29 Laplace Equation: Consequence of spatial moving average

A solution of the Laplace equation is called a harmonic function. This must be a function invariant under spatial moving averaging as we discussed in **1.13**. This property almost determines the important features of the solutions of the Laplace equation and guarantees its well-posedness, etc.

key words: harmonic function, Green's formula, mean-value theorem, its converse, maximum principle, analyticity of solution, Liouville's theorem

Summary:
(1) Solutions to the Laplace equation must be invariant under spatial moving average; a precise statement is the spherical mean-value theorem and its converse (**29.4-5**). The resulting smoothness can also be stated precisely (**29.10**).
(2) From this, we immediately know that harmonic functions cannot have any local extremum inside the domain (**29.6**, **29.8**). This denies the existence of any stable electrostatic structure (**29.7**).
(3) Typical potential problems are well-posed (**29.9**).

29.1 Elementary summary. We have learned where the Laplace equation appears (→**1.2**, **1.14**, **a1B.3**, **a1F.6**), and physically argued what auxiliary conditions can ensure the uniqueness of the solution (→**1.19**). The most important boundary conditions are Dirichlet conditions in which the value of the function ψ on the boundary of the domain is fixed, and Neumann conditions in which the normal derivative of ψ on the boundary is given.

Discussion.
The Cauchy problem of the Laplace equation is not well-posed. This was seen in Discussion **2B.4**(7). Physically, this is not surprising. To obtain the Laplace equation instead of the wave equation for electromagnetic wave, we must change the sign of Faraday's law (→**a1F.8**). This implies that we replace Lenz's law with 'anti-Lenz's' law'. Lenz's law is a manifestation of the stability of the world, so there is no surprise that the Laplace equation does not describe the well-behaved time evolution in our world.

29.2 Laplace equation and harmonic functions. Any classical solution to the Laplace equation is called a *harmonic function.* The electric potential due to point charges is a harmonic function where there is no charge (→**a1F.6**), and charges correspond to the singularities of the functions. The equilibrium drumhead is described by a harmonic function. The real and imaginary parts of an analytic function are harmonic functions (→**5.6**).

Discussion.
(1) There is no solution to the 3-Laplace equation on the unit ball centered at the origin with the origin removed with the boundary condition $u = 1$ on the $|x| = 1$ and $u(0) = 0$.
(2) Consider the 2D Laplace equation $\Delta u = 0$ on the half plane $x > 0$ with the 'initial condition' $u(0, y) = 0$ and $\partial_x u(0, y) = f(y)$. If f is analytic, then there is a local analytic solution, but if it is not, then there is not even a local solution.

29.3 Green's formula. Let $D \subset \boldsymbol{R}^n$ be a bounded region, and u and v be C^2-functions defined on the closure of D. Here, we record the formulas again for convenience (→**16A.19**).

$$\int_D (v\Delta u + grad\, u \cdot grad\, v)d\tau = \int_{\partial D} v\, grad\, u \cdot d\boldsymbol{S}, \tag{29.1}$$

and

$$\int_D (v\Delta u - u\Delta v)d\tau = \int_{\partial D} (v\, grad\, u - u\, grad\, v) \cdot d\boldsymbol{S}. \tag{29.2}$$

29.4 Spherical Mean-value theorem. Let u be harmonic on a region $D \subset \boldsymbol{R}^n$, and $B_r(x)$ be a ball of radius r centered at x such that $B_r(x) \subset D$. Then, we have

$$u(x) = \frac{1}{S_{n-1}(r)} \int_{\partial B_r(x)} u(y) d\sigma(y), \tag{29.3}$$

where $d\sigma(y) = |d\boldsymbol{S}(y)|$, the area of the surface element, and $S_{n-1}(r)$ is the surface area of $(n-1)$-sphere (i.e., the skin of the n-ball) of radius r.[376] □

This should be intuitively expected from the interpretation of the Laplacian (→**1.13**).
[Demo] Set $v(y) = 1/|x-y|^{n-2}$ $(n > 2)$ or $\ln|x - y|$ $(n = 2)$ in (29.2), and $D = B_r(x) \setminus B_\epsilon(x)$ $(r > \epsilon)$.[377] Since v is harmonic in $\boldsymbol{R}^n \setminus \{x\}$ as a function of

[376] $S_{n-1}(r) = 2\pi^{n/2} r^{n-1}/\Gamma(n/2)$.
[377] $A \setminus B$ is the set of all the points in A but not in B: $A \setminus B \equiv \{x | x \in A, x \notin B\}$.

y, $v(y)$ is harmonic on D. To calculate the RHS of (29.2) we need the normal derivatives on $\partial B_r(x)$:

$$\frac{\partial v}{\partial n} = (2-n)r^{1-n}. \tag{29.4}$$

Since both $u(y)$ and $v(y)$ are harmonic on D, (29.2) reads

$$\begin{aligned} 0 &= \int_{\partial D} (v\partial_n u - u\partial_n v)d\sigma(y) \\ &= \int_{\partial B_r(x)} (v\partial_n u - u\partial_n v)d\sigma(y) - \int_{\partial B_\epsilon(x)} (v\partial_n u - u\partial_n v)d\sigma(y). \end{aligned} \tag{29.5}$$

Using (29.4) and (16.35), we can rewrite this as

$$0 = -(2-n)\left[r^{1-n}\int_{\partial B_r(x)} ud\sigma(y) - \epsilon^{1-n}\int_{\partial B_\epsilon(x)} ud\sigma(y)\right], \tag{29.6}$$

which implies

$$\lim_{\epsilon\to 0} \epsilon^{1-n}\int_{\partial B_\epsilon(x)} ud\sigma(y) = S_{n-1}(r)u(x). \tag{29.7}$$

The converse of this theorem is also true:

29.5 Theorem [Converse of mean-value theorem]. Let u be a continuous function on a region D. If the mean value theorem **29.4** holds for any $r > 0$ and x such that $B_r(x) \subset D$, then u is C^∞ and harmonic on D. □[378]

29.6 Maximum principle. Let D be an open region and u be harmonic ($\to$**29.2**) there. Suppose $\sup_{x\in D} u(x) \equiv A < \infty$. If $u \not\equiv A$ for $\forall x \in D$, then $u(x) < A$ for $\forall x \in D$. □

This should be obvious from the mean-value theorem **29.4**. Also, since a harmonic function is a steady solution of a diffusion equation, from the maximum principle for the diffusion equation ($\to$**28.2**), this should be physically sensible. Changing u to $-u$ gives the minimum counterpart. This theorem implies:

Corollary. Let D be a compact set, and u be a harmonic function on the open kernel of D and continuous on D, then the extremum of u on D is achieved on ∂D. □

This implies that static electric potential cannot have its extreme values where there is no charge. A grave consequence is the collapse of classical physics.

Discussion.

Consider

$$\Delta u = u - u^3 \tag{29.8}$$

in 3-space on a bounded region Ω. Assume $u = 0$ on $\partial\Omega$. Show that $-1 \le u \le 1$.

[378] For a proof, see Folland p91 (2.5).

29.7 Classical physics cannot explain atoms: Earnshaw's theorem. It is impossible to have a stable static configuration of charges in any static electric field. □
Unstable stationary configurations are not impossible (give an example). This theorem and electromagnetic radiation inevitable from accelerated charges conclusively killed the possibility of explaining atoms within classical physics.

29.8 Strong maximum principle. Let Ω be a bounded region in $\boldsymbol{R}^n$, and u be harmonic there. If u attains its maximum value M at an inner point of Ω, then u is constant on Ω.

This is obvious from the mean value theorem.

29.9 Uniqueness and well-posedness. The solution of the Laplace equation on a bounded domain D, if exists,[379] is unique and depends continuously on the boundary data (→**29.11**).□
The proof is quite parallel to that for the diffusion equation (→**28.4**).
[Demo] Let u_1 and u_2 be two solutions of the same problem. Then, due to the linearity of the problem, the difference $u = u_1 - u_2$ obeys the Laplace equation with the homogeneous Dirichlet boundary condition (i.e., $u = 0$ at the boundary of the domain). From the maximum principle (→**29.6**) u cannot be larger than 0, and $-u$ cannot be larger than 0. Hence, $u_1 = u_2$. That is, if there is a solution, it is unique. Now, we compare two different problems 1 and 2 with the auxiliary conditions different slightly. Let the solutions of 1 and 2 be u_1 and u_2, respectively. Then, the maximum principle tells us that the maximum value of $|u_1 - u_2|$ in the region cannot be larger than the differences in the boundary data.

Discussion.
The existence of a solution in a domain in 3 or higher dimensional space is a very difficult problem, even if the boundary condition is continuous.

29.10 Smoothness of the solution. Since a harmonic function is, roughly speaking, invariant under spatial moving average, it must be smooth. Actually,
Theorem. All the solutions of the Laplace equation are real analytic (→**13C.6(2)** for $d = 2$. Here the assertion is for all $d \geq 2$. Analyticity means the convergence of the Taylor series.). □

Discussion.
(1) A solution to $\Delta u = f$ is analytic if f is analytic (Courant-Hilbert).
(2) Hadamard's example
Let D be a bounded region. There exists a continuous function $F : \partial D \to \boldsymbol{R}$

[379] We have not yet constructed the solution!

such that it becomes the boundary value of a harmonic function ϕ on D for which $\int_D |grad\phi|^2 d\sigma$ is not bounded. In this case although ϕ is C^∞, its derivatives behave wilder and wilder as the point approaches the boundary of the domain.

If the boundary value is continuous, then the corresponding Dirichlet problem of the Laplace equation on a bounded domain has at most one solution.

29.11 Well-posedness of Poisson's equation. The general Poisson problem has the following form

$$\Delta u = F \text{ in } \Omega,\ u = f \text{ on } \partial\Omega. \tag{29.9}$$

Here Ω is a bounded region. If we are interested in smooth solution (for example, C^2), then

$$\|u_1 - u_2\|_\Omega \leq c_1\|f_1 - f_2\|_{\partial\Omega} + c_2\|F_1 = F_2\|_\Omega, \tag{29.10}$$

where $\|\ \|_D$ is the L_2-norm on D, and c_1, c_2 are positive constants. This inequality clearly implies the well-posedness of our problem.

It is a good occasion to learn something about the so-called a priori estimate.

The inequality can be demonstrated as follows.

(1) First, the problem is split into v and w: $\Delta v = F$ in Ω, $v = 0$ on $\partial\Omega$ and $\Delta w = 0$ in Ω, $w = f$ on $\partial\Omega$.

(2) From the properties of the algebraic and geometric averages we get

$$2|(v, w)| = 2|(\epsilon^{-1/2}v, \epsilon^{1/2}w)| \leq \epsilon\|v\| + (\epsilon)^{-1}\|w\| \tag{29.11}$$

for any positive ϵ.

(3) Therefore,

$$\|v + w\|^2 \leq (1+\epsilon)\|v\| + (1+\epsilon^{-1})\|w\|. \tag{29.12}$$

That is, we have only to find bounds for v and w, respectively.

(4) With the aid of the variational problem (→**34C.13**) for the eigenvalue of the Laplacian $-\Delta$:

$$0 < \lambda_1 = \inf_{v|_{\partial\Omega}=0} \frac{\int_\Omega v(-\Delta)v dx}{\int_\Omega v^2 dx}. \tag{29.13}$$

Hence, with the aid of the Schwarz inequality (→**20.7**)

$$\|v\|_\Omega^2 \leq \frac{1}{\lambda}\left(\int_\Omega v(-\Delta)v dx\right) \leq \frac{1}{\lambda}\left(\int_\Omega v^2 dx\right)^{1/2}\left(\int_\Omega (\Delta v)^2 dx\right)^{1/2} \tag{29.14}$$

Hence,

$$\|v\|^2 \leq \frac{1}{\lambda_1^2}\int_\Omega F^2 dx. \tag{29.15}$$

(5) Introduce an auxiliary function φ such that $\Delta\varphi = w$ on Ω and the homogeneous Dirichlet condition on $\partial\Omega$ (the existence of the solutions w and φ is a prerequisite of our argument). With the aid of Green's formula (→**29.3**)

$$\int_{\partial\Omega} w\frac{\partial\varphi}{\partial n}d\sigma - \int_{\partial\Omega}\varphi\frac{\partial w}{\partial n}d\sigma \int_\Omega w\Delta\varphi dx - \int_\Omega \varphi\Delta w dx. \tag{29.16}$$

Hence,

$$\int_\Omega w^2 dx = \int_{\partial\Omega} w \frac{\partial \varphi}{\partial n} d\sigma \le \left(\int_{\partial\Omega} w^2 dx\right)^{1/2} \left(\int_{\partial\Omega} \left(\frac{\partial \varphi}{\partial n}\right)^2 d\sigma\right)^{1/2}. \tag{29.17}$$

We have used the Schwarz inequality.
(6) For a function vanishing on the boundary

$$\int_{\partial\Omega} \left(\frac{\partial \varphi}{\partial n}\right)^2 d\sigma \le C \int_\Omega (\Delta\varphi)^2 dx. \tag{29.18}$$

Hence, $||w||^2$ is bounded by $||F||^2$. [380]

Discussion.
Partial derivatives of a harmonic function with respce to the Cartesian coordinates are again harmonic. However, the partial derivatives with respect to curvilinear coordinates are not necessarily so.

29.12 Comparison theorem. Let u and v be harmonic functions on a bounded domain Ω, and $u \ge v$ on $\partial\Omega$. Then, $u \ge v$ throughout Ω.

29.13 Liouville's theorem.[381] If u is a bounded harmonic function on the whole space $\boldsymbol{R}^n$, then u is a constant.□[382]

29.14 More general elliptic equation. The essence of the Laplacian is that it is an operator giving the deviation of the value of the function from its local average. The Laplacian is obtained when we assume that the weight for the average is everywhere uniform (→**1.13**). We should be able to choose a weighted average. Then, a more general equation

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[380] The following theorem is also relevant.
Aleksandrov's theorem. The solution to Poisson's equation smoothly depends on the charge distribution. Or, more precisely: Let D be a bounded domain and u be a solution of $\Delta u = f$ in D with a homogeneous Dirichlet condition and is continuous up to the boundary of D. Then,

$$\sup_D u \le C||f||_d, \tag{29.19}$$

where C is a constant dependent on the spatial dimensionality and the radius of D, and $||\cdot||_d$ is the L^d-norm. [L^p-norm for any positive p is defined by $||f||_p \equiv \left(\int |f|^p dx\right)^{1/p}$, where the integral is the Lebesgue integral (→**19**).] See Egorov-Shubin, p93.

[381] Joseph Liouville, 1809-1882.

[382] Folland p94 (2.11).

like

$$a_{ij}\frac{\partial^2 u}{\partial x_i x_j} + b_i\frac{\partial u}{\partial x_i} + c(x)u = 0 \tag{29.20}$$

with the positive definite matrix $Matr(a_{ij})$ appears. We may expect that the key properties of the Laplacian should be true even for $a_{ij}\partial_i\partial_j$, because they are due to the averaging principle. Indeed the maximum principle is true if $c \le 0$ as intuitively expected. (The most statements above hold if $c \le 0$[383]).

[383]See, Yu. V. Egorov and M. A. Shubin (eds) *Partial Differential Equations III*, Chapter 2 (Springer, 1991)