26 Spherical Harmonics

Separation of variables of the Laplace equation in the spherical coordinates requires the spherical harmonic functions which make a complete orthonormal set of functions of spatial directions (i.e., functions on a unit sphere). Derivation of functional forms, the orthonormal relation, addition theorem related to the multipole expansion, and the application to PDE boundary value problems (potential problems) are discussed.

Key words: spherical harmonics, spherical harmonic function, addition theorem, multipole expansion, interior problem, exterior problem, annular problem

Summary:
(1) The angular part of the Laplacian in the spherical coordinates have the orthonormal eigenfunctions called spherical harmonics Y_n^m (**26A.8-9**). They are simultaneous eigenfunctions of the total and the z-component of the quantum mechanical angular momentum (**26A.10**).
(2) The addition theorem is used to decouple two spatial directions (**26A.12**), and applied to the multipole expansion of the electrostatic potential (**26A.14-15**).
(3) Spherical potential problems have different general expansion forms depending on the domain of the problem (**26B.2-5**).

26.A Basic Theory

26A.1 Separating variables in spherical coordinates. In the polar coordinate system, the 3-Laplacian reads (→**2D.10**)

$$\Delta = \frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{1}{r^2}L, \tag{26.1}$$

where

$$L = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \tag{26.2}$$

Separating the solution as $u(r,\theta,\varphi) = R(r)Y(\theta,\varphi)$, we get

$$\frac{d^2}{dr^2}rR(r) = l(l+1)\frac{R(r)}{r}, \tag{26.3}$$

$$LY(\theta,\varphi) \quad = \quad -l(l+1)Y(\theta,\varphi). \tag{26.4}$$

L is essentially the Laplacian on the unit sphere, and is a negative definite operator.

26A.2 Further separation of angular variables. Let us further assume $Y(\theta,\varphi) = \Theta(\theta)\Phi(\varphi)$. The φ-direction must be the periodic direction, so the equation for Φ must be an eigenvalue problem (cf. **23.9** or **18.2**). Hence,

$$\frac{d^2\Phi}{d\varphi^2} = -m^2\Phi, \tag{26.5}$$

and the rest is

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(l(l+1) - \frac{m^2}{\sin^2\theta}\right)\Theta = 0. \tag{26.6}$$

26A.3 Legendre's equation. If we introduce $x = \cos\theta$, the (26.6) reads

$$\frac{d}{dx}\left((1-x^2)\frac{d\Theta}{dx}\right) + \left(l(l+1) - \frac{m^2}{1-x^2}\right)\Theta = 0, \tag{26.7}$$

which is called (modified) *Legendre's equation.*

26A.4 $\boldsymbol{m} = 0$. For $m = 0$ Legendre's equation reads (→**24C.1**)

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] + l(l+1)\Theta = 0, \tag{26.8}$$

The general solution to this can be written as (→**24C.3**)

$$\Theta = AP_l(x) + BQ_l(x), \tag{26.9}$$

where P_l and Q_l are Legendre functions of first and second kind, respectively. Q_l is divergent at $x = \pm 1$, so that for a sphere problem this function should not appear. Furthermore, P_l is not finite at $x = 1$ if l is not an integer. Hence, we need P_n ($n \in \boldsymbol{N}$), the Legendre polynomials (→**21B.2**, **24C.2**(3)). That is, l must be a nonnegative integer (the eigenvalue problem has been solved).

26A.5 $m \neq 0$. For convenience **24C.5** is repeated here. If we define $Z(x)$ by

$$\Theta = (1-x^2)^{m/2}Z(x), \tag{26.10}$$

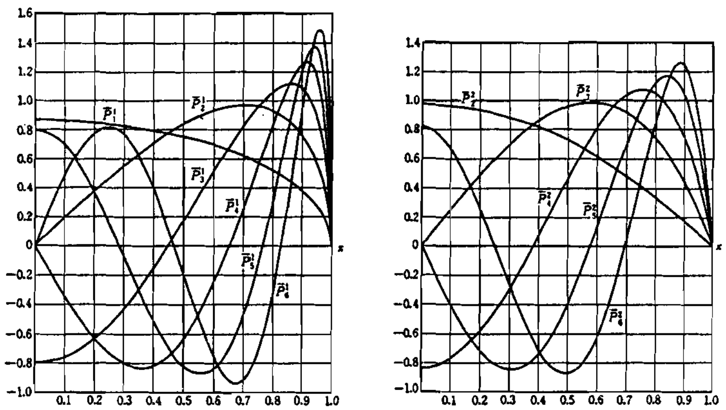
(26.7) becomes

$$(1-x^2)\frac{d^2Z}{dx^2} - 2(m+1)x\frac{dZ}{dx} + (n-m)(n+m+1)Z = 0. \quad (26.11)$$

This equation can be obtained by differentiating (26.7) m times. Therefore, the general solution of (26.7) is given by ($\rightarrow$**24C.5**)

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x), \; Q_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}Q_n(x). \quad (26.12)$$

These functions are called *associate functions* of P_n and Q_n. If we require that the solution is finite at $x = 1$, then P_n^m is the functions appearing in the solution.



$$\overline{P_n^m}(x) \equiv \sqrt{\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}}\, P_n^m(x) \qquad \text{cf } (26.24)$$

26A.6 Associate Legendre functions. If m is odd, then P_n^m is not a polynomial:

$$P_1^1(x) = (1-x^2)^{1/2} = \sin\theta, \quad (26.13)$$

$$P_2^1(x) = 3(1-x^2)^{1/2}x = 3\sin\theta\cos\theta = \frac{3}{2}\sin\theta, \quad (26.14)$$

$$P_2^2(x) = 3(1-x^2) = 3\sin^2\theta\frac{3}{2}(1-\cos 2\theta), \quad (26.15)$$

$$P_3^1(x) = \frac{3}{2}(1-x^2)^{1/2}(5x^2-1) = \frac{3}{8}(\sin\theta + 5\sin 3\theta), \quad (26.16)$$

$$P_3^2(x) = 15(1-x^2)x = \frac{15}{4}(\cos\theta - \cos 3\theta), \qquad (26.17)$$

$$P_3^3(x) = 15(1-x^2)^{3/2} = 15\sin^3\theta = \frac{15}{4}(3\sin\theta - \sin 3\theta), \qquad (26.18)$$

etc., where $x = \cos\theta$.

26A.7 Orthonormalization of associate Legendre functions. We have

$$\int_{-1}^{1} P_k^m(x) P_l^m(x) dx = \frac{(l+m)!}{(l-m)!}\frac{2}{2l+1}\delta_{k,l}. \qquad (26.19)$$

[Demo]. The LHS is, for $l > m$, $k > m$

$$f(m) \equiv \int_{-1}^{1}(1-x^2)^m \frac{d^m P_k}{dx^m}\frac{d^m P_l}{dx^m}dx \qquad (26.20)$$

$$= -\int_{-1}^{1}\frac{d^{m-1}P_k}{dx^{m-1}}\frac{d}{dx}\left((1-x^2)m\frac{d^m P_l}{dx^m}\right)dx. \qquad (26.21)$$

On the other hand, replacing m with $m-1$ and n with l in (26.11) and multiplying $(1-x^2)^{m-1}$, we get

$$\frac{d}{dx}(1-x^2)^m\frac{d^m P_l}{dx^m} = -(l+m)(l-m+1)(1-x^2)^{m-1}\frac{d^{m-1}P_l}{dx^{m-1}}. \qquad (26.22)$$

Hence, (26.21) implies

$$f(m) = (l+m)(l-m+1)f(m-1) = \cdots = \frac{(l+m)!}{(l-m)!}f(0). \qquad (26.23)$$

$f(0) = 2/(2l+1)$ is obtained from **21A.5**.

26A.8 Spherical harmonics. Now we can construct a complete orthonormal set of $L_2(S_2, \sin\theta)$ (S_2 is the unit 2-sphere) (→**20.19**). Let us define the kets $\{|l,m\rangle\}$ by (→**20.21**-)

$$\langle\theta,\varphi|l,m\rangle = Y_l^m(\theta,\varphi)$$

$$= (-)^{\{1+(-1)^m\}/2}\sqrt{\frac{2l+2}{2}\frac{(l-|m|)!}{(l+|m|)!}}P_l^{|m|}(\cos\theta)\frac{1}{\sqrt{2\pi}}e^{im\varphi}, \qquad (26.24)$$

where the ket $|\theta,\varphi\rangle$ satisfies (→**20.23**, **20.25**)

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta |\theta,\varphi\rangle \sin\theta \langle\theta,\varphi| = 1. \qquad (26.25)$$

and

$$\langle\theta,\varphi|\theta',\varphi'\rangle = \delta(\theta-\theta')\delta(\varphi-\varphi')/\sin\theta. \tag{26.26}$$

26A.9 Orthonormal relation for spherical harmonics. The decomposition of unity (→**20.15**) reads

$$1 = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} |l,m\rangle\langle l,m| \tag{26.27}$$

with the normalization

$$\langle l,m|l',m'\rangle = \delta_{l,l'}\delta_{m,m'}. \tag{26.28}$$

In the ordinary notation these formulas read (→**20.26-27**)

$$\frac{\delta(\theta-\theta')\delta(\varphi-\varphi')}{\sin\theta} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_l^m(\theta,\varphi)\overline{Y_l^m(\theta',\varphi')}, \tag{26.29}$$

and

$$\int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \sin\theta \overline{Y_l^m(\theta,\varphi)} Y_{l'}^{m'}(\theta,\varphi) = \delta_{l,l'}\delta_{m,m'}. \tag{26.30}$$

26A.10 Angular momentum. Quantum mechanically, $-\hbar^2 L^2$ is the total angular momentum operator. $|l,m\rangle$ is the simultaneous eigenket of the total angular momentum operator and the z-component of the momentum M_z:

$$(i\hbar)^2 L|l,m\rangle = \hbar^2 l(l+1)|l,m\rangle, \tag{26.31}$$

$$M_z|l,m\rangle = m|l,m\rangle. \tag{26.32}$$

26A.11 Spherical harmonic function. A function X of angular coordinates θ and φ is called a *spherical harmonic function* of order n, if $r^n X$ becomes a harmonic function (→**2C.11**). X satisfies

$$LX + n(n+1)X = 0, \tag{26.33}$$

where L is in **26A.1**. Because of the completeness (→**17.3**) of the spherical harmonics (essentially, its proof is in **37.1**), any spherical harmonic function of order n can be written as

$$X(\theta,\varphi) = \sum_{m=-n}^{n} A_m Y_m^n(\theta,\varphi). \tag{26.34}$$

26A.12 Addition theorem. Let γ be the angle between the directions specified by the angular coordinates (θ, φ) and (θ', φ').[356] Then,

$$P_n(\cos\gamma) = \frac{4\pi}{2n+1} \sum_{m=-n}^{n} \overline{Y_n^m(\theta', \varphi')} Y_n^m(\theta, \varphi). \tag{26.35}$$

This theorem allows us to decouple two directions.
[Demo]. Notice that $P_n(\cos\gamma)$ is a spherical harmonic function of order n (due to spherical symmetry), so that we can expand it as

$$P_n(\cos\gamma) = \sum_{m=-n}^{n} Y_n^m(\theta, \varphi) A_m(\theta', \varphi'). \tag{26.36}$$

The coefficients are fixed immediately from the following formula and the orthogonality of $\{Y_n^m\}$.

26A.13 Lemma. Let X be a spherical harmonic function of order n, and γ is the angle in **26A.12**. Then,

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta X(\theta, \varphi) P_n(\cos\gamma) = \frac{4\pi}{2n+1} X(\theta', \varphi'). \tag{26.37}$$

[Demo]. The integration is all over the sphere, so we can freely choose the $\theta = 0$ direction. Let us choose it to be the direction of (θ', φ'), and write the new angular coordinates as (γ, ψ). The integral we wish to compute becomes

$$I = \int_0^{2\pi} d\psi \int_0^{\pi} d\gamma \sin\gamma \hat{X}(\gamma, \psi) P_n(\cos\gamma), \tag{26.38}$$

where $\hat{X}$ is X in new variables. $\hat{X}$ is again a spherical harmonic function of order n (look at the spherical symmetry of (26.33)), so that it can be expanded as

$$\hat{X}(\gamma, \psi) = \sum_{m=-n}^{n} B_m Y_n^m(\gamma, \psi). \tag{26.39}$$

Hence,

$$I = \sqrt{\frac{4\pi}{2n+1}} B_0. \tag{26.40}$$

To calculate B_0 note the fact that $Y_n^m(0, \varphi) = 0$ if $m \neq 0$ (see the definition of P_n^m in **26A.5**), and $Y_n^0(0, \varphi) = \sqrt{(2n+1)/4\pi}$ ($P_n(1) = 1$ →**21B.5**(1)). Hence, from (26.39) we obtain

$$B_0 = \hat{X}_n(0, \psi) \sqrt{\frac{4\pi}{2n+1}} = X(\theta', \varphi') \sqrt{\frac{4\pi}{2n+1}}. \tag{26.41}$$

[356] We have

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi').$$

26A.14 Multipole expansion. Let $\rho(x)$ be the charge distribution. Then the potential due to this charge distribution with respect to the zero potential at infinity is given by

$$V(x) = \int dy \frac{\rho(y)}{4\pi\epsilon_0 |x-y|}. \tag{26.42}$$

If $\rho(x)$ vanishes for $|x| \ge R$, then

$$V(x) = \sum_{n=0}^{\infty} \frac{1}{\epsilon_0 R^{n+1}} \left[\sum_{m=-n}^{n} \frac{1}{2m+1} q_n^m Y_n^m(\theta, \varphi) \right], \tag{26.43}$$

where

$$q_n^m = \int_0^R dr \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi \, r^n \overline{Y_n^m(\theta,\varphi)} \rho(r,\theta,\varphi). \tag{26.44}$$

The expansion (26.43) is called the *multipole expansion.* □

[Demo]. Let the angle between x and y be γ, $R = |x|$ and $r = |y|$. Then

$$|x-y| = R\sqrt{1 - 2\zeta\cos\gamma + \zeta^2}, \tag{26.45}$$

where $\zeta = r/R$ (< 1). With the aid of the generating function of the Legendre polynomials (→**21A.9**), we get

$$\frac{1}{|x-y|} = \frac{1}{R}\sum_{n=0}^{\infty} P_n(\cos\gamma)\zeta^n. \tag{26.46}$$

Now we use the addition theorem **26A.12** to separate the x and y directions as

$$\frac{1}{|x-y|} = \frac{1}{R}\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n \left[\sum_{m=-n}^{n} \overline{Y_n^m(\theta',\varphi')} Y_n^m(\theta,\varphi)\right]. \tag{26.47}$$

Putting this into (26.42) and exchanging the order of summation and integration (→**19.11**), we get the desired formula.

26A.15 Lower order multipole expansion coefficients. For low order expansions, the Cartesian expression is much more popular. It reads

$$V(\boldsymbol{R}) = \frac{q}{R} + \frac{\boldsymbol{p}\cdot\boldsymbol{R}}{r^3} + \frac{1}{2}\frac{\sum_{i,j} Q_{ij} R_i R_j}{R^5} + \cdots, \tag{26.48}$$

where $\boldsymbol{R}$ is the position vector from the center of the charge distribution, q is the total charge, $\boldsymbol{p}$ is the dipole moment

$$\boldsymbol{p} = \int d\boldsymbol{x} \rho(\boldsymbol{x})\boldsymbol{x}, \tag{26.49}$$

and Q_{ij} is the *quadrupole moment tensor*

$$Q_{ij} = \int d\boldsymbol{x}(3x_i x_j - x^2\delta_{ij})\rho(\boldsymbol{x}). \tag{26.50}$$

In terms of these more familiar moments, we can write

$$\begin{aligned}
q_0^0 &= \frac{1}{\sqrt{4\pi}}q, && (26.51)\\
q_1^1 &= -\sqrt{\frac{3}{8\pi}}(p_x - ip_y), && (26.52)\\
q_1^0 &= \sqrt{\frac{3}{4\pi}}p_z, && (26.53)\\
q_1^{-1} &= \sqrt{\frac{3}{8\pi}}(p_x + ip_y), && (26.54)\\
q_2^2 &= \frac{1}{12}\sqrt{\frac{15}{2\pi}}(Q_{11} - 2iQ_{12} - Q_{22}), && (26.55)\\
q_2^1 &= -\frac{1}{3}\sqrt{\frac{15}{8\pi}}(Q_{13} - iQ_{23}), && (26.56)\\
q_2^0 &= \frac{1}{2}\sqrt{\frac{5}{4\pi}}Q_{33}, && (26.57)\\
q_2^{-1} &= \frac{1}{3}\sqrt{\frac{15}{8\pi}}(Q_{13} + iQ_{23}), && (26.58)\\
q_2^{-2} &= \frac{1}{12}\sqrt{\frac{15}{2\pi}}(Q_{11} + 2iQ_{12} - Q_{22}). && (26.59)
\end{aligned}$$

Note that, generally

$$q_n^m = \overline{q_n^{-m}}. \tag{26.60}$$

26.B Application to PDE

26B.1 Formal expansion of harmonic function in 3-space. **26A.1-3** and **26A.9** tell us that a harmonic function ψ (→**2C.11**) can have the following (formal)[357] expansion in 3-space in terms of spherical har-

[357] If we wish, we could say an expansion as a generalized function (→**14**).

monic functions:

$$\psi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} R_{lm}(r) Y_l^m(\theta, \varphi), \tag{26.61}$$

where $R_{lm}(r)$ obeys (→**26A.1**)

$$\frac{d^2}{dr^2} r R_{lm} = l(l+1)\frac{R}{r}. \tag{26.62}$$

Hence, R_{lm} has the following general solution (→**11B.14**)

$$R_{lm}(r) = A_{lm} r^l + B_{lm} r^{-l-1}. \tag{26.63}$$

That is, we get the following formal expansion:

$$\psi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (A_{lm} r^l + B_{lm} r^{-l-1}) Y_l^m(\theta, \varphi). \tag{26.64}$$

26B.2 Interior problem. A harmonic function on 3-ball of radius a centered at the origin must be finite at the origin, so its general form must be

$$\psi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} r^l Y_l^m(\theta, \varphi). \tag{26.65}$$

for $r \in [0, a]$.
(1) Dirichlet condition on the sphere. The solution to the Lapalce equation on the sphere with the boundary condition at the surface

$$\psi(a, \theta, \varphi) = V(\theta, \varphi) \tag{26.66}$$

must have the form of (26.65). Hence we must have

$$V(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} a^l Y_l^m(\theta, \varphi). \tag{26.67}$$

With the aid of the orthonormality in **26A.9**, we obtain

$$A_{lm} a^l = \int_0^{\pi} d\theta \, \sin\theta \int_0^{2\pi} d\varphi \overline{Y_m^l(\theta, \varphi)} V(\theta, \varphi). \tag{26.68}$$

(2) Neumann condition on the sphere. The solution to the Lapalce equation on the sphere with the boundary condition at the surface

$$\left.\frac{\partial \psi}{\partial r}\right|_{r=a} = E(\theta, \varphi). \tag{26.69}$$

Differentiating (26.65), we obtain

$$\frac{\partial\psi}{\partial r} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} l r^{l-1} A_{lm} Y_l^m(\theta,\varphi). \tag{26.70}$$

Hence, it is easy to obtain an explicit formula analogous to (1).

26B.3 Exterior problem. If the harmonic function outside of a sphere is bounded, then the solution must have the following form

$$\psi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} B_{lm} r^{-l-1} Y_l^m(\theta,\varphi). \tag{26.71}$$

B_{lm} are determined with the aid of orthonormality of spherical harmonics just as the interior problem.

26B.4 Uniqueness condition for exterior problem. We have discussed that if the domain D is not bounded, then the uniqueness condition is not trivial (→**1.19**, **29.9**). To study this, first we study the problem in the domain $D \cap V$, where V is a sphere of radius R. Suppose ψ_1 and ψ_2 are solutions to a given Dirichlet problem. Let $\psi = \psi_1 - \psi_2$. Then, it is a solution to a homogeneous Dirichlet problem. Green's formula tells us that

$$\int_{D\cap V} (grad\,\psi)^2 d\tau = \int_{\partial(D\cap V)} \psi\, grad\,\psi \cdot d\boldsymbol{S} = \int_{\partial V\cap D} \psi\, grad\,\psi \cdot d\boldsymbol{S}. \tag{26.72}$$

Hence, for the integral to vanish a sufficient condition is

$$|\psi| < const. R^{-1/2-\epsilon} \tag{26.73}$$

Boundedness of ψ is generally not enough to guarantee the unique solution.

26B.5 Annular problem. If the domain is a concentric sphere, the problem is called an annular problem. In this case both terms in R_{lm} in **26B.1** are needed. The boundary conditions on two spherical boundary surfaces allow us to determine the coefficients uniquely.

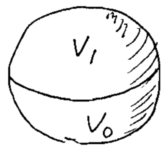
Exercise.
Find the harmonic function on the annular region $r \in [a, 3a]$ with the boundary conditions $u = \cos\phi$ on $r = a$ and $u = \cos 3\phi$ on $r = 3a$.

26B.6 Cylindrically symmetric case. If the system under consideration is independent of φ (→**24C.1**), then the general solution has the following formal expansion:

$$\psi(r,\theta,\varphi)=\sum_{l=0}^{\infty}(A_l r^l + B_l r^{-l-1})P_l(\cos\theta). \qquad (26.74)$$

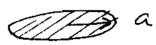
This is certainly a solution of the Laplace equation as can be seen from the result in **26B.1** (also **26B.8**). The uniqueness of the solution tells us that this is the general solution.

26B.7 Examples.
(1) A conducting sphere of radius a is separated into the upper and the lower halves. The upper half is maintained at potential V_1, and the lower at V_0. The electric potential outside the sphere is given by



$$V+\frac{V_1-V_0}{2}\frac{a}{r}-(V_1-V_0)\sum_{\mathrm{odd}\,l}(-1)^{(l-1)/2}\frac{2l+1}{\sqrt{2}}\left(\frac{a}{r}\right)^{l+1}\frac{(l-2)!!}{(l+1)!!}P_l(\cos\theta). \qquad (26.75)$$

(2) The electric potential due to uniformly charged disk of radius a. For $r>a$

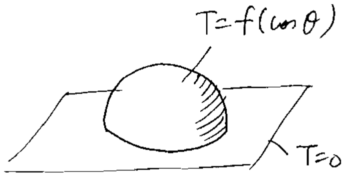


$$V=\frac{Q}{2\pi\epsilon_0 r}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2n-3)!!}{(2n)!!}\left(\frac{a}{r}\right)^{2n}P_{2n-1}(\cos\theta). \qquad (26.76)$$

Here Q is the total charge on the disk. For $r<a$ there is an extra complication, because $\theta=\pi/2$ is in the disk. However, for $\theta\in[0,\pi/2)$ there is no problem, and the solution is

$$V=\frac{Q}{2\pi\epsilon_0 r}\left[1-\frac{r}{a}P_1(\cos\theta)+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}(n-1)!n!}\left(\frac{r}{a}\right)^{2n}P_{2n-1}(\cos\theta)\right]. \qquad (26.77)$$

For $\theta>\pi/2$ we use the symmetry $V(r,\theta,\varphi)=V(r,\pi-\theta,\varphi)$.
(3) The equilibrium temperature distribution of a half ball of radius a with the surface temperature specified as $T=f(\cos\theta)$ and the bottom disk is maintained at $T=0$. In this case we use the reflection principle (→**16A.10**) to extend the problem to the whole ball. The boundary condition for the extended problems is given by $T_{r=a}=g(\cos\theta)$, where $g(x)=sgn(x)f(x)$. From the symmetry, the boundary condition on the bottom surface is automatically satisfied. The formal expansion of the interior problem with cylindrical symmetry (→**26B.6**) is given by **26B.3**, so the answer reads



$$T=\sum_{l=0}^{\infty}A_l r^l Y_l^0(\theta,\varphi) \qquad (26.78)$$

with

$$A_l a^l = \sqrt{(2l+1)\pi}\{1-(-1)^n\}\int_0^1 dx P_l(x) f(x). \tag{26.79}$$

Exercise.
(1) Find the gravitational potential due to a sphere of radius R with the density distribution given by

$$\rho = r^k X_m(\theta, \varphi), \tag{26.80}$$

where X_m is a spherical harmonics of order m ($\to$**26A.11**).
In this case due to the superposition principle, the potential V is given by

$$V(\boldsymbol{x}) = \int d^3\boldsymbol{y} \frac{\rho(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}. \tag{26.81}$$

Use (26.46) in **26A.14** to expand the Green's function. Then, use **26A.13** to perform the angular integral. In this way, we arrive at

$$V(\boldsymbol{x}) = \frac{4\pi}{2m+1}\frac{R^{m+k+3}}{m+k+3}\frac{1}{|\boldsymbol{x}|^{m+1}} X_m(\theta, \varphi). \tag{26.82}$$

(2) Discuss the waves in a thin spherical layer of radius R. The equation of motion is the wave equation written in the spherical coordinates with r suppressed ($r = R$).