t}e^{-i\mu s}\langle x(t-s)x(0)\rangle \\ &= 2\pi\delta(\nu-\mu)\int_{-\infty}^{\infty} dt e^{i\nu t}C(t). \end{aligned}\tag{32.55}$$

That is, $\langle x_\nu x_{-\mu}\rangle = \delta(\nu-\mu)\sigma(\nu)$ so that

$$\sigma(\nu) = 2\pi\int_{-\infty}^{\infty} dt e^{i\nu t}C(t). \tag{32.56}$$

[405] Actually, in this case we only need the absolute time independence of the correlation function. A process with this property is called a *weak stationary process.*

[406] Actually, if we normalize $C(t)$ so that $C(0) = 1$ (simply regard $C(t)/C(0)$ as $C(t)$), then we have probability measure σ (→**a19.19**) such that

$$C(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\nu t}d\sigma(\nu).$$

However, in practice, the numerical constant and normalization are not crucial.

32B.11 Discrete Fourier transformation. Let $\boldsymbol{X} \equiv \{X_n\}_{n=0}^{N-1}$ be a sequence of complex numbers, and

$$e(x) \equiv \exp(-2\pi i x). \tag{32.57}$$

The following sequence $\hat{\boldsymbol{X}} \equiv \{X^n\}$ is called the *discrete Fourier transform* of $\boldsymbol{X}$:

$$X^k = \sum_{n=0}^{N-1} e\left(\frac{kn}{N}\right) X_n. \tag{32.58}$$

Its inverse transform is given by

$$X_n = \frac{1}{N} \sum_{k=0}^{N-1} e\left(\frac{-kn}{N}\right) X^k. \tag{32.59}$$

Notice that a straightforward calculation of these sums (N of them) costs $O[N^2]$ operations and is costly.

Exercise.
Demonstrate the above inverse transform formula by showing

$$\frac{1}{N} \sum_{k=0}^{N-1} e^{ik(m-n)/N} = \delta_{mn}. \tag{32.60}$$

32B.12 Principle of fast Fourier transform.[407] Let $N = N_1 N_2$. $n, k \in \{0, 1, \cdots, N-1\}$ can be uniquely written as[408]

$$n = n_1 + n_2 N_1, \quad k = k_1 N_2 + k_2, \tag{32.61}$$

where $n_i, k_i \in \{0, 1, \cdots, N_i - 1\}$ ($i = 1$ or 2). Notice that

$$e(kn/N) = e(k_1 n_1/N_1) e(k_2 n_2/N_2) e(k_2 n_1/N). \tag{32.62}$$

n_i and k_i are uniquely determined, so we may write, e.g., $(n_1 n_2)$ instead of n. Then, (32.58) can be calculated as

$$\begin{aligned} X^{(k_1 k_2)} &= \sum_{n=0}^{N_1 N_2 - 1} e(k_1 n_1/N_1) e(k_2 n_2/N_2) e(k_2 n_1/N) X_{(n_1 n_2)}, \\ &= \sum_{n_1=0}^{N_1-1} e(k_1 n_1/N_1) \left\{ e(k_2 n_1/N) \left[\sum_{n_2=0}^{N_2-1} e(k_2 n_2/N_2) X_{n_1 n_2} \right] \right\}. \end{aligned} \tag{32.63}$$

[407] The algorithm, known sometimes as the Cooley-Tukey algorithm (J W Cooley and J W Tukey, Math. Comp. **19**, 297 (1965)), was actually known to Gauss, but the importance was widely recognized after this paper.

[408] This is an example of the so-called *Chinese remainder theorem.*

Consequently, the calculation of discrete Fourier transfrom can be decomposed into the following three steps:
(1) Compute for any k_2

$$X_{n_1}{}^{k_2} = \sum_{n_2=0}^{N_2-1} e(k_2 n_2/N_2) X_{n_1 n_2}. \tag{32.64}$$

(2) Then, rotate the phase:

$$\tilde{X}_{n_1}{}^{k_2} = e(k_2 n_1/N) X_{n_1}{}^{k_2}. \tag{32.65}$$

(3) Finally compute for any k_1

$$\hat{X}^{k_1 k_2} = \sum_{n_1=0}^{N_1-1} e(k_1 n_1/N_1) \tilde{X}_{n_1}{}^{k_2}. \tag{32.66}$$

Now the number of necessary operations is $O[N_1 \times N_2^2] + O[N_1^2 \times N_2]$; if $N_1 = N_2 = \sqrt{N}$, then $O[2N^{3/2}]$. If we can decompose N into m factors of similar order, then the number of necessary operations is roughly $N^{1-1/m}N^{2/m} = N \times N^{1/m}$. Hence, asymptotically, we can guess $N \ln N$ is the best possibility for the discrete Fourier transform of N numbers.

Exercise.
Find the autocorrelation function of the signal

$$f(t) = \Theta(t+T) - \Theta(t-T). \tag{32.67}$$

Then illustrate the Wiener-Khinchine theorem with the example.

32.C Fourier Analysis of Generalized Function

Generalized functions can be Fourier transformed and physicists' favorite formulas like $\int e^{ikx} dk = 2\pi\delta(x)$ or the Plemelj formula $1/(x+i0) = P(1/x) - i\pi\delta(x)$ can be demonstrated. Fourier expansion of δ-function gives us the Poisson sum formula which may be used to accelerate the convergence of series.

Key words: Fourier expansion of unity, Poisson sum formula, Euler-MacLaurin sum formula, Plemelj formula

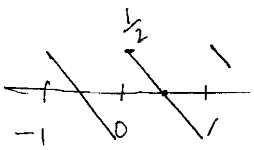
Summary:
(1) Not convergent Fourier series may be interpreted as a generalized

function. A typical example is Poisson's sum formula (**32C.2**).
(2) Formal calculation of Fourier transform of generalized functions often works, but whenever there is some doubt, return to the definition (**32C.6**, **32C.8**).

32C.1 Delta function.

$$\delta(x) = \sum_{n=-\infty}^{\infty} e^{i2n\pi x} \tag{32.68}$$

for $x \in (-1, 1)$.
[Demo] We know as an ordinary Fourier series



$$\frac{1-2x}{2} = \sum_{n=1}^{\infty} \sin(2n\pi x)/n\pi \tag{32.69}$$

for $x \in (0 , 1)$. We may use the RHS to extend the LHS periodically for all $\boldsymbol{R}$. Differentiate this termwisely, interpreting this as a formula for generalized functions (→**14.14**). We get

$$-1 + \delta(x) = 2\sum_{n=1}^{\infty} \cos 2n\pi x \tag{32.70}$$

for $x \in (-1/2, 1/2)$.
The decomposition of unity (→**20.27**) can also be used to obtain (32.68).

32C.2 Poisson's sum formula.

$$\sum_{k=-\infty}^{\infty} \delta(x-k) = \sum_{n=-\infty}^{\infty} e^{i2n\pi x} \tag{32.71}$$

for $x \in \boldsymbol{R}$.

This can be obtained easily from (32.68) by 'tessellating' the formula for $(-1/2, 1/2)$ over the whole range of $\boldsymbol{R}$. From (32.71) we get

$$|\lambda| \sum_{k=-\infty}^{\infty} \delta(x - \lambda k) = \sum_{n=-\infty}^{\infty} e^{i2\pi n x/\lambda} \tag{32.72}$$

(cf. **14.11**). Applying a test function φ to this, we get the following *Poisson sum formula*:

$$|\lambda| \sum_{k=-\infty}^{\infty} \varphi(\lambda k) = \sum_{n=-\infty}^{\infty} \hat{\varphi}(2n\pi/\lambda). \tag{32.73}$$

(Be careful with the normalization constant.) Also we can make a cosine version of the Poisson sum formula

$$\sum_{k=-\infty}^{\infty} \delta(x-k) = 1 + 2\sum_{n=1}^{\infty} \cos(2n\pi x). \qquad (32.74)$$

If $f(x)$ is a gently decaying function, then its Fourier transform decays rapidly, and vice versa. The Poisson sum formula is useful because it may help accelerating the convergence of the series.

Exercise.
Demonstrate

$$\sum_{n=1}^{\infty} \frac{\cos na}{1+n^2} = \frac{\pi}{2}\frac{\cosh(\pi - a)}{\sinh \pi} - \frac{1}{2}.$$

32C.3 Applications of Poisson sum formula.

(1)

$$\sum_{n\in \boldsymbol{Z}} \frac{1}{1+a^2n^2} = \frac{\pi}{a}\coth\frac{\pi}{a}. \qquad (32.75)$$

The key formulas are

$$\hat{\varphi}(k) = \frac{1}{1+a^2k^2/4\pi^2}, \quad \varphi(x) = \frac{\pi}{a}e^{-2\pi|x|/a}. \qquad (32.76)$$

(2)

$$\sum_{n=1}^{\infty} \frac{\cos na}{1+n^2} = \frac{\pi}{2}\frac{\cosh(\pi - a)}{\sinh \pi} - \frac{1}{2}. \qquad (32.77)$$

32C.4 Euler-MacLaurin sum formula.

$$\sum_{n=0}^{\infty} f(n) = \int_0^{\infty} f(x)dx + \frac{1}{2}f(0) - \frac{1}{12}f'(0) + \frac{1}{720}f^{(3)}(0) - \frac{1}{30240}f^{(5)}(0) + \cdots. \qquad (32.78)$$

[Demo] Let f be a function defined on the positive real axis. Extend it to the whole $\boldsymbol{R}$ as an even function ($f(x) = f(-x)$). Apply the cosine version of the Poisson sum formula (32.74) and integrate from 0 to ∞. Using the evenness of the function, we get

$$-\frac{1}{2}f(0) + \sum_{k=0}^{\infty} f(k) = \int_0^{\infty} f(x)dx + 2\sum_{n=1}^{\infty}\int_0^{\infty} f(x)\cos(2n\pi x)dx. \qquad (32.79)$$

Integrating by parts the last integrals containing cosines, we get

$$\sum_{k=0}^{\infty} f(k) = \frac{1}{2}f(0) + \int_0^{\infty} f(x)dx - \sum_{n=1}^{\infty}\int_0^{\infty} f'(x)\frac{\sin 2n\pi x}{2n\pi}dx. \qquad (32.80)$$

Keep applying integration by parts to get

$$\sum_{n=1}^{\infty}\int_0^{\infty} f'(x)\frac{\sin 2n\pi x}{2n\pi}dx = -\sum_{n=1}^{\infty}\left[f'(x)\frac{\cos 2n\pi x}{2(n\pi)^2}\right]_0^{\infty} + \sum_{n=1}^{\infty}\int_0^{\infty} f''(x)\frac{\cos 2n\pi x}{2(n\pi)^2}dx. \tag{32.81}$$

Thus

$$\sum_{k=0}^{\infty} f(k) = \frac{1}{2}f(0) + \int_0^{\infty} f(x)dx - f'(0)\sum_{n=1}^{\infty}\frac{1}{2n^2\pi^2} + \cdots. \tag{32.82}$$

This gives the $f'(0)$ term of the formula.

32C.5 Mulholland's formula for the canonical partition function for the rotational motion of a heteronuclear diatomic molecule. The rotational partition function $r(T)$ at temperature T is given by

$$r(T) = \sum_{\ell=0}^{\infty}(2\ell+1)\exp\left[-\frac{\hbar^2\ell(\ell+1)}{2Ik_BT}\right], \tag{32.83}$$

where I is the moment of inertia of the molecule, and k_B is the Boltzmann constant. Introduce $\sigma \equiv \hbar^2/2Ik_BT$, and let

$$f(x) = (2x+1)\exp[-x(x+1)\sigma]. \tag{32.84}$$

Apply (32.78) to this function, we get the following Mulholland's formula

$$r(T) = \frac{1}{\sigma} + \frac{1}{3} + \frac{\sigma}{15} + \frac{4\sigma^2}{315} + O[\sigma^3]. \tag{32.85}$$

The first term on the RHS is the classical value.

32C.6 Fourier transform of generalized functions. The crucial observation is (for ^ see **32A.1**): if f and φ both have well-defined Fourier transforms,

$$\langle \hat{f}, \varphi\rangle = \int dk\left[\int dx f(x)e^{-ikx}\right]\varphi(k) = \langle f, \hat{\varphi}\rangle \tag{32.86}$$

The Fourier transform $\hat{\tau} \equiv \mathcal{F}[\tau]$ of a generalized function τ is defined by

$$(\hat{\tau}, \varphi) = (\tau, \hat{\varphi}), \text{ or } (\mathcal{F}[\tau], \varphi) = (\tau, \mathcal{F}[\varphi]), \tag{32.87}$$

where $\varphi \in \mathcal{D}$, a test function.

Exercise.
Demonstrate

$$\lim_{\lambda\to\infty}\frac{\sin\lambda x}{x} = \pi\delta(x). \tag{32.88}$$

$$\lim_{\lambda\to\infty}\int_a^b \sin\lambda x = 0. \tag{32.89}$$

32C.7 Convenient test function space. For this definition it is desirable that the set of test functions $\mathcal{D}$ (→**14.8**) and the set of their Fourier transforms $\hat{\mathcal{D}}$ are identical. For the set of Schwartz class functions (→**14.8** footnote) this holds (→**32A.11**). [If we choose $\mathcal{D}$ to be the set of all the functions with compact supports, then $\hat{\mathcal{D}}$ becomes very large, so that the class of generalized functions (for which $(\tau,\hat{\varphi})$ must be meaningful) must be severely restricted, and is not very convenient.]

32C.8 Fourier transform of unity = delta function.

$$\hat{1} = 2\pi\delta(k). \tag{32.90}$$

This is the *true* meaning of the physicists' favorite

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}dk = \delta(x). \tag{32.91}$$

Obviously, $\hat{\delta} = 1$ (direct calculation). That is, $\mathcal{F}^2$ implies multiplication of 2π as we know in **32A.10**.

[Demo] $(\hat{1},\varphi) = (1,\hat{\varphi}) = \int\hat{\varphi}(k)dk = \mathcal{F}^2[\varphi](0)$. Here $\mathcal{F}[\varphi]$ is a function on the configuration space (that is, a function of x) and is equal to $2\pi\varphi(x)$. Therefore we have obtained

$$(\hat{1},\varphi) = 2\pi\varphi(0) = \int 2\pi\delta(x)\varphi(x)dx = (2\pi\delta,\varphi). \tag{32.92}$$

Exercise.
Show

$$\delta(t) = \frac{1}{\pi}\int_0^\infty \cos\omega t d\omega. \tag{32.93}$$

Cf. **32A.8**.

32C.9 Translation. The following formulas should be obvious

$$\mathcal{F}[\delta(x-a)] = e^{-iak}, \quad \mathcal{F}[e^{iax}] = 2\pi\delta(a-k). \tag{32.94}$$

30C.10 Fourier transform of x, $d/dx \leftrightarrow +ik$. (→**32A.3**)

$$\hat{x} = +2\pi i\delta'(k). \tag{32.95}$$

In other words, since $\mathcal{F}^2 \equiv 2\pi$,

$$\hat{\delta}' = +ik. \tag{32.96}$$

[Demo] Start with the definition $(\hat{x}, \varphi) = (x, \hat{\varphi})$ (→**32C.6**) which is equal to

$$\int dx x\hat{\varphi}(x) = \int dx x \left[\int e^{-ikx}\varphi(k)dk\right] = \int dx \int dk \left(-\frac{d}{dik}e^{-ikx}\right)\varphi(k). \quad (32.97)$$

Integrating this by parts, taking into account that the test function φ decays sufficiently quickly, we get

$$-\int dx \int dk ie^{-ikx}\varphi'(k) = -i\int dk\hat{1}(k)\varphi'(k) = -2\pi i\int dk\delta(k)\varphi'(k) = 2\pi i\int dk\delta'(k)\varphi(k), \quad (32.98)$$

where we have used (32.90) in **32C.8**, and the definition of δ' (→**14.14**).

A more formal and direct 'demonstration' is

$$\hat{x} = \int xe^{-ikx}dx = \int \left(i\frac{d}{dk}\right)e^{-ikx}dx = 2\pi i\frac{d}{dk}\delta(k). \quad (32.99)$$

Convolution of the derivative of delta function is differentiation (→**14.23**(2)), and the Fourier transform of a convolution is the product of the Fourier transforms, i.e., $\mathcal{F}(f*g) = \mathcal{F}(f)\mathcal{F}(g)$ (→**32A.2**), so that we easily get)cf. **32A.3**)

$$\hat{f'} = +ik\hat{f}. \quad (32.100)$$

32C.11 Fourier transform of x^n.

$$\hat{x^n} = 2\pi\left(+i\frac{d}{dk}\right)^n\delta(k). \quad (32.101)$$

In other words,

$$\hat{\delta^{(n)}} = (+ik)^n. \quad (32.102)$$

Since $\delta' * f = f'$, $\delta^{(n)} = \delta' * \delta^{(n-1)} = \delta' * \delta' * \cdots \delta' * \delta$ (n δ' are convoluted) (this is well defined →**14.23**(2)). This and (32.96) immediately imply (32.102).

32C.12 Fourier transform of sign function.

$$\hat{sgn}(k) = \frac{2}{i}P\frac{1}{k}, \quad (32.103)$$

where P denotes the Cauchy principal value (→**14.17**).

[Demo] We have demonstrated (→**14.15**)

$$\frac{d}{dx}sgn(x) = 2\delta(x). \quad (32.104)$$

Fourier-transforming this, we get (→(32.100) and $\hat{\delta} = 1$)

$$+ik\mathcal{F}(sgn)(k) = 2. \quad (32.105)$$

With the aid of (2) in **14.17**, we can solve this equation for $\hat{sgn}$ as

$$\hat{sgn}(k) = 2iP\frac{1}{k} + c\delta(k), \tag{32.106}$$

where c is a constant not yet determined. To fix this constant we apply this equality to an even test function, say e^{-k^2}. Since sgn is an odd generalized function, and since the Fourier transform of a Gaussian function is again Gaussian,

$$(\hat{sgn}, e^{-k^2}) \propto (sgn, e^{-x^2}) = 0. \tag{32.107}$$

$P(1/k)$ is also an odd function, so that this implies $c = 0$.

32C.13 Plemelj formula.

$$w\text{-}\lim_{\epsilon\to+0} \frac{1}{x \pm \epsilon i} = P\frac{1}{x} \mp i\pi\delta(x), \tag{32.108}$$

where $w\text{-}\lim_{\epsilon\to+0}$ is the weak limit, that is, the limit is taken *after* integration in which the function appears is completed (→**8B.12**).
[Demo] Obviously,

$$\lim_{\epsilon\to0+} e^{-\epsilon x}\Theta(x) = \Theta(x), \tag{32.109}$$

If we interpret this equation as an equation for generalized functions, then integration and the limit can be freely exchanged. Therefore, we get

$$\hat{\Theta}(k) = w\text{-}\lim_{\epsilon\to0+} \int_0^\infty e^{-(ik-\epsilon)x} = \lim_{\epsilon\to0+} \frac{1}{ik+\epsilon}. \tag{32.110}$$

Since $sgn(x) = 2\Theta(x) - 1$, (32.103), (32.90) and (32.110) imply

$$2iP\frac{1}{k} = \lim_{\epsilon\to0+} \frac{-2}{ik-\epsilon} - 2\pi\delta(k). \tag{32.111}$$

32C.14 Initial value problem for wave equation. Thie is the second method to solve the wave equation in d-space (→**30.9**, **32D.9**). Consider

$$\Box u = \partial_t^2 u - \Delta u = 0 \tag{32.112}$$

with the intial condition $u(\boldsymbol{x}, 0) = f(\boldsymbol{x})$ and $\partial_t u(\boldsymbol{x}, 0) = g(\boldsymbol{x})$ on $\boldsymbol{R}^d$. Here we assume f and g are with compact supports (i.e., vanish far from the origin). Applying spatial Fourier transformation, we get

$$\partial_t^2 \hat{u}(\boldsymbol{k}, t) = -k^2 \hat{u}(\boldsymbol{k}, t), \tag{32.113}$$

so that we obtain

$$\hat{u}(\boldsymbol{k}, t) = \hat{f}\cos kt + \hat{g}\frac{\sin kt}{k}. \tag{32.114}$$

Therefore,

$$u(\boldsymbol{x},t) = \frac{1}{(2\pi)^d}\int d^d\boldsymbol{k}\left(\hat{f}\cos kt + \hat{g}\frac{\sin kt}{k}\right)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{32.115}$$

If we introduce the following Fourier transform (in the generalized function sense)

$$K(\boldsymbol{x},t) = \frac{1}{(2\pi)^d}\int d^d\boldsymbol{k}\frac{\sin kt}{k}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{32.116}$$

We obtain (cf (30.28))

$$u(\boldsymbol{x},t) = \frac{\partial}{\partial t}\int d^d\boldsymbol{y}K(\boldsymbol{x}-\boldsymbol{y},t)f(\boldsymbol{y}) + \int d^d\boldsymbol{y}K(\boldsymbol{x}-\boldsymbol{y},t)g(\boldsymbol{y}). \tag{32.117}$$

Discussion.
We can further transform the result with the aid of ($d \geq 2$)

$$\int d^d\boldsymbol{y}K(\boldsymbol{x}-\boldsymbol{y},t)f(\boldsymbol{y}) = \frac{1}{(d-2)!}\frac{\partial^{d-2}}{\partial t^{d-2}}\int_0^{|t|}dr(t^2-r^2)^{(d-3)/2}rM_f(\boldsymbol{x},r), \tag{32.118}$$

where M_f is the same as in **30.9** (the spherical average).

Exercise.
Demonstrate that the solution to a wave equation can be written as a superposition of plane waves. Or, demonstrate the following statement. If we introduce

$$\hat{h}_{\pm}(\boldsymbol{k}) \equiv \frac{1}{2}\left(\hat{f}(\boldsymbol{k}) \pm i\frac{\hat{g}(\boldsymbol{k})}{|\boldsymbol{k}|}\right), \tag{32.119}$$

Then, (32.115) can be written as

$$u(\boldsymbol{x},t) = \frac{1}{(2\pi)^d}\int d^d\boldsymbol{k}e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-kt)}\hat{h}_+(\boldsymbol{k}) + \frac{1}{(2\pi)^d}\int d^d\boldsymbol{k}e^{i(\boldsymbol{k}\cdot\boldsymbol{x}+kt)}\hat{h}_-(\boldsymbol{k}) \tag{32.120}$$

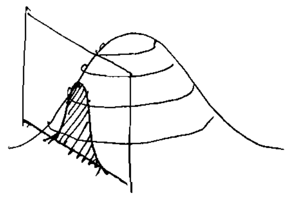
32.D Radon Transformation

Radon transformation is a theoretical basis of various tomographies. Its inverse transformation is constructed with the aid of Fourier transformation. Radon transformation allows us to solve the Cauchy problem of the wave equation in any dimensional space. The explicit formula clearly demonstrates the marked difference of even and odd dimensional spaces.

Key words: Radon's problem, Radon transform, modified Radon transform, tomography, wave equation, afterglow.

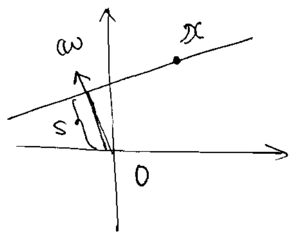
Summary:
(1) The mathematical principle of tomography is Radon transformation (**32D.3**) whose inverse transformation is essentially calculable by Fourier transformation (**32D.4-5**).
(2) Radon transform gives a general method to solve d-wave equation (**32D.9**). The resultant solution clearly exhibits the afterglow effect in even dimensional spaces (**32D.10**).

32D.1 Radon's problem. Radon (1917) considered the following problem: Reconstruct a function $f(x, y)$ on the plane from its integral along all lines in the plane. That is, the problem is to reconstruct the shape of a hill from the areas of all its vertical cross-sections.



32D.2 Radon transform. Let f be a function defined on a region in $\boldsymbol{R}^2$.[409]

$$\mathcal{R}f(s, \boldsymbol{\omega}) \equiv \int_{\boldsymbol{R}^2} d\boldsymbol{x}\delta(\boldsymbol{x} \cdot \boldsymbol{\omega} - s) f(\boldsymbol{x}) \qquad (32.121)$$



is called the *Radon transform* of f, where $\boldsymbol{\omega}$ is the directional vector $|\boldsymbol{\omega}| = 1$ specifying a line normal to it, and $s \in \boldsymbol{R}$ is the (signed) distance between the line and the origin. The Radon problem **32D.1** is to find f from $\mathcal{R}f$.

That (32.121) is the integral of f along the line specified by $\boldsymbol{\omega} \cdot \boldsymbol{x} = s$ can easily be seen if we introduce the rotated Cartesian coordinate system O-x_1x_2 such that the x_2 axis is parallel to the line and x_1 perpendicular to it. The integral now reads $\int \delta(x_1 - s) f(x_1, x_2) dx_1 dx_2 = \int f(s, x_2) dx_2$.

32D.3 Some properties of Radon transform. Note that
(1) $\mathcal{R}f(s, \boldsymbol{\omega})$ is an even homogeneous function (→**13B.1**) of s and $\boldsymbol{\omega}$ of degree -1:

$$\mathcal{R}f(\lambda s, \lambda \boldsymbol{\omega}) = |\lambda|^{-1} \mathcal{R}f(s, \boldsymbol{\omega}). \qquad (32.122)$$

[409] The definition given here can easily be extended to general d-space. See **32D.7-8**. A good introduction to the topic may be found in I. M. Gel'fand, M. I. Graev and N. Ya. Vilenkin, *Generalized Functions*, vol.5 *Integral Geometry and Representation Theory* (Academic Press, 1966). See also R. S. Strichartz, Am. Math. Month. 1982 June-July.

(2) The Radon transform of a convolution (→**14.22**) is a convolution of Radon transforms:

$$\left(\mathcal{R}\left[\int_{\boldsymbol{R}^2} f_1(\boldsymbol{y})f_2(\boldsymbol{x}-\boldsymbol{y})d\boldsymbol{y}\right]\right)(s,\boldsymbol{\omega}) = \int_{-\infty}^{\infty} dt\,[\mathcal{R}f_1(t,\boldsymbol{\omega})]\,[\mathcal{R}f_2(s-t,\boldsymbol{\omega})]. \tag{32.123}$$

32D.4 Fourier transform of Radon transform.

$$\hat{f}(\rho\boldsymbol{\omega}) = \mathcal{F}(\mathcal{R}f)(\rho,\boldsymbol{\omega}) \equiv \int_{-\infty}^{\infty} \mathcal{R}f(s,\boldsymbol{\omega})e^{-i\rho s}ds. \tag{32.124}$$

That is, the Fourier transform of $Rf(s,\boldsymbol{\omega})$ with respect to s is the Fourier transform of the function f itself with the '$\boldsymbol{k}$-vector' parallel to $\boldsymbol{\omega}$.

[Demo] Using the definition (32.121), we have only to perform a straightforward calculation:

$$\int_{-\infty}^{\infty} \mathcal{R}f(s\boldsymbol{\omega})e^{-i\rho s}ds = \int_{-\infty}^{\infty} ds \int d\boldsymbol{x}\, f(\boldsymbol{x})\delta(s-\boldsymbol{x}\cdot\boldsymbol{\omega})e^{-i\rho s} = \int d\boldsymbol{x}\, f(\boldsymbol{x})e^{-i\rho\boldsymbol{\omega}\cdot\boldsymbol{x}}. \tag{32.125}$$

Thus f can be reconstructed by

$$f(\boldsymbol{r}) = \frac{1}{(2\pi)^d}\int \hat{f}(\rho\boldsymbol{\omega})e^{i\rho\boldsymbol{\omega}\cdot\boldsymbol{r}}d\rho d\boldsymbol{\omega}. \tag{32.126}$$

32D.5 Theorem [Radon inversion formula]. Let f be a piecewise C^1-function defined on a region in $\boldsymbol{R}^2$. Then

$$f(\boldsymbol{x}) = \int \tilde{\mathcal{R}}f(\boldsymbol{x}\cdot\boldsymbol{\omega},\boldsymbol{\omega})d\sigma(\boldsymbol{\omega}), \tag{32.127}$$

where $d\sigma$ is the arc length element of the unit circle, and $\tilde{\mathcal{R}}f$ is the *modified Radon transform* defined by

$$\tilde{\mathcal{R}}f(s,\boldsymbol{\omega}) \equiv \frac{1}{8\pi^2}\int_{-\infty}^{\infty} d\rho e^{-i\rho s}\rho\hat{\mathcal{R}}f(\rho,\boldsymbol{\omega}). \tag{32.128}$$

32D.6 X-ray tomography. The Radon transformation is the theoretical underpinning of the particle beam tomographies. These are applied not only medically, but also, e.g., to the anatomical study of fossils such as trilobites.

32D.7 *d*-space version. In d-space the Radon transform is defined as

$$\mathcal{R}f(s,\boldsymbol{\omega}) = \int_{\boldsymbol{R}^d} f(\boldsymbol{x})\delta(s-\boldsymbol{\omega}\cdot\boldsymbol{x})d\boldsymbol{x}, \tag{32.129}$$

where $\boldsymbol{\omega}$ is the position vector on the unit $d-1$-sphere S^{d-1} (the skin of the d-unit ball). The d-dimensional version of **32D.5** reads:

32D.8 Theorem.

$$f(\boldsymbol{x}) = \int_{S^{d-1}} d\sigma(\boldsymbol{\omega})\tilde{R}f(\boldsymbol{x}\cdot\boldsymbol{\omega},\boldsymbol{\omega}), \tag{32.130}$$

where

$$\tilde{\mathcal{R}}f(s,\boldsymbol{\omega}) \equiv \frac{1}{2(2\pi)^d}\int_{-\infty}^{\infty} e^{-i\rho s}|\rho|^{d-1}\hat{\mathcal{R}}f(\rho,\boldsymbol{\omega})d\rho, \tag{32.131}$$

$$\hat{\mathcal{R}}f(\rho,\boldsymbol{\omega}) \equiv \int_{-\infty}^{\infty} \mathcal{R}f(s,\boldsymbol{\omega})e^{i\rho s}ds \,(= \hat{f}(\rho\boldsymbol{\omega})\,) \tag{32.132}$$

with σ being the area element of S^{d-1}.

32D.9 Solving d-wave equation using Radon transform. Consider a wave equation in the whole d-space

$$(\partial_t^2 - \Delta)u = 0 \tag{32.133}$$

with the initial condition $u = f$ and $\partial_t u = g$ at $t = 0$. If the initial data are constant on all the hyperplanes perpendicular to the direction $\boldsymbol{\omega}$, i.e., $f(\boldsymbol{x}) = F(\boldsymbol{x}\cdot\boldsymbol{\omega})$ and $g(\boldsymbol{x}) = G(\boldsymbol{x}\cdot\boldsymbol{\omega})$, where F and G are functions defined on $\boldsymbol{R}$, then we can apply the method to solve the 1-space problem ($\to$**2B.4**) to get the solution as

$$u(\boldsymbol{x},t) = \frac{1}{2}[F(\boldsymbol{x}\cdot\boldsymbol{\omega}+t) + F(\boldsymbol{x}\cdot\boldsymbol{\omega}-t)] + \frac{1}{2}\int_{\boldsymbol{x}\cdot\boldsymbol{\omega}-t}^{\boldsymbol{x}\cdot\boldsymbol{\omega}+t} G(s)ds. \tag{32.134}$$

Therefore, if we can decompose the initial data into a superposition of data depending only on $\boldsymbol{x}\cdot\boldsymbol{\omega}$, the superposition principle ($\to$**1.4**) allows us to reconstruct the solution from the pieces like (32.134). As can be seen from (32.130), d-dimensional Radon transformation is the very tool to accomplish the desired decomposition.
The strategy is as follows:
(1) Calculate the modified Radon transform (32.132) for f and g,
(2) Solve the wave equation for $\tilde{\mathcal{R}}u$.
(3) Use (32.130) to reconstruct u:

$$u(\boldsymbol{x},t) = \frac{1}{2}\int_{S^{d-1}} d\sigma(\boldsymbol{\omega})\left\{\frac{1}{2}[\tilde{\mathcal{R}}f(\boldsymbol{x}\cdot\boldsymbol{\omega}+t,\boldsymbol{\omega}) + \tilde{\mathcal{R}}f(\boldsymbol{x}\cdot\boldsymbol{\omega}-t,\boldsymbol{\omega})] + \frac{1}{2}\int_{\boldsymbol{x}\cdot\boldsymbol{\omega}-t}^{\boldsymbol{x}\cdot\boldsymbol{\omega}+t}\tilde{\mathcal{R}}g(s,\boldsymbol{\omega})\right\}ds. \tag{32.135}$$

32D.10 Waves in odd and even dimensional spaces behave very

differently. Let us calculate the modified Radon transform (32.132) explicitly. If d is odd, then $|\rho|^{d-1} = \rho^{d-1}$, so that multiplying ρ can be interpreted as differentiation with respect to s as

$$\tilde{\mathcal{R}}f(s,\boldsymbol{\omega}) = \frac{1}{2}(-1)^{(q-1)/2}\left(\frac{1}{2\pi}\right)^{d-1}\frac{\partial^{d-1}}{\partial s^{d-1}}\mathcal{R}f(s,\boldsymbol{\omega}). \tag{32.136}$$

In contrast, if d is even then the non-analyticity of $|\rho|$ must be dealt with as $|\rho|^{d-1} = sgn(\rho)\rho^{d-1}$, so that

$$\tilde{\mathcal{R}}f(s,\boldsymbol{\omega}) = \frac{1}{2}(-1)^{(q-1)/2}\left(\frac{1}{2\pi}\right)^{d-1} H\left[\frac{\partial^{d-1}}{\partial s^{d-1}}\mathcal{R}f(s,\boldsymbol{\omega})\right], \tag{32.137}$$

where H is the *Hilbert transform* (→**8B.15**) defined by

$$Hf(x) = P\int \frac{f(s)}{x-s}ds, \tag{32.138}$$

where P denotes the Cauchy principal value (→**14.17**). This can be obtained from the convolution formula and the Fourier transform of sgn (→**32C.12**).

Look at the use of the modified Radon transform in the solution (32.135) when the initial velocity is everywhere zero. This applies to the case of an instantaneous flash of light emitted from a point (that is, $f = \delta(x)$). If $\tilde{\mathcal{R}}f(s\boldsymbol{\omega})$ is determined by $\mathcal{R}f(s,\boldsymbol{\omega})$ only, then the observer at distance sees only a flash of light. That is, the wave is localized in time in odd-dimensional (≥ 3) spaces. On the other hand, if the spatial dimensionality is even, then the Hilbert transform implies that the wave is not localized in time. Thus, after watching a flash, the observer must feel that the world becomes brighter (the *afterglow effect* in even dimensional spaces) (→**16C.4**).

APPENDIX a32 Bessel Transform

a32.1 Theorem [Hankel]. Let $f \in L_1([0,\infty), r)$ and be piecewise continuous. Then

$$\frac{1}{2}[f(r+0)+f(r-0)] = \int_0^\infty J_\nu(\sigma r)\sigma d\sigma \int_0^\infty f(\rho)J_\nu(\sigma\rho)\rho d\rho \quad (32.139)$$

for $\nu \geq 1/2$. This may also be expressed as

$$\int_0^\infty J_\nu(\sigma r)J_\nu(\sigma r')\sigma d\sigma = \delta(r-r')/r. \quad (32.140)$$

Notice that the RHS is the delta function adapted to the weight r (i.e., $\delta_r(r-r')$ →**18.25**).[410] □
[Demo] Here (32.139) is proved for continuous L_1 (→**19.8**) functions and integer $\nu = n$. Let

$$F(x,y) = f(r)e^{in\varphi}, \quad (32.141)$$

where $x = r\cos\varphi$ and $y = r\sin\varphi$. With the aid of the Fourier expression of the delta function (→**32C.8**), we can write

$$F(x,y) = \frac{1}{(2\pi)^2}\int dk_x \int dk_y \int d\xi \int d\eta F(\xi,\eta)e^{ik_x(x-\xi)+ik_y(y-\eta)}. \quad (32.142)$$

Introduce polar coordinates as

$$\xi = r'\cos\psi, \quad \eta = r'\sin\psi, \quad (32.143)$$
$$k_x = k\cos\theta, \quad k_y = k\sin\theta. \quad (32.144)$$

(32.142) is rewritten as ($F(\xi,\eta) = f(r')e^{in\psi}$)

$$f(r)e^{in\varphi} = \int_0^\infty dkk \int_0^\infty dr'r'f(r')\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta e^{ikr\cos(\theta-\varphi)}\frac{1}{2\pi}\int_{-\pi}^{\pi} d\psi e^{in\psi}e^{-ikr'\cos(\psi-\theta)}\right\}. \quad (32.145)$$

Setting $\psi - \theta = t$, we get

$$\frac{1}{2\pi}\int_{\pi}^{\pi} e^{in\psi}e^{-ikr'\cos(\psi-\theta)}d\psi = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikr'\cos t}e^{in(t+\theta)}dt \quad (32.146)$$
$$= e^{in\pi/2}e^{in\theta}J_n(-kr') = e^{in\pi/2+in\theta}(-1)^n J_n(kr'). \quad (32.147)$$

Here the generating function of Bessels functions (→**27A.5**) has been used. Analogously, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikr\cos(\theta-\varphi)}e^{in\theta}d\theta = e^{in\pi/2+in\varphi}J_n(kr). \quad (32.148)$$

410 More generally, f may be of bounded variation. See G. N. Watson, *A Treatise on the Theory of Bessel Function* (Cambridge UP, 1962) p456–.

Hence, (32.145)–(32.148) implies (32.139) for $\nu = n$.
A more convenient formulas may be

a32.2 Bessel transform and its inverse.

$$g(r) = \int_0^\infty h(r')J_\nu(r'r)r'dr', \tag{32.149}$$

$$h(r) = \int_0^\infty g(r')J_\nu(r'r)r'dr'. \tag{32.150}$$

Note that these are the formulas for the Fourier sine (or cosine) transform (→**32A.8**) for $\nu = \pm 1/2$ (→**27A.19**).

a32.3 Examples. See **27A.15**.

$$\int_0^\infty e^{-ax}J_0(xy)dx = \frac{1}{\sqrt{a^2+y^2}} \leftrightarrow \int_0^\infty \frac{y}{\sqrt{a^2+y^2}}J_0(xy)dy = \frac{e^{-ax}}{x}. \tag{32.151}$$

$$\int_0^\infty \cos ax J_0(xy)dx = \frac{1}{\sqrt{y^2-a^2}} \leftrightarrow \int_0^\infty \frac{y}{\sqrt{y^2-a^2}}J_0(xy)dy = \frac{\cos ax}{x}. \tag{32.152}$$

$$\int_0^\infty e^{-a^2x^2}x^{\nu+1}J_\nu(xy)dx = \frac{y^\nu}{(2a^2)^{\nu+1}}e^{-y^2/4a^2} \leftrightarrow \int_0^\infty \frac{y^{\nu+1}}{(2a^2)^{\nu+1}}e^{-y^2/4a^2}J_\nu(xy)dy = e^{a^2x^2}x^\nu. \tag{32.153}$$