27 Cylinder Functions

Separation of variables of the Laplace equation in the cylindrical coordinates requires Bessel and modified Bessel functions, which may perhaps be the most representative special functions. Bessel and Neumann functions make a fundamental system for the radial part of the separated equation called the Bessel equation. Classical results about Bessel and Neumann functions are summarized such as orthonormal relations (Fourier-Bessel-Dini expansion), generating functions, integrals containing Bessel functions. Bessel functions with half odd integer parameter (or their streamlined version: spherical Bessel functions) are required to solve the Helmholtz equation in the spherical coordinates.

Key words: Bessel equation, Bessel function, Bessel's integral, generating function, recurrence relations, cylinder functions, zeros of Bessel functions, Neumann function, Hankel function, Fourier-Bessel-Dini expansion, Modified Bessel function, spherical Bessel function, partial wave expansion.

Summary:
(1) The Laplace equation in the cylindrical coordinates requires Bessel and Neumann functions (**27A.1**, **27A.2**, **27A.16**). Pay attention to the general shapes of these functions (**27A.4**, **27A.16**).
(2) Bessel functions make an orthonormal eigenfunction set for the radial part of the Laplacian (**27A.21-22**).
(3) The Helmholtz equation in the spherical cooridnates requires spherical Bessel functions (**27A.25-26**).
(4) Many second order linear ODE can be solved in terms of cylinder functions (**27A.28**).

27.A General Theory

27A.1 Bessel's equation. In terms of $z = \alpha r$, the equation (23.17) ($\rightarrow$**23.9**(1)) becomes

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{m^2}{z^2}\right)u = 0. \tag{27.1}$$

$z = 0$ is a regular singular point ($\to$**24B.2**(1)), and $z = \infty$ is an irregular singular point ($\to$**24B.2**(2)).[358]

27A.2 Series solution to Bessel's equation around $z = 0$. The indicial equation (24.24) ($\to$**24B.4**) is $\phi(\mu) = \mu^2 - m^2 = 0$. Choose $\mu = m$, $a_1 = 0$, $a_0 = 1/2^m\Gamma(m+1)$ and follow **24B.3**. We get

$$J_m(z) = \left(\frac{z}{2}\right)^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(m+k+1)} \left(\frac{z}{2}\right)^{2k}. \tag{27.2}$$

This is called the *Bessel function of order m* (of the first kind). If $\mu_1 - \mu_2 = 2m$ is not an integer ($\to$**24B.6[1]**), then J_{-m} is a partner ($\to$**24A.13**) of J_m in a fundamental system of solutions ($\to$**24A.11**) of (27.1). If m is a half odd integer, then J_m and J_{-m} are still functionally independent (that is, this is the case with no logarithmic term in **24B.7[22]**).

If m is a positive integer, then J_m and J_{-m} are not functionally independent:

$$J_m = (-1)^m J_{-m}. \tag{27.3}$$

This can be demonstrated from (27.2) with the aid of $\Gamma(-m+k+1) = \infty$ for $k < m$ ($\to$**9.1** or **27A.7**). In this case we need a different partner: Neumann functions ($\to$**27A.15**).

Exercise.
(A)
(1) Show that

$$\frac{(-1)^k}{k!\Gamma(m+k+1)}\left(\frac{z}{2}\right)^{m+2k} = \left(\frac{z}{2}\right)^m \frac{(-1)^k}{\Gamma(m+1/2)\Gamma(1/2)} \frac{z^{2k}}{(2k)!} B\left(m+\frac{1}{2}k+\frac{1}{2}\right). \tag{27.4}$$

(2) With the aid of the integral expression of the Beta function (9.22), show that formally[359]

$$J_m(z) = \frac{1}{\Gamma(1/2)\Gamma(m+1/2)}\left(\frac{z}{2}\right)^m \int_0^1 t^{m-1/2}(1-t)^{-1/2}\cos[z(1-t)^{1/2}]dt \tag{27.5}$$

for $m + 1/2 > 0$.
(3) Now, changing the integration variable as $t = \sin^2\theta$, this formula can be rewritten as

$$J_m(z) = \frac{1}{\Gamma(1/2)\Gamma(m+1/2)}\left(\frac{z}{2}\right)^m \int_0^\pi \cos(z\cos\theta) sin^{2m}\theta d\theta. \tag{27.6}$$

Notice that the integration from 0 to $\pi/2$ and from $\pi/2$ to π are identical in this case, so $\int_0^\pi$ can be replaced by $2\int_0^{\pi/2}$. This formula is called *Poisson's integral*

[358] Therefore, the Bessel function is a special function of confluent type ($\to$**23.5**).
[359] The exchange of the order of summation and integration can be justified.

representation.
(4) If $1-t=x^2$ is introduced, then (27.5) can be rewritten as

$$J_m(z)=\frac{1}{\Gamma(1/2)\Gamma(m+1/2)}\left(\frac{z}{2}\right)^m\int_{-1}^{1}(1-x^2)^{m-1/2}e^{izx}dx. \tag{27.7}$$

(B) Demonstrate the following *Whittaker's integral representation*

$$J_{n+1/2}(z)=(-i)^n\sqrt{\frac{\pi}{2z}}\int_{-1}^{1}e^{izx}P_n(x)dx. \tag{27.8}$$

Here P_n is the Legendre polynomial.
(C) Show

$$\int_0^{\pi/2}J_0(x\cos\theta)\cos\theta d\theta=\frac{\sin x}{x}. \tag{27.9}$$

[Hint. Use the integral of $\cos^n\theta$.]

27A.3 Definition. The *Bessel function* of order ν can also be defined by

$$J_\nu(z)=\frac{1}{2\pi i}\int_C e^{\frac{z}{2}(t-\frac{1}{t})}t^{-(\nu+1)}dt, \tag{27.10}$$

where C can be a unit circle centered at the origin, and ν any real number. Obviously,

$$J_0(0)=1,\quad J_n(0)=0 \text{ for positive integer } n. \tag{27.11}$$

For integer ν this definition and the result in **27A.1** are identical as seen in **27A.4**.

Discussion: Where did the Bessel functions appear first?
The position of the earth (x,y) on the rotation plane can be written as

$$x=a(\cos\phi-e),\ y=a\sqrt{1-e^2}\sin\phi, \tag{27.12}$$

where e is eccentricity, a the long radius, and ϕ the excentric angle measured from the perihelion, and

$$\phi-e\sin\phi=vt, \tag{27.13}$$

where t is the time since the earth passed the perihelion, and v is the average angular velocity. Hence, if we can write down ϕ as a function of t, then we can explicitly obtain $x(t)$ and $y(t)$. Consider

$$\frac{d\phi}{dvt}=\frac{1}{1-e\cos\phi}, \tag{27.14}$$

which is an even periodic function of vt. Hence, we can Fourier-expand it as

$$\frac{d\phi}{dvt}=1+\sum_{n=1^\infty}a_n\cos nvt. \tag{27.15}$$

The coefficient can be computed as

$$a_n = \frac{1}{2\pi}(1 - e\cos\phi)^{-1}\cos nvtd(vt), \tag{27.16}$$

$$= \frac{2}{\pi}\int_0^{\pi}\cos[n(\phi - e\cos\phi)]d\phi. \tag{27.17}$$

Comparing this with the generating equation in **27A.5**, we obtain

$$a_n = 2J_n(ne) \tag{27.18}$$

Exercise.
Demonstrate

$$J_n(z) = \frac{1}{\pi i^n}\int_0^{\pi} e^{iz\cos\theta}\cos n\theta d\theta. \tag{27.19}$$

27A.4 Series expansion. With the change of variables from t to $u = tz/2$, we rewrite the RHS of (27.10) as

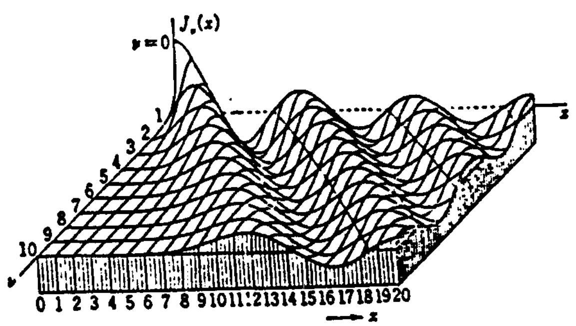
$$\frac{1}{2\pi i}\left(\frac{z}{2}\right)^{\nu}\int_{C'} e^{u-z^2/4u}u^{-(\nu+1)}du = \frac{1}{2\pi i}\left(\frac{z}{2}\right)^{\nu}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left(\frac{z}{2}\right)^m\int_{C'} e^u u^{-(\nu+m+1)}du. \tag{27.20}$$

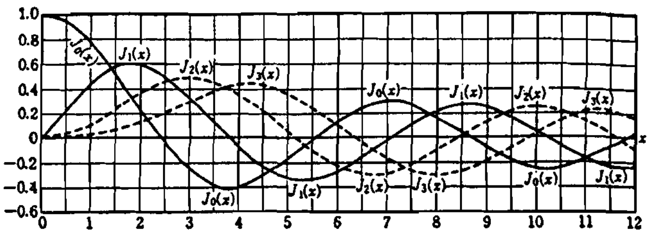
The integral can be computed with the residue theorem (→**8B.2**) as

$$J_\nu(z) = \left(\frac{z}{2}\right)^{\nu}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(\nu+m+1)}\left(\frac{z}{2}\right)^{2m}, \tag{27.21}$$

which is in agreement with (27.2). The series is convergent on the whole complex plane. Due to the factor $(-1)^m$ it is clear that J_ν cannot have any pure imaginary zero. From the formula, near the origin

$$J_\nu(z) \sim z^\nu. \tag{27.22}$$





Exercise.
(This problem need not be here.) Demonstrate

$$\int_0^1 t^{n+1} J_n(t)dt = J_{n+1}(1). \tag{27.23}$$

27A.5 Generating function.

$$e^{\frac{1}{2}z(t-\frac{1}{t})} = \sum_{n=-\infty}^{+\infty} J_n(z)t^n. \tag{27.24}$$

This is from (27.10). This equation implies that J_n for integer n is the coefficients of the Laurent expansion (→**8A.11(2)**) of $\exp[z(t-t^{-1})/2]$ around $t = 0$.

27A.6 Bessel's integral. Replacing t in (27.10) with $e^{i\theta}$, we have *Bessel's integral*

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)d\theta. \tag{27.25}$$

Exercise.
Show

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\theta}d\theta. \tag{27.26}$$

27A.7 $J_{-n}(z) = (-1)^n J_n(z)$. This can be obtained by replacing θ with $\pi - \theta$ in (27.25). (We have already shown this in **27A.2**.)

27A.8 Sine of sine → Bessel functions.

$$\sin(z\sin\theta) = 2J_1(z)\sin\theta + 2J_3(z)\sin 3\theta + 2J_5(z)\sin 5\theta + \cdots. \tag{27.27}$$

To show this rewrite (27.24) with the aid of **27A.7** as

$$e^{\frac{1}{2}z(t-\frac{1}{t})} = J_0(z) + \sum_{n=1}^{\infty} J_n(z)[t^n + (-)^n t^{-n}]. \tag{27.28}$$

Now replace t with $e^{i\theta}$, and we get

$$e^{iz\sin\theta} = J_0(z) + 2iJ_1(z)\sin\theta + 2J_2(z)\cos 2\theta + 2iJ_3(z)\sin 3\theta + \cdots. \tag{27.29}$$

Splitting this into real and imaginary part, we get (27.27) and

$$\cos(z\sin\theta) = J_0(z) + 2J_2(z)\cos 2\theta + 2J_4(z)\cos 4\theta + \cdots. \tag{27.30}$$

Thus, when sine appears inside a trigonometric function, recall J_n.

27A.9 Recurrence relations. Differentiating (27.24) with respect to z and comparing the coefficients of the power of t, we get

$$2J'_m(z) = J_{m-1}(z) - J_{m+1}(z). \tag{27.31}$$

In particular, with the aid of **27A.7** we have

$$J'_0(z) = -J_1(z) \tag{27.32}$$

If we differentiate (27.24) with respect to t and then compare the coefficients of t^n, we get

$$\frac{2m}{z}J_m(z) = J_{m-1}(z) + J_{m+1}(z). \tag{27.33}$$

27A.10 Cylinder function. Any function $f(z, \nu)$ satisfying the following relations is called a *cylinder function*:

$$f(z,\nu-1) + f(z,\nu+1) = 2\frac{\nu}{z}f(z,\nu), \tag{27.34}$$

$$f(z,\nu-1) - f(z,\nu+1) = 2\frac{\partial}{\partial z}f(z,\nu). \tag{27.35}$$

(27.31) and (27.33) thus imply that Bessel functions are cylinder functions.

Exercise.
(1) These relations can be rewritten as

$$\frac{d}{dz}[z^\nu J_\nu(z)] = z^\nu J_{\nu-1}(z), \tag{27.36}$$

$$\frac{d}{dz}[z^{-\nu} J_\nu(z)] = -z^{-\nu} J_{\nu+1}(z). \tag{27.37}$$

(2) Derive

$$z^{-(\nu+m)}J_{\nu+m}(z) = (-1)^m\left(\frac{1}{z}\frac{d}{dz}\right)^m [z^{-\nu}J_\nu]. \tag{27.38}$$

Similarly, we can obtain

$$z^{\nu-m}J_{\nu-m}(z) = \left(\frac{1}{z}\frac{d}{dz}\right)^m [z^{\nu}J_\nu]. \tag{27.39}$$

(3) Integral related to the Fraunhofer diffraction through a circular aperture:

$$J = \int_0^a \int_0^{2\pi} e^{ibr\cos\theta} d\theta r dr = \frac{2\pi a}{b} J_1(ab). \tag{27.40}$$

[Hint. Use **27A.6** and **27A.10**.]

27A.11 Zeros of Bessel functions.

(1) There are infinitely many zeros of $J_n(z)$.
(2) All the zeros of J_n for $n > -1$ are real and of multiplicity one except $z = 0$.
(3) $z^{-n}J_n(z)$ has no zero of multiplicity larger than one.
(4) The zeros of $J_n(z)$ separate the zeros of $J_{n\pm 1}(z)$.
[Demo] From the Bessel equation (27.1) by scaling with a constant α we get

$$\frac{d^2 J_n(\alpha z)}{dz^2} + \frac{1}{z}\frac{dJ_n(\alpha z)}{dz} + \left(\alpha^2 - \frac{n^2}{z^2}\right) J_n(\alpha z) = 0. \tag{27.41}$$

Hence,

$$\frac{1}{z}\frac{d}{dz}\left\{ z\frac{dJ_n(\alpha z)}{dz}J_n(\beta z) - z\frac{dJ_n(\beta z)}{dz}J_n(\alpha z)\right\} = (\beta^2 - \alpha^2)J_n(\alpha z)J_n(\beta z), \tag{27.42}$$

where β is another constant. Multiplying x and integrationg (27.42) gives

$$(\beta^2 - \alpha^2)\int_0^b xJ_n(\alpha x)J_n(\beta x)dx = \left[x\frac{dJ_n(\alpha x)}{dx}J_n(\beta x) - x\frac{dJ_n(\beta x)}{dx}J_n(\alpha x)\right]_0^b. \tag{27.43}$$

Since $J_n(x)$ is of order x^n near $x = 0$ ($\rightarrow$(27.22)), if $n > -1$, the contribution from $x = 0$ of the RHS of (27.43) vanishes. Choose α to satisfy $J_n(\alpha b) = 0$, and set $\beta = \overline{\alpha}$. Then $J_n(\beta b) = 0$ since all the coefficients in (27.21) are real ($\overline{J_n(z)} = J_n(\overline{z})$). For these choices, the RHS of (27.43) is zero, so we have

$$(\beta^2 - \alpha^2)\int_0^b x|J_n(\alpha x)|^2 dx = 0. \tag{27.44}$$

This implies $\beta^2 = \alpha^2$, that is $\alpha = \overline{\alpha}$, since there is no pure imaginary zeros ($\rightarrow$**27.4**). That is, the zeros of J_n are all real if $n > -1$.

The multiplicity of the zeros is known from the general property of the fundamental system ($\rightarrow$**24A.12**). At $z = 0$ the coefficient function is not C^1, so this argument is not applicable to $z = 0$. Thus (2) and (3) have been demonstrated. Bessel's equation can be rewritten as

$$\frac{d^2 U}{dz^2} + HU = 0 \tag{27.45}$$

with $U(z) = J_n/\sqrt{z}$ and $H(z) = 1 + (1/4 - n^2)/z^2$. Let $x \in \boldsymbol{R}$ be sufficiently large to make $H(x) > 0$. Suppose $U > 0$. Then, irrespective of the sign of dU/dx (27.45) tells $d^2U/dx^2 < 0$. Hence, as long as $U > 0$, dU/dx decreases with the increase of x. This continues until $U = 0$, but there $dU/dx < 0$ so U becomes eventually $U < 0$.

This argument can be continued indefinitely. Thus, there must be infinitely many zeros for J_n. This is (1). (4) follows from

$$\frac{d}{dz}(z^{\pm\nu}J_\nu) = \pm z^\nu J_{\nu\mp1}, \tag{27.46}$$

which can be derived from the nature of J_ν as cylinder functions (→**27A.10**).

27A.12 Proposition. For $x \in \boldsymbol{R}$

$$|J_0(x)| \le 1 \tag{27.47}$$
$$|J_n(x)| \le 1/\sqrt{2} \text{ for } n = 1, 2, \cdots. \tag{27.48}$$

The first inequality follows from (27.25). The second inequality follows from the Gegenbauer-Neumann formula (→**27A.14**. Expand the LHS and compare it with the RHS. cf. **27A.15**(2).).

27A.13 Addition theorem.

$$J_n(x+y) = \sum_{s=-\infty}^{\infty} J_{n-s}(x)J_s(y), \tag{27.49}$$
$$= \sum_{s=0}^{n} J_s(x)J_{n-s}(y) + \sum_{s=1}^{\infty}(-)^s[J_s(x)J_{n+s}(y) + J_{n+s}(x)J_s(y)]. \tag{27.50}$$

This follows from the generating function (27.24):

$$\sum_{n=-\infty}^{\infty} J_n(x+y)t^n = e^{(x+y)(t-1/t)/2} = \sum_{n=-\infty}^{\infty} J_n(x)t^n \sum_{n=-\infty}^{\infty} J_n(y)t^n. \tag{27.51}$$

27A.14 Gegenbauer-Neumann formula.

$$J_0(\sqrt{R^2+2Rr\cos\gamma+r^2}) = J_0(R)J_0(r) + 2\sum_{m=1}^{\infty}(-1)^m J_m(R)J_m(r)\cos m\gamma,$$
$$J_0(\sqrt{R^2-2Rr\cos\gamma+r^2}) = J_0(R)J_0(r) + 2\sum_{m=1}^{\infty} J_m(R)J_m(r)\cos m\gamma. \tag{27.52}$$

[Demo] The second formula can be obtained from the first by $r \to -r$ and **27A.7**. With the aid of the generating function **27A.5**, we obtain

$$\sum_{n=-\infty}^{\infty} \lambda^n J_n(x)t^n = \exp\left\{\frac{x}{2t}\left(\lambda - \frac{1}{\lambda}\right)\right\} \sum_{n=-\infty}^{\infty} J_n(\lambda x)t^n. \tag{27.53}$$

Setting $\lambda = e^{i\theta}$, this equation becomes

$$\sum_{n=-\infty}^{\infty} J_n(e^{i\theta}x)t^n = e^{-(ix/t)\sin\theta} \sum_{n=-\infty}^{\infty} e^{in\theta} J_n(x)t^n. \tag{27.54}$$

Following the demonstration of the addition theorem **27A.13**, we obtain

$$\sum_{n=-\infty}^{\infty} J_n(e^{i\theta}x+e^{i\varphi}y)t^n = e^{-(i/t)(x\sin\theta+y\sin\varphi)} \left[\sum_{n=-\infty}^{\infty} e^{in\theta}J_n(x)t^n\right]\left[\sum_{n=-\infty}^{\infty} e^{in\varphi}J_n(y)t^n\right]. \tag{27.55}$$

Let x, y, θ and φ be real and $xe^{i\theta} + ye^{i\varphi}$ be real. Then, $x\sin\theta + y\sin\varphi = 0$. Compare the coefficients of t^0:

$$J_0(x\cos\theta + y\cos\varphi) = \sum_{m=-\infty}^{\infty} e^{im(\theta-\varphi)} J_m(x)J_{-m}(y) \tag{27.56}$$

Notice that if the imaginary part of $Re^{i\theta} + re^{i\varphi}$ vanishes, we can write

$$\sqrt{R^2 + 2Rr\cos\gamma + r^2} = Re^{i\theta} + re^{i\varphi} \tag{27.57}$$

with $\gamma = \theta - \varphi$. This concludes the proof. See **27A.2**.

Exercise.
Show

$$J_n(z)J_n(z') = \frac{1}{\pi}\int_0^{\pi} J_0(\sqrt{z^2 + z'^2 - 2zz'\cos\theta})\cos n\theta d\theta \tag{27.58}$$

for $n = 0, 1, 2, \cdots$..

27A.15 Integrals containing Bessel functions.

(1) From Bessel's integral **27A.6** for $a \geq 0$ and $b > 0$

$$\int_0^{\infty} e^{-ax} J_0(bx)dx = \frac{1}{\sqrt{a^2+b^2}}. \tag{27.59}$$

Especially $\int_0^{\infty} J_0(bx)dx = 1/b$. Replacing a in (27.59) with ia we get ($b > a$ assumed)

$$\int_0^{\infty} J_0(bx)\cos ax dx = \frac{1}{\sqrt{b^2-a^2}}. \tag{27.60}$$

Differentiating these equations w.r.t. a, we compute similar integrals with insertions of powers of x. With the aid of (27.32) and integration by parts, we obtain

$$\int_0^{\infty} e^{-ax} J_1(bx)x dx = \frac{b}{(b^2+a^2)^{3/2}}. \tag{27.61}$$

(2) From the Gegenbauer-Neumann formula **27A.14** with the aid of the orthogonality of $\{\cos n\gamma\}$ we obtain ($\rightarrow$**17.16** with $l = \pi$)

$$\frac{1}{\pi}\int_0^\infty J_0(x\sqrt{R^2 - 2Rr\cos\gamma + r^2})\cos n\gamma d\gamma = J_n(Rx)J_n(rx). \quad (27.62)$$

From this and (27.59), we get

$$\int_0^{2\pi} J_n(Rx)J_n(rx)dx = \frac{1}{\pi}\int_0^\pi \frac{\cos\gamma}{\sqrt{R^2 - 2Rr\cos\gamma + r^2}}d\gamma. \quad (27.63)$$

Notice that this formula contains the generating function for the Legendre polynomials ($\rightarrow$**21A.9**), so that expansion in terms of r/R can be calculated with the aid of $P_n(\cos\gamma)$.
(3) [Weber's integral] Expanding $J_\nu(bx)$ as in **27A.2** and termwise integration ($\rightarrow$**19.11**) give for $a > 0$, $b > 0$ and for $Re\,\nu > -1$

$$\int_0^\infty e^{-a^2x^2}J_\nu(bx)x^{\nu+1}dx = \frac{b^\nu}{(2a^2)^{\nu+1}}e^{b^2/4a^2}. \quad (27.64)$$

(4) [Lommel's integral]

$$\int_0^x J_n(\alpha x)J_n(\beta x)xdx = \frac{x}{\alpha^2 - \beta^2}\{\alpha J_n(\beta x)J_{n+1}(\alpha x) - \beta J_n(\alpha x)J_{n+1}(\beta x)\}, \quad (27.65)$$

$$= \frac{x}{\alpha^2 - \beta^2}\{\beta J_n(\alpha x)J_{n-1}(\beta x) - \alpha J_n(\beta x)J_{n+1}(\alpha x)\}. \quad (27.66)$$

This follows from (27.43) and recurrence relations ($\rightarrow$**27A.9**).

Exercise.
Show

$$J_0(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos xt}{\sqrt{1-t^2}}dt. \quad (27.67)$$

27A.16 Neumann function of order m. When $m \in N\backslash\{0\}$, we may use the general theory or the procedure in **24B.7[22]**, but traditionally, the following partner is chosen:

$$N_m(z) = [J_m(z)\cos m\pi - J_{-m}(z)]/\sin m\pi, \quad (27.68)$$

which is called the *Neumann function of order* m. For non-integer m (27.68) is well defined and obviously a partner of J_m in **24B.1**. If

$m \in \boldsymbol{N}$, (27.68) becomes $0/0$, so we interpret the formula with the aid of l'Hospital's rule:[360]

$$N_m(z) = \frac{1}{\pi}\left[\frac{\partial J_m(z)}{\partial m} - (-1)^m \frac{\partial J_{-m}(z)}{\partial m}\right] \quad \text{for } m \notin \boldsymbol{N} \setminus \{0\}. \tag{27.69}$$

The general solution of Bessel's equation (27.1) is given by

$$AJ_m(z) + BN_m(z). \tag{27.70}$$

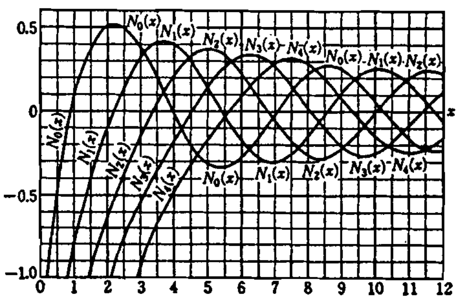
Notice that Neumann functions are cylinder functions as easily explicitly checked (→**27A.10**).

Exercise.
Demonstrate that

$$W(J_\nu(x), N_\nu(x)) = \frac{2}{\pi x}. \tag{27.71}$$

27A.17 $N_n(z)$ is singular at $z = 0$. This follows from explicit formulas in the $z \to 0$ limit (see the footnote of the previous entry):

$$\begin{aligned} N_0(z) &\sim (2/\pi)\, ln(z/2), && (27.72)\\ N_n(z) &\sim -(n-1)!(x/2)^{-n}/\pi \text{ for } n = 1, 2, \cdots. && (27.73)\end{aligned}$$



[360]If we explicitly compute (27.69), we get

$$N_m(z) = \frac{2}{\pi} J_m(z) \ln\left(\frac{z}{2}\right) - \frac{1}{\pi}\sum_{k=0}^{m-1} \frac{(m-k-1)!}{k!}\left(\frac{z}{2}\right)^{-m+2k} - \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{\psi(k+1)+\psi(m+k+1)}{k!(m+k)!}(-1)^m \left(\frac{z}{2}\right)^{m+2k},$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$. Thus this form is in conformity with the general theory **24B.7[22]**.

27A.18 Lommels' formula. Since J_ν and N_ν make a fundamental system of Bessel's equation (→**27A.16** and **24A.4**), their Wronskian (→**24A.6**) W must satisfy **24A.13**, i.e.,

$$W(x) = W_0 e^{-\ln x} = \frac{W_0}{x}. \tag{27.74}$$

To calculate W_0 we may use $\lim_{x\to 0} xW(x) = W_0$. If ν is not an integer, J_ν and $J_{-\nu}$ make a fundamental system (→**27A.2**), so

$$\lim_{x\to 0} xW(J_\nu(x), J_{-\nu}(x)) = \lim_{x\to 0} x[J_\nu(x)J'_{-\nu}(x) - J'_\nu(x)J_{-\nu}(x)] = -\frac{2\sin\nu\pi}{\pi}, \tag{27.75}$$

where we have used the formula of complementary arguments for the Gamma function **9.5**. Thus, we obtain

$$W(J_\nu(x), J_{-\nu}(x)) = -\frac{2\sin\nu\pi}{\pi x}. \tag{27.76}$$

The result is correct even if ν is an integer due to continuity. With the aid of this formula and the definition of N_n in **27A.16**, we obtain

$$W(J_n(x), N_n(x)) = \frac{2}{\pi x}. \tag{27.77}$$

This is called *Lommel's formula.*

Exercise.
Show
(1)

$$J_n N_{n+1} - J_{n+1} N_n = -\frac{2}{\pi x}. \tag{27.78}$$

(2)

$$J_n J_{-n+1} + J_{-n} J_{n-1} = \frac{2\sin n\pi}{\pi x}. \tag{27.79}$$

27A.19 Bessel function with half odd integer parameters (See spherical Bessel functions in **27A.25**). The Bessel and Neumann functions with half odd integer parameters can be written in terms of elementary functions:

$$J_{1/2}(z) \;=\; \sqrt{\frac{2}{\pi}}\frac{\sin z}{\sqrt{z}},\; J_{-1/2}(z) = \sqrt{\frac{2}{\pi}}\frac{\cos z}{\sqrt{z}} \tag{27.80}$$

$$J_{3/2} = \sqrt{\frac{2}{z\pi}}\left(\frac{\sin z}{z} - \cos z\right), \tag{27.81}$$

$$N_{1/2}(z) = -\sqrt{\frac{2}{z\pi}}\cos z, \quad N_{-1/2}(z) = \sqrt{\frac{2}{z\pi}}\sin z, \tag{27.82}$$

$$N_{3/2}(z) = 1\sqrt{\frac{1}{z\pi}}\left(\sin z + \frac{\cos z}{z}\right). \tag{27.83}$$

Exercise.
Derive

$$J_{m+1/2} = (-1)^m \frac{z^{m+1/2}\sqrt{2}}{\sqrt{\pi}}\left(\frac{1}{z}\frac{d}{dz}\right)^m\left(\frac{\sin z}{z}\right). \tag{27.84}$$

27A.20 Hankel functions. Hankel functions are defined as follows:

$$H_n^{(1)}(z) \equiv J_n(z) + iN_n(z), \tag{27.85}$$

$$H_n^{(2)}(z) \equiv J_n(z) - iN_n(z). \tag{27.86}$$

$H_n^{(1)}$ and $H_n^{(2)}$ make a fundamental system of solutions (→**24A.11**) for the Bessel equation (→**27A.1**).

Exercise.
Show

$$H_n^{(2)}(x)H_{n+1}^{(1)}(x) - H_n^{(1)}(x)H_{n+1}^{(2)}(x) = \frac{4}{\pi x}. \tag{27.87}$$

27A.21 Orthonormal basis in terms of Bessel functions. The set of kets $|i,\nu\rangle$ defined as follows is an orthonormal basis (→**20.10**) of $L_2([0,a],x)$ (→**20.19**)

$$\langle x|i,\nu\rangle = \frac{\sqrt{2}}{aJ_{\nu+1}(r_i^{(\nu)})}J_\nu(r_i^{(\nu)}x/a), \tag{27.88}$$

where $r_i^{(\nu)}$ is the i-th zero of $J_\nu(x)$ (→**27A.11**). That $\{|i,\nu\rangle\}$ is a basis follows from the corresponding eigenvalue problem and the general theory (→**36.3**). That this is normalized (orthogonality follows from the general theory) is seen with the aid of Lommel's integral (→**27A.15(4)**). Using l'Hospital's rule, we take the $\alpha \to \beta$ limit to obtain

$$\int_0^\alpha xJ_\nu(r_i^{(\nu)}x/a)^2dx = \frac{a^2}{2}[J_\nu'(r_i^{(\nu)})]^2. \tag{27.89}$$

This can be further transformed into the desired result with the aid of a recurrence relation in **27A.9**.

The corresponding decomposition of unit operator $1 = \sum_{i=1}^{\infty} |i,\nu\rangle\langle i,\nu|$ (**20.15**) implies (cf. **20.26** for the delta function with a weight)

$$\frac{\delta(x-y)}{x} = \sum_{i=1}^{\infty} \frac{2}{[aJ_{\nu+1}(r_i^{(\nu)})]^2} J_\nu(r_i^{(\nu)}x/a)J_\nu(r_i^{(\nu)}y/a). \tag{27.90}$$

27A.22 Fourier-Bessel-Dini expansion. $f \in L^2([0,a],x)$ (→**20.19**) can be expanded as

$$f(x) = \sum_{m=1}^{\infty} C_m J_\nu(r_m^{(\nu)}x/a), \tag{27.91}$$

where

$$C_m = \frac{2}{[aJ_{\nu+1}(r_i^{(\nu)})]^2}\int_0^a f(x)J_\nu(r_i^{(\nu)}x/a)x\,dx. \tag{27.92}$$

Notice that this is nothing but a standard generalized Fourier expansion with a special choice of the orthonormal basis. Hence the analogues of three key facts (→**17.8**) holds.

27A.23 Modified Bessel functions. In terms of $z = \alpha r$, the equation (23.37) becomes (→**23.9**(3))

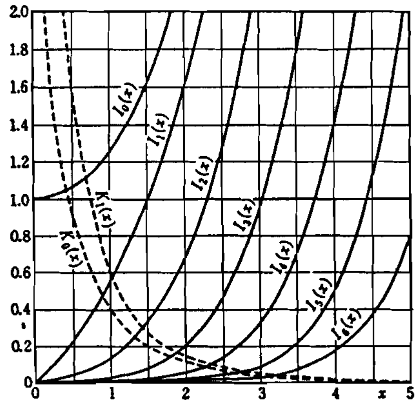
$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} - \left(1 + \frac{m^2}{z^2}\right)u = 0. \tag{27.93}$$

$z = 0$ is a regular singular point (→**24B.2**(1)), and $z = \infty$ is an irregular singular point (→**24B.2**(2)). If z in (27.1) is replaced with iz, we get this equation. Hence, $J_m(iz)$ and $N_m(iz)$ are solutions. However, with a suitable phase factor the following set is usually chosen as a fundamental system of solutions (→**24A.11**)

$$I_m(z) = e^{m\pi i/2}J_m(iz) = \sum_{n=0}^{\infty}\frac{1}{n!\Gamma(n+m+1)}\left(\frac{z}{2}\right)^{2n+m}, \tag{27.94}$$

$$K_m(z) = \frac{\pi}{2}\frac{I_{-m}(z) - I_m(z)}{\sin m\pi} = \frac{\pi}{2}\frac{e^{m\pi i/2}J_{-m}(iz) - e^{-m\pi i/2}J_m(iz)}{\sin m\pi}. \tag{27.95}$$

I and K are called *modified Bessel functions.* They are <u>not</u> cylinder functions.



Exercise.
(1) Show the leading singularities:

$$K_0(x) = -\ln x - \gamma + \ln 2 + \cdots, \tag{27.96}$$

$$K_n(x) = 2^{n-1}(n-1)!x^{-n} + \cdots. \tag{27.97}$$

(2) Demonstrate

$$\cosh x = I_0(x) + 2\sum_{n=1}^{\infty} I_{2n}(x). \tag{27.98}$$

(3) The solution to

$$u'' - zu = 0 \tag{27.99}$$

is called *Airy functions.* They become useful to study asymptotic behaviors of the Bessel functions for large $|z|$ and $|\nu|$. We can easily find a fundamental system for this equation, looking at the table in **27A.28**:

$$\begin{aligned} u_1 &= Ai(z) \equiv \frac{1}{\pi}\left(\frac{z}{3}\right)^{1/2} K_{1/3}\left(\frac{2z^{3/2}}{3}\right) = \frac{z^{1/2}}{3}\left[I_{-1/3}\left(\frac{2z^{3/2}}{3}\right) - I_{1/3}\left(\frac{2z^{3/2}}{3}\right)\right] \\ u_2 &= Bi(z) \equiv \frac{z^{1/2}}{3}\left[I_{-1/3}\left(\frac{2z^{3/2}}{3}\right) + I_{1/3}\left(\frac{2z^{3/2}}{3}\right)\right]. \end{aligned} \tag{27.100}$$

Ai (resp., *Bi*) is called the *Airy function of the first (resp., second) kind.*

27A.24 Helmholtz equation. For the equation of the type $L_t\psi = \Delta\psi$, where L_t is a differential operator with respect to time, the separation of variables gives us the *Helmholtz equation*

$$\Delta\psi = -\kappa^2\psi, \tag{27.101}$$

where $-$ is explicitly written, because the Laplacian is a non-positive operator. The separation of variables in the spherical coordinates $\psi =$

$R(r)Y(\theta,\varphi)$ gives

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)Y(\theta,\varphi)=-\ell(\ell+1)Y(\theta,\varphi), \quad (27.102)$$

and

$$\frac{1}{r}\frac{d^2}{dr^2}rR(r)=\left(-\kappa^2+\frac{\ell(\ell+1)}{r^2}\right)R(r). \quad (27.103)$$

(27.102) with the periodic boundary condition on the sphere gives eigenfunctions $Y_\ell^m(\theta,\varphi)$ (→**26A.8**) ($\ell=0,1,2,\cdots$ and $m=-\ell,-\ell+1,\cdots,-1,0,1,\cdots,\ell$ for each ℓ). We may assume

$$\psi=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}R_{lm}(r)Y_\ell^m(\theta,\varphi). \quad (27.104)$$

Here R_{lm} obeys (27.103).

27A.25 Spherical Bessel functions. Introducing $u=\sqrt{\kappa r}R(r)$ and $z=\kappa r$, (27.103) becomes

$$\frac{d^2u}{dz^2}+\frac{1}{z}\frac{du}{dz}+\left(1-\frac{(\ell+1/2)^2}{z^2}\right)u=0. \quad (27.105)$$

This is Bessel's equation (27.1) with $m=\ell+1/2$. Therefore, the fundamental system of solutions for (27.103) consists of $J_{\ell+1/2}(\kappa r)/\sqrt{\kappa r}$ and $N_{\ell+1/2}\kappa r)/\sqrt{\kappa r}$. Thus the following *spherical Bessel function* j_ℓ and *spherical Neumann function* n_ℓ are defined:

$$j_\ell(z)\equiv\sqrt{\frac{\pi}{2z}}J_{\ell+1/2}(z),\quad n_\ell(z)\equiv\sqrt{\frac{\pi}{2z}}N_{\ell+1/2}(z). \quad (27.106)$$

The general solution to (27.103) is given by

$$Aj_\ell(z)+Bn_\ell(z). \quad (27.107)$$

The *spherical Hankel function* is also defined analogously

$$h_l^{(1,2)}(x)=\sqrt{\frac{\pi}{2x}}H_{l+1/2}^{(1,2)}(x). \quad (27.108)$$

Exercise.
(1) Demonstrate

$$j_0(x)=\frac{\sin x}{x} \quad (27.109)$$

with the aid of the series expansion of the Bessel function. Also demonstrate

$$n_0(x) = -\frac{\cos x}{x} \tag{27.110}$$

(2) Show

$$j_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right). \tag{27.111}$$

(3) Show

$$j_n(x)n_n'(x) - j_n'(x)n_n(x) = \frac{1}{x^2}. \tag{27.112}$$

27A.26 Orthonormal basis in terms of spherical Bessel functions. There is nothing new in the present case, since we know the corresponding result for the Bessel function (→**27A.17**). Therefore,

$$\left\{\sqrt{\frac{2}{a^3}}\frac{1}{j_{l+1}(\rho_i^{(l)})}j_l(\rho_i^{(l)}r/a)\right\}_{i=1}^{\infty}, \tag{27.113}$$

where $\rho_i^{(l)} = r_i^{(l+1/2)}$ is the zeros of $J_{l+1/2}$ (→**27A.11**), is an orthonormal basis of $L_2([0,a],r^2)$ (→**20.19**). For example, the decomposition of unit operator reads (cf. **20.27**)

$$\frac{\delta(x-y)}{x^2} = \sum_{i=1}^{\infty}\frac{2}{a^3}\frac{1}{j_{l+1}(\rho_i^{(l)})^2}j_l\left(\frac{\rho_i^{(l)}x}{a}\right)j_l\left(\frac{\rho_i^{(l)}y}{a}\right). \tag{27.114}$$

27A.27 Partial wave expansion of plane wave.

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta). \tag{27.115}$$

[Demo] $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ satisfies the Helmholtz equation $(\Delta+k^2)u(r) = 0$ (→**27A.24**). Hence, we may assume

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty} c_l j_l(kr)P_l(\cos\theta). \tag{27.116}$$

Therefore, the problem is to determine the coefficients c_l. With the aid of the orthogonality of the Legendre polynomial (→**21A.5**), we obtain

$$c_l j_l(kr) = \frac{2l+1}{2}\int_{-1}^{1} dx e^{ikrx}P_l(x). \tag{27.117}$$

To evaluate the integral, integrate it by parts and ignore $o[1/r]$. We have

$$\int_{-1}^{1} dx e^{ikrx}P_l(x) \sim \frac{1}{ikr}[e^{ikr} - (-)^l e^{-ikr}] = \frac{2i^l}{kr}\sin\left(kr - \frac{l\pi}{2}\right). \tag{27.118}$$

Comparing this with the asymptotic formula for $r \to \infty$, we arrive at $c_l = (2l+1)i^l$.

27A.28 ODE solvable in terms of Cylinder functions. Many second order linear ODE can be solved in terms of cylinder functions. See the table.

$Z_\nu = J_\nu$ or N_ν

$u'' + \frac{1}{z}u' + \left(1 - \frac{\nu^2}{z^2}\right)u = 0$	$Z_\nu(z)$
$u'' + \frac{1}{z}u' + \left(\beta^2 - \frac{\nu^2}{z^2}\right)u = 0$	$Z_\nu(\beta z)$
$u'' + \frac{1}{z}u' + \left[(\beta\gamma z^{\gamma-1})^2 - \left(\frac{\nu\gamma}{z}\right)^2\right]u = 0$	$Z_\nu(\beta z^\gamma)$
$u'' + \frac{1-2\alpha}{z}u' + \left(\beta^2 + \frac{\alpha^2-\nu^2}{z^2}\right)u = 0$	$z^\alpha Z_\nu(\beta z)$
$u'' + \left(\beta^2 - \frac{4\nu^2-1}{4z^2}\right)u = 0$ (上の微分方程式の $\alpha=1/2$ の場合)	$\sqrt{z}\,Z_\nu(\beta z)$
$u'' + \frac{1-2\alpha}{z}u' + \left[(\beta\gamma z^{\gamma-1})^2 + \frac{\alpha^2-\nu^2\gamma^2}{z^2}\right]u = 0$	$z^\alpha Z_\nu(\beta z^\gamma)$
$u'' + \frac{1}{z}u' - \left(1 + \frac{\nu^2}{z^2}\right)u = 0$	$Z_\nu(iz)$
$u'' + \frac{1}{z}u' \pm \left(i \mp \frac{\nu^2}{z^2}\right)u = 0$ (複号同順)	$Z_\nu(e^{\pm\pi i/4}z)$
$u'' + \frac{1}{z}u' - \left[\frac{1}{z} + \left(\frac{\nu}{2z}\right)^2\right]u = 0$	$Z_\nu(2i\sqrt{z})$
$u'' + z^{-1}u' + \beta c^{i\alpha}u = 0$	$Z_0(\sqrt{\beta}\,e^{i\alpha/2}z)$
$u'' + [\beta c^{i\alpha} + (1/(4z^2))]u = 0$	$\sqrt{z}\,Z_0(\sqrt{\beta}\,e^{i\alpha/2}z)$
$u'' + \alpha z^\nu u = 0$	$\sqrt{z}\,Z_{\frac{1}{\nu+2}}\left(\frac{2\sqrt{\alpha}}{\nu+2}z^{\frac{\nu+2}{2}}\right)$
$u'' + (e^{2z} - \nu^2)u = 0$	$Z_\nu(e^z)$
$u'' + (e^{2/z} - \nu^2)z^{-4}u = 0$	$zZ_\nu(e^{1/z})$
$u'' + \left(\frac{1}{z} \mp 2i\right)u' - \left(\frac{\nu^2}{z^2} \pm \frac{i}{z}\right)u = 0$	$e^{\pm iz}Z_\nu(z)$
$u'' + u' - \frac{4\nu^2-1}{4z^2}u = 0$	$\sqrt{z}\,e^{-z/2}Z_\nu\left(\frac{iz}{2}\right)$
$u'' + \left(\frac{2\nu+1}{z} - \alpha\right)u' - \frac{\alpha(2\nu+1)}{2z}u = 0$	$z^{-\nu}e^{\alpha z/2}Z_\nu\left(\frac{i\alpha z}{2}\right)$
$u'' + \left(\frac{1-2\alpha}{z} \mp 2\beta\gamma i z^{\gamma-1}\right)u' + \left[\frac{\alpha^2-\nu^2\gamma^2}{z^2} \mp \beta\gamma(\gamma-2\alpha)iz^{\gamma-2}\right]u = 0$	$z^\alpha e^{\pm i\beta z^\gamma}Z_\nu(\beta z^\gamma)$
$u'' + \left(\frac{1}{z} - 2\tan z\right)u' - \left(\frac{\nu^2}{z^2} + \frac{\tan z}{z}\right)u = 0$	$\frac{1}{\cos z}Z_\nu(z)$
$u'' + \left(\frac{1}{z} + 2\cot z\right)u' - \left(\frac{\nu^2}{z^2} - \frac{\cot z}{z}\right)u = 0$	$\frac{1}{\sin z}Z_\nu(z)$
$u'' + \left(\frac{1}{z} - 2\psi\right)u' + \left(1 - \frac{\nu^2}{z^2} + \psi^2 - \psi' - \frac{\psi}{z}\right)u = 0$	$\exp\left(\int\psi dz\right)Z_\nu(z)$
$u'' + \left\{\frac{1}{2}\frac{\psi'''(z)}{\psi'(z)} - \frac{3}{4}\left(\frac{\psi''(z)}{\psi'(z)}\right)^2 + \left[(\psi(z))^2 + \frac{1}{4} - \nu^2\right]\left(\frac{\psi'(z)}{\psi(z)}\right)^2\right\}u = 0$	$\sqrt{\frac{\psi(z)}{\psi'(z)}}Z_\nu(\psi(z))$
$u'' - \frac{\varphi'(z)}{\varphi(z)}u' + \left\{\frac{3}{4}\left(\frac{\varphi'(z)}{\varphi(z)}\right)^2 - \frac{1}{2}\frac{\varphi''(z)}{\varphi(z)} - \frac{3}{4}\left(\frac{\psi''(z)}{\psi'(z)}\right)^2 + \frac{1}{2}\frac{\psi'''(z)}{\psi'(z)} + \left[(\psi(z))^2 + \frac{1}{4} - \nu^2\right]\left(\frac{\psi'(z)}{\psi(z)}\right)^2\right\}u = 0$	$\sqrt{\frac{\varphi(z)\psi(z)}{\psi'(z)}}Z_\nu(\psi(z))$
$u'' - \left[\frac{\psi''(z)}{\psi'(z)} + (2\alpha-1)\frac{\psi'(z)}{\psi(z)}\right]u' + [\alpha^2 - \nu^2 + (\psi(z))^2]\left(\frac{\psi'(z)}{\psi(z)}\right)^2u = 0$	$(\psi(z))^\alpha Z_\nu(\psi(z))$

(from 岩波公式)

Exercise.
Find the general solution to the following ODE

$$\frac{d^2u}{dz^2} + \left(\frac{1}{2} + \sinh^2 z - \frac{3}{4}(\tanh^2 z + \coth^2 z)\right) u = 0. \tag{27.119}$$

27.B Applications to Solving PDE

(1) A circular membrane of radius a is applied a uniform force $b \sin \omega t$ over the membrane. Find the forced oscillation.[361]
(2)[362] Consider a disc of radius a whose center is located at the origin in the xy-plane. The boundary is maintained at $T = 0$, and the initial temperature is given by

$$T(x, y, 0) = T_0 \left(1 - \frac{x^2 + y^2}{r^2}\right). \tag{27.120}$$

Assume that the thermal diffusivity is κ. Find $T(x, y, t)$. The solution is given in the form of Fourier-Bessel-Dini expansion (→**27A.22**). Compute the expansion coefficients explicitly with the aid of the following formula

$$\int_0^{\pi/2} d\phi \sin^{\mu+1}\phi \cos^{2\nu+1}\phi J_\mu(z \sin\phi) = \frac{2^\nu \Gamma(\nu+1)}{z^{\nu+1}} J_{\mu+\nu+1}(z). \tag{27.121}$$

(3) **Circular wave guide**: The equation for $\phi = B_z$ reads

$$(\Delta_2 + k^2)\psi = 0 \tag{27.122}$$

on $r = a$ with the boundary condition $\phi = 0$. The field can be separated as

$$\phi(r, \varphi) = B(r)e^{im\varphi}, \tag{27.123}$$

where $m \in \boldsymbol{Z}$ due to the univalency of the field. $B(r)$ obeys

$$\frac{d^2B}{dr^2} + \frac{1}{r}\frac{dB}{dr} + \left(k^2 - \frac{m^2}{r^2}\right) B = 0. \tag{27.124}$$

Therefore, $B = J_m(kr)$ is the eigenfunction.

[361] LSU82.
[362] L138.