30 Wave Equation: Finiteness of propagation speed

Wave equations are representative hyperbolic equations. With the aid of energy conservation, we discuss the well-posedness of wave equation problems. A general method to solve 3-space wave equation is given (method of spherical means due to Poisson), which clearly shows Huygens' principle. Finally, the characterization of hyperbolic equation with constant coefficients due to Gårding is summarized.

Key words: characteristic curve, domain of dependence (influence), energy conservation, Huygens' principle, method of spherical means, focusing, hyperbolicity in Gårding's sense, finiteness of propagation speed.

Summary:
(1) Wave equations have well-defined domains of dependence and influence: they are called the past and the future in relativity (**30.3**). Huygens' principle is correctly captured by the wave equation (**28.9**).
(2) Wave equations allow propagation of a solution which is not smooth along a special curve (characteristic curve) (**30.2**).
(3) Wave equations preserve energy. This implies well-posedness of wave equation problems (**30.4**, **30.6**).
(4) All the general methods to solve d-space wave equations are based on reducing them to 1D wave equations (**30.19**. For another, see **32D.9**). In $d(\geq 2)$-space, the time evolution due to wave equations may reduce the smoothness in the initial waves (**30.10**).
(5) Gårding conclusively characterized hyperbolicity (**30.12-14**), which implies finiteness of propagation speed (**30.15**).

30.1 Elementary summary. We have learned where the wave equations appear (→**1.2**, **a1D.9-11**, **a1F.8**), and physically argued what auxiliary conditions can ensure the uniqueness of the solution (→**1.20**). We know how to obtain the unique solution to the initial value problem in $\boldsymbol{R}$ as d'Alembert's formula (→**2B.4**) for the 1-space problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}. \tag{30.1}$$

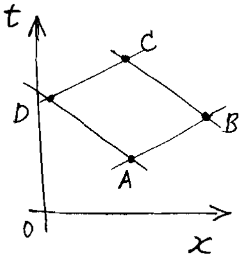
We know from the telegrapher's equation (→**a1F.17** or the Maxwell-Cattaneo equation →**28.10**) that the second order time derivative prohibits infinite speed propagation of the signal.

Exercise.
Solve

$$u_{tt} - u_{xx} = e^{-|x-t|} \tag{30.2}$$

on $\boldsymbol{R} \times \boldsymbol{R}$.

30.2 Characteristic curve. The solution method in **13C.6**(1) reduces the 1-wave equation (30.1) to two first order PDEs whose characteristic curves (→**13A.4**) are $x \pm ct = const.$ These curves (actually lines) are called the *characteristic curves* of the wave equation (→(C) below). If u is a solution to (30.1), then we can prove the following general identity:



$$u(A) + u(C) = u(B) + u(D), \tag{30.3}$$

where A-D are the apices of any parallelogram ABCD in space-time whose edges are parallel to the characteristic curves $x \pm ct = const.$ This equality can be shown easily with the aid of d'Alembert's solution (→**2B.4**). We may characterize a 'generalized solution' to (30.1) as any function u satisfying (30.3).

Discussion.
(A) Hyperbolic equations allow propagation of discontinuity without smoothing. Rewrite the wave equation (3.1) in the following form:

$$\frac{\partial v}{\partial t} = c\frac{\partial u}{\partial x}, \ \frac{\partial u}{\partial t} = c\frac{\partial v}{\partial x}. \tag{30.4}$$

Is there any curve $\phi(x,t) = 0$ on which u and v are continuous but their derivatives jump? [We have already discussed this in detail in **1.2a** Discussion.]
(B) Try the same thing as above for the telegrapher's equation.
(C) We have already discussed the meaning of the characteristic curve in **D2.2**. Let us continue the discussion for more general cases. Consider

$$\frac{\partial^2 u}{\partial t^2} - c^2(x)\frac{\partial^2 u}{\partial x^2} = 0 \tag{30.5}$$

where $c(x)$ is a positive valued function. Suppose there is a discontinuity of the solution of this equation along a curve $\varphi(x,t) = 0$. We assume the solution is smooth except on this curve. We rewrite the equation with the new coordinate $X = \varphi(x,t)$ and $Y = \psi(x,t)$, where ψ is chosen to make (X,Y) a well-behaved coordinate system.
(1) Show that the result can be written as

$$Q(\varphi,\varphi)\frac{\partial^2 u}{\partial X^2} + 2Q(\varphi,\psi)\frac{\partial^2 u}{\partial x\partial Y} + Q(\psi,\psi)\frac{\partial^2 u}{\partial Y^2} + L(\varphi)\frac{\partial u}{\partial X} + L(\psi)\frac{\partial u}{\partial Y} = 0, \tag{30.6}$$

where

$$Q(\varphi,\psi) = \frac{\partial\varphi}{\partial t}\frac{\partial\psi}{\partial t} - c^2(x)\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x}, \tag{30.7}$$

$$L(\varphi) = \frac{\partial^2\varphi}{\partial t^2} - c^2(x)\frac{\partial^2\varphi}{\partial x^2}. \tag{30.8}$$

(2) Suppose $\partial u/\partial X$ has a discontinuity across $\varphi(x,t) = 0$. Then, show that

$$Q(\varphi,\varphi) = 0 \tag{30.9}$$

must be satisfied. This equation is called the *characteristic equation*, and $\varphi =$ const. is called a *characteristic curve*.
(3) See that $x = \pm ct =$ const. are characteristic curves for the ordinary wave equation.
(4) There are two characteristic curves passing through a given point. The singularity we are discussing is constrained on them, so its propagating speed should be given by

$$\frac{dx}{dt} = -\left.\frac{\partial\varphi(x,t)}{\partial t}\right/\frac{\partial\varphi(x,t)}{\partial x} = \pm c(x). \tag{30.10}$$

(5) Notice that to solve the equation $Q = 0$ is equivalent to solving (30.10).

30.3 Domain of dependence, finite propagation speed. D'Alembert's solution (→**2B.3**) clearly shows that u at x at time t is completely determined by the initial data in the interval $[x - ct, x + ct]$. This interval is called the *domain of dependence.* Conversely, the initial data at ζ can influence the interval $[\zeta - ct, \zeta + ct]$ of the space at time t. This of course means that the disturbance can propagate at fastest with speed c in contradistinction to parabolic equations (→**28.9**).

Discussion: Characteristic initial value problem.
The light cone is a characteristic surface. If u is given on a characteristic surface as is shown in figure, then the solution is uniquely determined within its domain of influence. Hence, generally no boundary value problem in a closed domain has a solution for wave equations.

30.4 Energy conservation. The *energy integral*

$$E(t) = \frac{1}{2}\int_0^l \left\{\left(\frac{\partial u}{\partial t}\right)^2 + c^2\left(\frac{\partial u}{\partial x}\right)^2\right\} \tag{30.11}$$

is time independent for classical solutions (→**a1D.12**).
A formal calculation exchanging the order of differentiation with respect to time and integration is justifiable (→**19.17**).

Discussion.
Suppose that a vibrating string of length L with a fixed end condition is subjected

to a damping force $-a\frac{\partial\psi}{\partial t}$. Discuss how the energy conservation is violated.

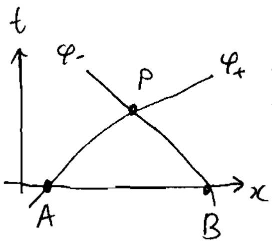
30.5 Uniqueness revisited. Although we already know the unique existence of the solution to the initial value problem of (30.1) in $\boldsymbol{R}$, let us reconsider the problem in terms of the energy integral. Since the equation is linear, to prove the uniqueness, we have only to consider that the homogeneous problem has only the zero solution: if v satisfies (30.1) and the auxiliary conditions $v(x,0)=0$ for $x \in D$, and $v(x,t)=0$ for $x \in \partial D$ for $t \geq 0$, then $v(x,t)=0$ in $D \times [0,t]$. For this initial condition the total energy (30.11) is zero, so that the constancy of energy integral implies that $\partial_t v(x,t) = \partial_x v(x,t) = 0$. This implies (with the aid of the mean value theorem) v is a constant. Since v is continuous, this implies that $v \equiv 0$.

Discussion.
(1) **Riemann's method.** Let

$$L \equiv \rho(x)\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}. \tag{30.12}$$

Consider :

$$
\begin{aligned}
Lv &= 0, &(30.13)\\
v|_{t=t_0,x=x_0} &= 1, &(30.14)\\
2\sqrt{\rho(x)}\frac{\partial v}{\partial s} + \frac{\partial\sqrt{\rho(x)}}{\partial s}v &= 0 \text{ on characteristic curves.} &(30.15)
\end{aligned}
$$



The solution v is called the *Riemann function* (fundamental solution). In terms of this function, the solution to the initial value problem can be obtained as

$$(\sqrt{\rho(x)}u)(P) = \frac{1}{2}\left[(\sqrt{\rho(x)}uv)(A) + (\sqrt{\rho(x)}uv)(B)\right] + \frac{1}{2}\int_{x_A}^{x_B} \rho(x)\left(\frac{\partial u}{\partial t}v - u\frac{\partial v}{\partial t}\right)dx, \tag{30.16}$$

where A, B, P are the points in the figure. The formula is called *Riemann's formula*, and d'Alembert's formula is its special case.
(2) How can we determine Riemann's function? The problem is to solve v for which the auxiliary conditions are given on the characteristic curves. Such a problem is called a *Goursa's problem* or *characteristic boundary value problem.* We change the independent variables from x, t to φ_+ and φ_- (characteristic curves ($\to$**30.2**). The problem now reads

$$\frac{\partial^2 v}{\partial\varphi_-\partial\varphi_+} - a\frac{\partial v}{\partial\varphi_-} - b\frac{\partial v}{\partial\varphi_+} = 0 \tag{30.17}$$

with the boundary conditions

$$v(\varphi_-,0) = f_-, \quad v(\varphi_+,0) = f_+. \tag{30.18}$$

Here a, b and $f_\pm$ are given functions. If we define

$$\Psi_\pm = \frac{\partial v}{\partial\varphi_\pm}, \tag{30.19}$$

then, the PDE can be cast in the following simultaneous Volterra integral equation:

$$\Psi_- = f'_- + \int_0^{\varphi_+} [a(\varphi_-, \eta)\Psi(\varphi_-, \eta) + b(\varphi_-, \eta)\Psi_+(\varphi_-, 0)]d\eta, \tag{30.20}$$

$$\Psi_+ = f'_+ + \int_0^{\varphi_-} [a(\xi, \varphi_+)\Psi_-(\xi, \varphi_+) + b(\xi, , \varphi_+)\Psi_+(\xi, \varphi_+)]d\xi. \tag{30.21}$$

This can be solved by an interative replacement method with the starting choice of $\Psi_- = f'_-$, $\Psi_+ = f'_+$.

30.6 Well-posedness. We consider two problems (30.1) with $u(x,0) = f_i(x)$ and $\partial_t u(x,0) = g_i(x)$ in $\boldsymbol{R}$ ($i = 1, 2$). Denoting each solution as u_i, we can easily get

$$|u_1(x,t) - u_2(x,t)| \le ||f_1 - f_2||_{max} + |t|\,||g_1 - g_2||_{max} \tag{30.22}$$

from d'Alembert's formula ($\to$**2B.4**). Hence, the solution depends on the data continuously. That is, small changes of the data cause a small change in the solution for any finite time.

30.7 Inhomogeneous wave equation. Consider

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = F(x,t) \tag{30.23}$$

in $\boldsymbol{R} \times \boldsymbol{R}$ with the initial condition $u(x,0) = f(x)$ and $\partial_t u(x,0) = g(x)$, where f is C^2 and g is C^1. The problem is a superposition of the homogeneous equation with the inhomogeneous initial conditions studied in **2B.4** and the following problem of inhomogeneous equation with homogeneous initial conditions:

$$\frac{\partial^2 v}{\partial t^2} - c^2 \frac{\partial^2 v}{\partial x^2} = F(x,t) \tag{30.24}$$

with $v(x,0) = 0$ and $\partial_t v(x,0) = 0$. The problem can be solved easily with the introduction of the new variables (a standard trick $\to$**2B.3**) $x \pm ct$ as in

$$v(x,t) = \frac{1}{2c} \int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\sigma, \tau) d\sigma. \tag{30.25}$$

Notice that if $F(x,t)$ is an odd function of x, then so is v for all t.

30.8 Wave equation in 3-space, Huygens' principle. The initial value problem

$$\partial_t^2 u = c^2 \Delta u \tag{30.26}$$

with the initial condition

$$u = f(x), \partial_t u = g(x) \text{ for } t = 0 \tag{30.27}$$

in 3-space has the following solution:

$$u(x,t) = \frac{1}{4\pi c^2 t}\int_{|y-x|=ct} g(y)d\sigma(y) + \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\int_{|y-x|=ct} f(y)d\sigma(y)\right). \tag{30.28}$$

This is an explicit expression of *Huygens' principle.* This equation can be a starting point of a numerical scheme. A demonstration of the equation follows.

Exercise.
Solve the following 3-wave equation:

$$u_{tt} = \Delta u \tag{30.29}$$

with the initial condition $u = x^2 + y^2 + z^2$ and $u_t = z$.

Needless to say, an inhomogeneous problem $\Box u = q$ can be solved by linear decomposition. The inhomogeneous problem with a homogeneous auxiliary conditions can be solved easily in terms of Green's functions (→**40**).

30.9 Method of spherical means [Poisson]. Define

$$M_h(\boldsymbol{x}, r) = \frac{1}{4\pi^2}\int_{|y|=1} h(\boldsymbol{x} + r\boldsymbol{y})d\sigma(\boldsymbol{y}), \tag{30.30}$$

where h is a C^2-function, and σ is the area element of the sphere. M_h is an even function of r. Using Gauss' theorem (→**2C.13**), we get the following *Darboux's equation*

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) M_h(\boldsymbol{x}, r) = \Delta M_h(\boldsymbol{x}, r). \tag{30.31}$$

Here Δ is the Laplacian acting on the function of $\boldsymbol{x}$. (30.26) is converted to

$$\frac{\partial^2}{\partial t^2}(rM_u) = c^2\frac{\partial^2}{\partial r^2}(rM_u), \tag{30.32}$$

where M_u is interpreted as a function of $\boldsymbol{x}$, r and t as $M_r u(\boldsymbol{x}, r, t)$, and the initial condition becomes

$$M_u = M_f, \quad \partial_t M_u = M_g \quad \text{for } t = 0. \tag{30.33}$$

Notice that $M_u(\boldsymbol{x}, 0, t) = u(\boldsymbol{x}, t)$. (30.32) can be solved as (→**2B.4**):

$$rM_u(\boldsymbol{x}, r, t) = \frac{1}{2}[(r+ct)M_f(\boldsymbol{x}, r+ct) + (r-ct)M_f(\boldsymbol{x}, r-ct)] + \frac{1}{2c}\int_{r-ct}^{r+ct} yM_g(\boldsymbol{x}, y)dy. \tag{30.34}$$

Using the fact that $M_r f$ and $M_r g$ are even functions of r, we can rewrite this as

$$M_u(\boldsymbol{x},r,t) = \frac{(ct+r)M_f(\boldsymbol{x},ct+r)-(ct-r)M_f(\boldsymbol{x},ct-r)}{2r} + \frac{1}{2cr}\int_{ct-r}^{ct+r} yM_g(\boldsymbol{x},y)dy. \tag{30.35}$$

Now, take the $r \to 0$ limit (l'Hospital's rule is used) and we finally arrive at (30.28). See **32C.14**.

30.10 Focusing effect. (30.28) implies that the smoothness of the solution u can be less than that of the initial data due to the derivative in the formula. This effect is called the *focusing effect.* This can happen when the initial condition is focussed into a small set, making caustics. This does not happen in 1-space.

30.11 What is the mathematical essence of the wave equation? Physically, that the singularity can be propagated without smoothing (propagation of shock waves, for example) is a remarkable distinction from the diffusion equation (parabolic equation). Also the finiteness of the speed of propagation is in striking contrast to the diffusion equation (→**28.9**). Since the wave equation is nothing but Newton's equation of motion of an idealized elastic body (→**a1D.9**), the Newton-Laplace determinacy should apply. That is, the Cauchy problem must be well-posed (→**28.3**). Gårding[384] answered the question decisively at least for the constant coefficient linear partial differential equations (of any order).

30.12 Hyperbolicity in Gårding's sense. Let $L \equiv L(\partial_t, \nabla)$ be a N-th order linear PDE operator with constant coefficients. If L contains $\partial^N/\partial t^N$[385] and if the real parts of the zeros $\lambda_i(\xi)$ of the characteristic equation $L(\lambda, i\xi) = 0$[386] considered as an equation for λ are bounded as a function of ξ, then we say $Lu = 0$ is a *hyperbolic equation in Gårding's sense.*

30.13 Example.
(1) Wave equation $(\partial_t^2 - c^2\Delta)u = 0$. $L(\lambda, i\xi) = \lambda^2 + c^2\xi^2$. Therefore, $\lambda(\xi) = \pm ic|\xi|$. That is, the characteristic roots are purely imaginary, so obviously the equation is hyperbolic in Gårding's sense.
(2) Diffusion equation $(\partial_t - D\Delta)u = 0$. $L(\lambda, i\xi) = \lambda + D\xi^2$, so that $\lambda(\xi) = -D\xi^2$ is real and is not bounded as a function of ξ.

[384] Gårding wrote a nice book on mathematics: L. Gårding, *Encounter with Mathematics* (Springer, 1977). Those who are interested in mathematics as a part of the modern culture will enjoy the book.

[385] If the highest order derivative is not ∂_t^N, then **30.15** below does not hold. That is, the propagation of front has infinite speed.

[386] Here not only the highest order terms but all the derivatives are taken into account. Furthermore i accompanies with ξ.

(3) However, if we add a second order time derivative term with a small positive coefficient as $(\epsilon\partial_t^2 + \partial_t - D\Delta)u = 0$, which is called the telegrapher's equation or Maxwell-Cattaneo equation (→**a1F.17**, **28.10**), the situation is drastically different from the diffusion equation. For this $L(\lambda, i\xi) = \epsilon\lambda^2 + \lambda + D\xi^2$, so that $\lambda(\xi) = (-1 \pm \sqrt{1 - 4\epsilon D\xi^2})/2\epsilon$. Hence its real part is bounded as a function of ξ. That is, the telegrapher's equation is hyperbolic in Gårding's sense.
(4) Certainly, the Laplace equation $\Delta u = 0$ is not hyperbolic.

Discussion.
The equation for transversal oscillations of a beam is given by

$$\frac{\partial^4 u}{\partial x^4} + c^{-4}\frac{\partial^2 u}{\partial t^2} = f(x,t), \tag{30.36}$$

where f is essentially the external load. This equation is hyperbolic.

30.14 Theorem [Gårding]. The Cauchy problem $Lu = 0$ under the Cauchy condition $\partial^k f/\partial t^k(0,x) = u_k(x)$ $(0 \le k \le N-1)$ is well-posed in the sense of Hadamard (→**28.3**) if and only if L is hyperbolic in Gårding's sense. □[387]
Hence, the determinacy (and more) for the wave equation is vindicated.

30.15 Theorem [Finiteness of the propagation speed]. Let Ω be the support of the Cauchy data for $Lu = 0$, where L is a linear partial differential operator with constant coefficients, and is hyperbolic in the sense of Gårding (→**30.12**). Then the support of the solution at time $t > 0$ is included in the set $\{x : \cup_{\xi\in\Omega}|x - \xi| \le ct\}$, where c is a finite number such that

$$c \ge \max_{i-1,\cdots,m;|\xi|=1} \bar{\lambda}_i(\xi). \tag{30.37}$$

Here $\overline{\lambda}_i$ are zeros of the symbol of the principal part of the differential operator (and are real for hyperbolic equations).□[388]

[387] See John, Section 5.2.
[388] S. Mizohata, *Partial Differential Equations* (Iwanami, 1965), Theorem 4.9.