25 Asymptotic Expansion

A formal expansion of a solution of a linear ODE discussed in the previous section around an irregular singular point gives generally a divergent series, but the series may still be useful as asymptotic series. Almost all the expansion series obtained by perturbation calculations in physics are divergent but asymptotic series. The famous perturbation series of QED are examples. We cannot uniquely reconstruct the function from its asymptotic series expansion in general, but we can with some auxiliary conditions. A famous example is the Borel summability.

Key words: asymptotic sequence, asymptotic series, optimal truncation, Watson's lemma, Laplace's method, Stirring's formula, acceleration of convergence, Borel sum, Borel transformation, Nevanlinna's theorem.

Summary:
(1) If Frobenius' method is blindly applied around an irregular singular point, we usually obtain divergent formal series, but they are often asymptotic (**25.1**). Most perturbation series in physics are only asymptotic (**25.17**).
(2) Divergence does not automatically mean asymptoticity; A series is an asymptotic expansion of a function, if the truncation error at the n-th order is smaller than the n-th order term (**25.3**). Therefore, its optimal truncation (**25.5**) is practically very useful.
(3) Computation involving asymptotic series can be performed termwisely except differentiation (**25.10**).
(4) There are several standard methods to obtain the asymptotic expansion of functions and integrals (**25.11-13**, **25.15**).
(5) The asymptotic expansion (in terms of a given asymptotic sequence) of a function is unique (**25.6**), but an asymptotic series cannot uniquely determine a function (**25.7**).
(6) However, if the function satisfies certain auxiliary conditions, then it can be recovered from the asymptotic series. The most important condition is the Borel summability (Nevanlinna's theorem **25.20**). In this case the Borel summation allows us to reconstruct the function (**25.18-20**).

25.1 Irregular singularity and divergence. Try to solve (24.8) following Frobenius (**24B**) blindly, assuming that $x = 0$ is an irregular singular point (→**24B.2**):

$$u(x) = x^\lambda \sum_{k=0}^{\infty} c_k x^k. \tag{25.1}$$

Formally, we get a set of formulas for c_k and λ as in **24B.3**. If, fortunately, $c_l = 0$ for all l larger than some N, we can get a regular solution. However, this is an accidental case, and usually we can prove that for some $k > 0$

$$\lim_{n\to\infty} \left| \frac{c_{n-k}}{c_n} \right| = 0, \tag{25.2}$$

that is, the series (25.1) is divergent.[348] However, the resultant divergent series may be used as an *asymptotic series* around $x = 0$.

25.2 Asymptotic sequence. Let $\{\phi_n(x)\}$ be an infinite sequence of continuous functions. If $\phi_{n+1}(x) = o[\phi_n(x)]$ around x_0, i.e.,

$$\lim_{x\to x_0} \phi_{n+1}(x)/\phi_n(x) = 0, \tag{25.3}$$

for all $n > 0$, the sequence is called an *asymptotic sequence* (around x_0).

25.3 Asymptotic series. Let $\{\phi_n\}$ be an asymptotic sequence around x_0. Then, the following formal series

$$a_0\phi_0(x) + a_1\phi_1(x) + \cdots + a_n\phi_n(x) + \cdots \tag{25.4}$$

is called an *asymptotic series* for a function f at x_0, if for each fixed n

$$f(x) = a_0\phi_0(x) + a_1\phi_1(x) + \cdots + a_n\phi_n(x) + o[\phi_n(x)] \tag{25.5}$$

as $x \to x_0$. That is, if

$$\lim_{x\to x_0} \frac{f(x) - \sum_{k=0}^{n} a_n\phi_n(x)}{\phi_n(x)} = 0 \tag{25.6}$$

for all n, we say (25.5) is an *asymptotic expansion* of f around x_0 in terms of asymptotic function sequence $\{\phi_j\}$, and write

$$f(z) \sim a_0\phi_0(z) + a_1\phi_1(z) + \cdots + a_n\phi_n(z) + \cdots. \tag{25.7}$$

Discussion.
(A) Let $A_a(\alpha,\beta)$ denote the angular region

$$A_a(\alpha,\beta) \equiv \{z \,|\, \alpha < Arg(z-a) < \beta\}, \tag{25.8}$$

[348] See E L Ince, *Ordinary Differential Equations* (Dover, 1956; original 1926), p422. Also see W R Wasow, *Asymptotic Expansions for Ordinary Differential Equations* (Intescience, 1965).

where Arg is the principal argument (→**a4.7**). We say a function f is expanded in the (generalized) asymptotic power series around a in the angular region $A_a(\alpha, \beta)$, if (25.5) hold when $z \to a$ is taken inside the angular region. The boundary of the maximal angular region where a given asymptotic expansion holds is called a Stokes line (→**25.8**).
(B) The Stirling formula **9.11** is admissible in the angular region $A_0(-\pi, \pi)$. This can be shown with the aid of

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \log \sqrt{2\pi} - \frac{1}{\pi} \int_0^\infty \frac{z}{z^2 + t^2} \log(1 - e^{-2\pi t}). \quad (25.9)$$

25.4 Example. A typical example is:

$$F(x) = \int_0^\infty \frac{e^{-t/x}}{1+t} dt \sim \sum_{n=0}^\infty (-1)^n n! x^{n+1}. \quad (25.10)$$

This is an asymptotic series around $x = 0$. If $x = 1/2$, then the series read

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{4} - \frac{3}{8} + \frac{3}{4} + \cdots. \quad (25.11)$$

This is hardly useful. However, if x is small, then the series should be usable as a numerical tool:[349]

$$F(0.1) \sim \frac{1}{10} - \frac{1}{100} + \frac{2}{1000} - \frac{6}{10000} + \cdots. \quad (25.12)$$

25.5 Optimal truncation of asymptotic series. As is clear from the definition **25.3**, to evaluate $f(\epsilon)$, if we truncate the asymptotic sequence at the n-th order, then the error (i.e., the difference between the true value and the estimate obtained from the truncated series) must be smaller than $a_n \phi_n(\epsilon)$. Hence, for a given ϵ we can find an optimal n to truncate the series by looking for n which minimizes $a_n \phi_n(\epsilon)$.

For example, for (25.10), with the aid of Stirling's formula (→**9.11**, also see **25.14**)

$$n! x^{n+1} \sim e^{(n+1)\, ln x + n \ln(n/e)}. \quad (25.13)$$

Hence, $n \sim 1/\epsilon$ gives the optimal truncation position.

Discussion: How to efficiently compute series.
(1) **Euler transformation.** Let

$$f(x) = \sum_{n=0}^\infty a_n x^n \quad (25.14)$$

[349] Read a conversation between a numerical analyst and an asymptotic analyst on p19 of N. G. de Bruijn, *Asymptotic Methods in Analysis* (Dover, 1958, 1981).

be a convergent series. Define the difference operator D as

$$\mathrm{D}a_n = a_{n+1} - a_n. \tag{25.15}$$

Then,

$$f(x) = (1-x)^{-1}a_0 + (1-x)^{-1}\sum_{n=0}^{\infty} \mathrm{D}a_n x^{n+1}. \tag{25.16}$$

This transformation is called the *Euler transformation.* Practically it is wise to use this beyond some finite terms.

(2) **Subtraction trick.** The above idea may be understood as subtracting the expansion of $(1-x)^{-1}a_0$ from $f(x)$. If we could find a function g which is close to f and easily expandable analytically, then considering $f-g$ may be a good idea to compute the series for f. For example, to compute

$$f = \sum_{n=0}^{\infty} \frac{1}{(1+n^2)}, \tag{25.17}$$

it is advantageous to use the knowledge

$$\sum_{n=0}^{\infty} \frac{1}{n(n+1)} = 1. \tag{25.18}$$

Hence,

$$f - 1 = \sum_{n=0}^{\infty} \frac{n-1}{n(n+1)(n^2+1)}. \tag{25.19}$$

(3) We wish to compute

$$S = \sum_{n=1}^{\infty} \frac{1}{1+n^2} \tag{25.20}$$

(i) The remainder satisfies the following inequalities

$$\int_N^{\infty} \frac{dx}{1+x^2} < S_N \equiv \sum_{n=N}^{\infty} \frac{1}{1+n^2} < \int_{N-1}^{\infty} \frac{dx}{1+x^2}. \tag{25.21}$$

Using this, find the necessary number of terms to obtain S within a 0.01% error.

(ii) Now we use the subtraction trick ($\rightarrow$**D7.5**) with the aid of

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}. \tag{25.22}$$

What N do you need to obtain the same accuracy?

(4) The same idea works for integrals as well. Consider

$$I(\epsilon) = \int_0^1 \frac{1}{\sqrt{\epsilon + x}} dx. \tag{25.23}$$

In this case $I(0) = 2$ is easy, so let us subtract $1/\sqrt{x}$:

$$\int_0^1 \left[\frac{1}{\sqrt{\epsilon + x}} - \frac{1}{\sqrt{x}}\right] dx. \tag{25.24}$$

Introducing $u = x/\epsilon$ (*rescaling trick*), we realize that this integral is of order $\epsilon^{1/2}$. The integration range may be replaced by $[0, \infty)$ to the lowest nontrivial order.

25.6 Uniqueness of asymptotic expansion. The asymptotic expansion up to a given number of terms of a given function is unique if an asymptotic sequence is specified. □
This follows from the explicit formula for the coefficients:

$$a_n = \lim_{x \to x_0} \frac{f(x) - \sum_{k=0}^{n-1} a_k \phi_k(x)}{\phi_n(x)}. \tag{25.25}$$

25.7 Warning. However, an asymptotic series cannot uniquely determine a function. $(1+x)^{-1}$, $(1+e^{-x})/(1+x)$ and $(1+e^{-\sqrt{x}}+x)^{-1}$ all have the same asymptotic expansion $\sum(-1)^{n-1}x^{-n}$ $(x \to \infty)$ (Demonstrate this statement). If we try to asymptotically expand $e^{-1/x}$ in terms of the asymptotic sequence $\{x^n\}$ $(x \to 0)$, all the coefficients vanish, but obviously the function is not equal to 0. Hence, we cannot generally recover a function from its asymptotic expansion, because transcendentally small terms are ignored by asymptotic expansion.

25.8 Stokes line. The transcendentally small term $e^{-1/x}$ $(x \to +0)$ cannot be seen through asymptotic expansions as seen in **25.7**. However, obviously this is no more small for $x < 0$. Hence, if we consider the function $f(x)$ as a function $f(z)$ of the complex variable z instead of x, then its 'expandability into asymptotic series' should change drastically according to the sectors or regions on the complex plane. The occurrence of this drastic change is called *Stokes' phenomenon* and the boundary of these regions is called a *Stokes line.* The existence of this phenomenon signifies nonconvergent asymptotic series.

25.9 Convergent power series is asymptotic. If $f(x)$ is Taylor-expandable at $x = a$ (i.e., is analytic (→**7.1**) around a), then the Taylor series is an asymptotic series. Conversely, if $f(x)$ is holomorphic (→**5.4**) and single valued in $0 < |x - a| < r$ for some positive r, then a is a removable singularity (→**8A.5(4)**(i)), and the asymptotic series is the Taylor series of f around a.[350]

25.10 Operations with asymptotic series.
(1) Termwise addition and subtraction of two asymptotic series (with the same asymptotic sequence **25.2**) is again an asymptotic series.
(2) In the case of power series $f \sim \sum a_n x^n$ and $g \sim \sum b_n x^n$, their product fg has the asymptotic power series $\sum c_n x^n$ with $c_n = \sum_{r=0}^{n} a_{n-r} b_r$.

[350] *Encyclopedic Dictionary of Mathematics* vol I p124-6.

(3) Also for power series the asymptotic series of $f(g)$ is obtained from that of f and g by substitution.
(4) The termwise integration of the power asymptotic series is the asymptotic series of the integral:

$$\int_0^x f(x)dx \sim \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}. \tag{25.26}$$

(5) However, termwise differentiation may not be allowed. A famous counter example is $e^{-1/x}\sin(e^{1/x})$, which has 0 as its asymptotic power series as guessed easily from **25.7**, but its derivative cannot be expanded in powers.
(6) Termwise differentiation is allowed if the derivative of the function also has an asymptotic expansion. See the Discussion below.

Discussion.
In this case, if f is holomorphic near a in the angular region and has an asymptotic power series, then termwise differentiation is allowed so long as a is reached within $A_a(\alpha, \beta)$ ($\rightarrow$**25.3** Discussion (A)).

25.11 How to obtain expansion I: Integration by parts

(1) Let us estimate the tail of the normal distribution

$$G(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-y^2/2} dy. \tag{25.27}$$

Integrating by parts, we get

$$\sqrt{2\pi} G(x) = \frac{1}{x} e^{-x^2/2} - \int_x^{\infty} e^{-y^2/2} dy. \tag{25.28}$$

From this we easily get

$$\frac{x}{1+x^2} e^{-x^2/2} \le \sqrt{2\pi} G(x) \le \frac{1}{x} e^{-x^2/2}. \tag{25.29}$$

This suggests that $G(x)\exp x^2/2$ can be asymptotically expanded in powers of x^{-1}. See also **25.12**.
(2)

$$-Ei(-t) \equiv \int_t^{\infty} \frac{e^{-s}}{s} ds \sim \frac{e^{-t}}{t} - \int_t^{\infty} \frac{e^{-s}}{s^2} ds \tag{25.30}$$

etc., gives an asymptotic expansion.
(3) The decay rate of the Fourier expansion coefficients of a C^k- function discussed in **17.14** is an application of this method thanks to the

Riemann-Lebesgue lemma (→**17.11**).
(4) Fourier expansion of piecewise C^k-functions. To compute

$$\int_{-\infty}^{\infty} f(x)e^{i\omega x}dx \tag{25.31}$$

we decompose the integration range into piecewise C^k sections, and then estimate the integral asymptotically by integration by parts (again thanks to the Riemann-Lebesgue lemma) in each section.

Exercise.
(A) Find the asymptotic expansion of Fresnel integrals

$$C(x) \equiv \int_0^x \cos\frac{\pi u^2}{2}du;\ S(x) \equiv \int_0^x \sin\frac{\pi u^2}{2}du; \tag{25.32}$$

[Hint. Use $\int_0^\infty$ →**8B.8**.]
(B) Approximate estimation of integrals[351]
(1)

$$I(x) = \int_0^x e^{t^2}\frac{dt}{\sqrt{x^2-t^2}}. \tag{25.33}$$

For $x \ll 1$, we may replace $e^{t^2} \simeq 1$. For $x \gg 1$, we introduce $\xi = x - t$, and

$$I(x) = e^{x^2}\int_0^x e^{-2\xi x+\xi^2}\frac{d\xi}{\sqrt{2\xi x-\xi^2}}. \tag{25.34}$$

Plotting the exponent in the integrand, we realize that the exponential factor is the largest when $\xi = 0$, so that

$$I(x) \simeq e^{x^2}\int_0^x e^{-2\xi x}\frac{d\xi}{\sqrt{2x\xi}} \simeq \frac{e^{x^2}}{2x}\int_0^\infty e^{-z}\frac{dz}{\sqrt{z}} \sim \frac{e^{x^2}}{2x}. \tag{25.35}$$

(2)

$$I(a,b) = \int_0^\infty e^{-ax^2}\sin^2 bx dx. \tag{25.36}$$

This can be rewritten as

$$I(a,b) = \frac{1}{\sqrt{a}}\int_0^\infty e^{-z^2}\sin^2\left(\frac{b}{\sqrt{a}}z\right)dz. \tag{25.37}$$

If $b \gg \sqrt{a}$, then the sine factor oscillates very rapidly, so we may replace it with its average value 1/2. Therefore, $\simeq 1/\sqrt{a}$. Compare this with the exact value of I.

25.12 How to obtain asymptotic series II: Watson's lemma.

Consider the following Laplace integral[352]

$$F(s) = \int_0^\infty e^{-st}f(t)dt. \tag{25.38}$$

[351] Migdal
[352] F is the Laplace transform of f (→**33**).

Assume that $f(t)$ has a power series expansion

$$f(t) = \sum_{n=0}^{\infty} a_n t^n \tag{25.39}$$

with the radius of convergence R. Replace f in the integral (25.38) with its series expansion (25.39), and perform the integration termwisely. Then we get the following formal result:

$$F(s) \text{ `} = \text{'} \sum_{n=0}^{\infty} \frac{a_n}{s^{n+1}}. \tag{25.40}$$

Watson's Lemma. If there is $a > 0$ such that $|f(t)| = O[e^{at}]$ for sufficiently large t, then (25.40) is actually an asymptotic expansion of F around $s = \infty$. □[353]

Example. An asymptotic expansion of the error function may easily be obtained with the help of Watson's lemma:

$$Erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt = \frac{2}{\sqrt{\pi}} e^{-x^2} \int_0^{\infty} e^{-2xt-t^2} dt. \tag{25.41}$$

Now introduce $u = xt$ and expand e^{-u^2/x^2} in power series.

$$Erfc(x) = \frac{2e^{-x^2}}{x\sqrt{\pi}} \int_0^{\infty} e^{-2u} \left(1 - \frac{u^2}{x^2} + \frac{u^4}{2x^4} - \frac{u^6}{6x^6} + \cdots\right) du. \tag{25.42}$$

This lemma can be used to estimate the asymptotic form of Fourier transforms as well.

Exercise.
(1) Show for $x > 0$

$$\int_0^{\infty} \frac{e^{-xt}}{1+t^2} dt = \frac{1}{x} - \frac{2!}{x^2} + \frac{4!}{x^4} - \cdots. \tag{25.43}$$

[Hint. Use $s = xt$ as a new integration variable.]
(2) Show

$$\int_0^{\infty} \frac{e^{-xt}}{1+t^3} dt \sim \sum_{n=0}^{\infty} (-1)^n \frac{(3n)!}{x^{3n+1}}. \tag{25.44}$$

(3) The asymptotic expansion of Ci and si:

$$Ci(x) \equiv \int_x^{\infty} \frac{\cos t}{t} dt, \; si(x) \equiv \int_x^{\infty} \frac{\sin t}{t} dt. \tag{25.45}$$

[353] For a proof see B. Friedman, *Lectures on Application-Oriented Mathematics* (Wiley, 1969), p78.

This is the real and imaginary parts of

$$J(x) = \int_x^\infty \frac{e^{it}}{t} dt = \int_0^\infty \frac{e^{i(x+u)}}{x+u} du. \tag{25.46}$$

Apply Watson's theorem to obtain the asymptotic expansions of these functions. Of course, repeated integration by parts should also work as can be guessed from the example in **D25.11**.

(4) (This problem need not be here.) Find the asymptotic expansion in the $x \to \infty$ limit of

$$E_n(x) = \int_1^\infty t^{-n} e^{-xt} dt \tag{25.47}$$

in powers of $1/x$ For $n = 1$, what is the optimum truncation of the resultant asymptotic series to compute $E_1(N)$?

25.13 How to obtain asymptotic series III: Laplace's method. Consider

$$F(\theta) = \int_{-\infty}^{+\infty} e^{\theta h(x)} dx, \tag{25.48}$$

where h is a real C^2-function with the following properties:
(i) $h(0) = 0$ is an absolute maximum of h, and $h < 0$ for any nonzero x.
(ii) There are positive constant a and b such that $h \le -a$ for $|x| \ge b$.
We must of course assume that the integral converges for sufficiently large θ. Then, in the $\theta \to \infty$ limit, we get

$$F(\theta) \sim \sqrt{2\pi}(-\theta h''(0))^{-1/2}. \tag{25.49}$$

25.14 Gamma function and Stirling's formula. Although we can apply Watson's lemma to get the asymptotic expansion of Gamma function (→**9.1**), it is not very easy, so we use the Laplace method. Substituting $t = z(1+x)$ in (9.3), we get

$$\Gamma(z+1) = e^z z^{z+1} \int_{-1}^\infty \left[e^{-x}(1+x)\right]^z dx. \tag{25.50}$$

h in **25.13** reads $-x + \ln(1+x)$, so it satisfies the condition of Laplace's method, and $h''(0) = -1$. Hence, we get

$$n! \sim \sqrt{2\pi} e^{-n} n^{n+1/2}, \tag{25.51}$$

which is the famous *Stirling's formula* (→**9.11**) obtained by Laplace in this way.

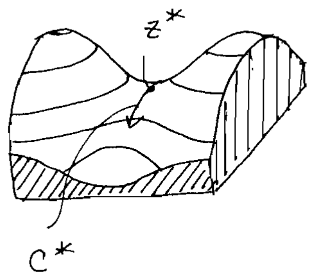
25.15 How to obtain asymptotic series IV: Method of steepest

descent. This is perhaps the most famous method to obtain asymptotic expansions of integrals. The principle is explained as follows. We wish to compute the following contour integral on the complex plane

$$I = \int_C G(z)e^{tf(z)}dz, \tag{25.52}$$

where C is a contour from infinity to infinity on the complex plane such that on both ends the holomorphic function (→**5.4**) f goes to $-\infty$. G is also assumed to be holomorphic and t is a large positive constant. Let us split f into its real and imaginary parts as $f = \phi + i\psi$. Since ϕ is a harmonic function (→**5.6**), it can have a saddle point (→**29.6**) z^*, which satisfies $f'(z^*) = 0$. Modify the contour C to C^* so that it can pass through z^* and parallel to $grad\,\phi$ near z^*. Along this pass

$$f(z) = f(z^*) + \frac{1}{2}(z - z^*)^2 f''(z^*) + \cdots \tag{25.53}$$



and ψ must be almost constant, because the Cauchy-Riemann equation (→**5.3**) tells us that gradients of ϕ and ψ are orthogonal. Hence the second term in the above expansion along C^* near z^* must be real non-positive. We may introduce a real coordinate ζ such that $f(z) = f(z^*) - \zeta^2/2 + \cdots$. Let α be the angle between the real axis and the tangent of C^* at z^*. Then

$$\frac{d\zeta}{dz} = e^{i\alpha}\sqrt{|f''(z)|}. \tag{25.54}$$

Changing the integration variable, we get

$$I = e^{tf(z^*)}G(z^*)e^{-i\alpha}\left|\frac{2\pi}{tf''(z^*)}\right|^{1/2}. \tag{25.55}$$

25.16 Acceleration or improvement of asymptotic series. If we could convert the asymptotic series around 0 in powers of x into another asymptotic sequence which is in terms of an asymptotic sequence converging much more quickly to 0 than x^n, then the asymptotic estimate should become much more accurate. An example is given here.[354] Consider (25.10)

$$F(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} + \cdots. \tag{25.56}$$

[354]See, for example, C. N. Moore, *Summable Series and Convergence Factors* (Dover, 1966).

We wish to convert this into the power series in $y = \phi(x)$. We assume $y/x \to 1$ in the $x \to \infty$ limit, and the Taylor-expandability: $x/y = 1 + a/y + b/y^2 + \cdots$. Substituting this into (25.56), we get

$$\begin{aligned} F(x) &= \frac{1}{y+a+b/y} - \frac{1}{(y+a)^2} + \cdots \\ &= \frac{1}{y}\left(1 - \frac{a}{y} + \frac{a^2}{y^2} - \frac{b}{y^2} + \cdots\right) - \frac{1}{y^2}\left(1 - \frac{2a}{y} + \cdots\right) + \cdots. \end{aligned} \tag{25.57}$$

Hence, choosing $a = -1$, we can kill the $1/y^2$ term. That is, we get

$$F(x) = \frac{1}{x+1} + O\left[\frac{1}{(x+1)^3}\right]. \tag{25.58}$$

This is much better than the original expansion for $x \gg 1$. Of course, one should not believe that the improvement is increasingly better if we continue this procedure indefinitely; the outcome is still an asymptotic expansion.

Discussion.
Consider the summation

$$S = \sum_{r=1}^{\infty} f(r), \tag{25.59}$$

where f is well-behaved. Let $S(n)$ be the partial sum up to the n-th term. Then, often

$$S(n) = S + \frac{B}{n} + \frac{C}{n^2} + o[n^{-2}]. \tag{25.60}$$

This can be used to estimate S from partial sums.

A variant of this idea is the estimation of integral from numerical integration with increment h. Let the integral be I and its approximately computed value with the increment h be $I(h)$. Then, often

$$I(h) = I + Bh + Ch^2 + o[h^2]. \tag{25.61}$$

25.17 Most perturbation series in physics are at best asymptotic. In field theory and statistical mechanics, in many cases we can perform analytical work only with the aid of some sort of perturbation techniques. The resultant perturbation series are usually divergent. Physicists often claim that they are asymptotic, but divergence does not automatically mean that the series is asymptotic. Hence, we have two problems: (1) To show that the series is asymptotic and (2) To recover the desired quantity from the asymptotic series. As we have seen in **25.7**, (2) is impossible without some auxiliary knowledge about the function. Read Fejer's theorem (→**17.10**) for Fourier series. A certain summation method may recover the original function from a divergent series under an appropriate auxiliary conditions. (For Fejer's theorem the needed auxiliary condition is the continuity of the function.) Thus, we may expect that a function satisfying certain auxiliary condition could be recovered from its asymptotic series by a particular summation method. A representative method is the Borel summation (→**25.18**). Often the perturbation series in field theory are proved to be *Borel summable* (i.e., the original quantity can be recovered from its asymptotic series as a Borel sum).

25.18 Borel transform. Even if the RHS of

$$f(z) \sim \sum_{n=0}^{\infty} a_n z^n \tag{25.62}$$

diverges, its "Borel sum"

$$B(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \tag{25.63}$$

may converge. $B(t)$ is called the *Borel transform* of the series (25.62).

25.19 Heuristics. Consider

$$\frac{1}{z}\int_0^\infty \frac{t^n}{n!} e^{-t/z} dt = z^n. \tag{25.64}$$

Inserting this into (25.62), and formally changing the order of intergration and summation, we obtain

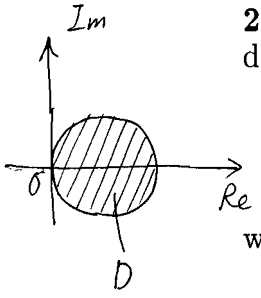
$$f(z) = \frac{1}{z}\int_0^\infty B(t)e^{-t/z} dt = \int_0^\infty B(\lambda z)e^{-\lambda} d\lambda. \tag{25.65}$$

Essentially, the Laplace transform (→**33**) of $B(t)$ is the desired function.

Exercise.
(1) Apply this to $(1+x)^{-1} \sim \sum(-)^n x^n$.
(2) We can asymptotically expand as (show it)

$$Erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2} dt \sim \frac{2}{\sqrt{\pi}}\frac{e^{-x^2}}{x}\sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{n!(2x)^{2n}}. \tag{25.66}$$

Apply the Borel summation method to this series and recover the error function.



25.20 Nevanlinna's theorem. Let $f(z)$ be analytic on the open disc D in the figure, and its asymptotic expansion satisfies

$$f(z) = \sum_{k=0}^{n-1} a_k z^k + R_n(z) \tag{25.67}$$

with

$$|R_n(z)| \le \text{const.}\sigma^n n! |z|^n \tag{25.68}$$

uniformly for all n and all $z \in D$ for some positive σ. Then (25.67) is Borel summable (→**25.17**). That is, the Borel transform $B(t)$ of the series converges for $|t| < \sigma^{-1}$ and can be analytically continued to an analytical function $B(t)$ (→**7.10**) on the strip containing the entire positive real axis. From this f can be recovered as[355]

$$f(z) = \frac{1}{z}\int_0^\infty B(t)e^{-t/z}dt. \tag{25.69}$$

[355] For an elegant proof see A D Sokal, J. Math. Phys. **21**, 261-3 (1980). However, this is not the general form given by the original author. For applications, see, for example, Itzykson and Zuber, *Quantum Field Theory* (McGraw-Hill, 1980), Section 9.4. J. Zinn-Justin. *Quantum Field Theory and Critical Phenomena* (Clarendon Press, 1989) Section 27.