24 General Linear ODE

The theory of general linear ODE is summarized, and then a constructive solution method (Frobenius' method) is outlined. This series method is best implemented with the aid of symbol manipulation programs. The reader should practice the method for one or two representative examples by hand or a step by step application of mathematics softwares.

Key words: analyticity of solution, fundamental system of solutions, fundamental matrix, Wronskian, separation theorem, Frobenius' theory, (regular and irregular) singular point, indicial equation, index.

Summary:
(1) First-order n-vector continuous ODE preserves the linear independence of the initial condition vectors (the existence of fundamental systems **24A.4**, **24A.11**).
(2) If the coefficient functions are holomorphic around x, then the solution around x is Taylor-expandable, so a series form fundamental system can be constructed (**24B.1**). Even if the coefficients are not holomorphic, if their singularities are not very bad (regular **24B.2**), then still a series form fundamental system can be constructed (Frobenius' theory) (**24B.3-7**).
(3) The Frobenius method is best implemented by a computer. See **24B.8** for a 'practical Frobenius.'
(4) Separation of variables of the Laplace equation in the spherical coordinates requires Legendre polynomials (**24C.1-2**) and associate Legendre functions (**24C.5**, examples in **26B**).

24.A General Theory

24A.1 The problem. We must be able to solve separated equations (→**23**) which are usually ODE. They are linear but with nonconstant coefficients. We know we have only to consider (→**11A.5**)

$$\frac{d\boldsymbol{u}(x)}{dx} = A(x)\boldsymbol{u}(x), \tag{24.1}$$

where $A(x)$ is a $n \times n$ matrix which is continuous[339] on an interval $I \subset \boldsymbol{R}$.

24A.2 Theorem [Unique existence of solution]. If $A(x)$ is continuous[340] in an open interval $I \subset \boldsymbol{R}$, then for any $\boldsymbol{u}_0 \in \boldsymbol{R}^n$ and $x_0 \in I$, there is a unique solution $\boldsymbol{u}(x)$ passing through $(\boldsymbol{u}_0, x_0)$ whose domain is I. □
This follows directly from the Cauchy-Peano and Cauchy-Lipschitz theorems (→**11A.8**, **11A.10**).

24A.3 Analyticity of solution. $A(x)$ may be considered to be a matrix consisting of functions on $\boldsymbol{C}$ as $A(z)$.
Theorem. Assume $A(z)$ to be analytic (i.e., all the components are analytic functions →**7.1**, **7.10**) in $D \subset \boldsymbol{C}$. Then, a solution analytic around $a \in D$ can be analytically continued (→**7**) to any point in D along any curve in D. □

This implies that the singular points of a solution, if any, appear where there are singularities (→**8A.2-7**) of $A(z)$.

Discussion.
For 1D Schrödinger equation, the wave function is finite at a point which is not a singularity of the potential. For example, the wave function of the harmonic oscillator is finite for finite x. For the Coulomb potential, the singularity can exist only at the origin.

24A.4 Theorem [Fundamental system of solutions]. The totality of solutions of (24.1) makes a n-vector space. Any basis set of this space is called the *fundamental system of solutions.* □
[Demo] Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ be linearly independent vectors and $x_0 \in I$. Write the solution passing through $(\boldsymbol{v}_j, x_0)$ as $\boldsymbol{\phi}_j(x)$ $(j = 1, \cdots, n)$. Let $\boldsymbol{u}_0 = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n$, and

$$\boldsymbol{u}(x) = c_1\boldsymbol{\phi}_1(x) + c_2\boldsymbol{\phi}_2(x) + \cdots + c_n\boldsymbol{\phi}_n(x). \tag{24.2}$$

It is obvious that the space cannot have a dimension larger than n. If there is x such that $\boldsymbol{u}(x) = 0$, then due to the uniqueness of the solution (→**24A.2**) it must agree with the solution starting from 0, which is obviously identically zero, so that $\boldsymbol{u}(x)$ can never be 0. Hence, the dimension of the solution space cannot be less than n. □
Notice that this theorem implies that $\boldsymbol{\phi}_1(x), \boldsymbol{\phi}_2(x), \cdots, \boldsymbol{\phi}_n(x)$ are functionally independent: the identity for $x \in I$

$$c_1\boldsymbol{\phi}_1(x) + c_2\boldsymbol{\phi}_2(x) + \cdots + c_n\boldsymbol{\phi}_n(x) \equiv 0 \tag{24.3}$$

[339] We say that $A(x)$ is continuous, analytic, etc., if all its components are, as functions, continuous, analytic, etc.
[340] Our problem is a linear problem, so this is enough. A related discussion is in **11A.10** Discussion (B).

implies $c_j = 0$ for all j.[341]

24A.5 Fundamental matrix. The matrix $\Phi(x) = (\boldsymbol{\phi}_1(x), \boldsymbol{\phi}_2(x), \cdots, \boldsymbol{\phi}_n(x))$ is called a *fundamental matrix* of (24.1), if $\{\boldsymbol{\phi}_1(x), \boldsymbol{\phi}_2(x), \cdots, \boldsymbol{\phi}_n(x)\}$ is a fundamental system of solutions (→**24A.4**).[342]

24A.6 Wronskian. Let $\boldsymbol{u}_1(x), \boldsymbol{u}_2(x), \cdots, \boldsymbol{u}_n(x)$ be n solutions to (24.1). The determinant of the matrix $(\boldsymbol{u}_1(x), \boldsymbol{u}_2(x), \cdots, \boldsymbol{u}_n(x))$ is called the *Wronskian* of the set of solutions $\{\boldsymbol{u}_1(x), \boldsymbol{u}_2(x), \cdots, \boldsymbol{u}_n(x)\}$.

If the Wronskian of the set $\{\boldsymbol{u}_1(x), \boldsymbol{u}_2(x), \cdots, \boldsymbol{u}_n(x)\}$ is nonzero, then this set is a fundamental system of solutions.

The converse is also true according to **24A.4**. In other words:

24A.7 Theorem. A regular matrix $X(x)$ satisfying

$$\frac{dX(x)}{dx} = A(x)X(x) \tag{24.4}$$

is a fundamental matrix of (24.1).

24A.8 Theorem. Let $W(x)$ be the Wronskian of the set of (any) n solutions to (24.1). Then,

$$\frac{dW(x)}{dx} = [Tr\, A(x)]W(x). \tag{24.5}$$

This should be obvious from

$$det[(1 + At)X] = detX + t\,Tr A\, detX + O[t^2]. \tag{24.6}$$

This formula follows from

$$detX = \exp[Tr \ln X], \tag{24.7}$$

which is a very important formula and essentially follows from $detX = \prod \lambda_i$, where λ_i are eigenvalues of X.

24A.9 Theorem. Let $\Phi(x)$ be a fundamental matrix (→**24A.5**) of (24.1). Then, for any non-singular matrix P, $\Phi(x)P$ is again a fundamental matrix of (24.1). Conversely, if $\Phi(x)$ and $\Psi(x)$ are two fundamental matrices of (24.1), then there is a constant non-singular matrix P such that $\Psi(x) = \Phi(x)P$. □

[341] This is of course a stronger condition that $\boldsymbol{u} \neq 0$.

[342] The evolution operator $T(x, y)$ such that $\boldsymbol{u}(x) = T(x, y)\boldsymbol{u}(y)$ is given by $T(x, y) = \Phi(x)\Phi(y)^{-1}$.

[Demo] Obviously, $\Phi(x)P$ satisfies (24.4) and non-singular, so it is a fundamental matrix. Next, let $P = \Phi(x)^{-1}\Psi(x)$, then a straightforward calculation shows $dP/dx = 0$. Hence, P must be a constant matrix, and non-singular by definition.

24A.10 Second order linear ODE. Separation of variables (→**23**) of linear second order PDE often gives second order linear ODE of the following type:

$$\frac{d^2u}{dx^2} + P(x)\frac{du}{dx} + Q(x)u = 0, \tag{24.8}$$

where P and Q are functions of $x \in \boldsymbol{R}$. This can be transformed into the first order ODE of the form discussed in **24A.1**:

$$\frac{d\boldsymbol{u}}{dx} = A(x)\boldsymbol{u} \tag{24.9}$$

with $\boldsymbol{u}(x) = (u, du/dx)^T$ and

$$A(x) = \begin{pmatrix} 0 & 1 \\ -Q & -P \end{pmatrix}. \tag{24.10}$$

24A.11 Fundamental system of solutions. Let u_1 and u_2 be two solutions for (24.8). The Wronskian $W(x)$ (→**24A.6**) for these solutions is defined as

$$W(x) = \begin{vmatrix} u_1 & u_2 \\ u_1'(x) & u_2'(x) \end{vmatrix}. \tag{24.11}$$

Tha is, W is the Wronskian of (24.9). If we can find u_1 and u_2 with $W(x) \neq 0$, then the set $\{u_1, u_2\}$ is called a *fundamental system of solutions.* The general solution to (24.8) is $c_1u_1 + c_2u_2$ for arbitrary constants c_1 and c_2 (cf. **24A.4**).

24A.12 Theorem [Separation theorem]. Let u and v make a fundamental system of solutions of (24.8). Then
(1) The zeros of u and v are all of multiplicity one.
(2) The zeros of u and v separate each other.□

[Demo] Suppose u has a zero of multiplicity larger than one. Then u and u' can vanish simultaneously, so that the Wronskian W (→**24A.6**) of u and v can vanish. This contradicts the assumption. Thus (1) must be true. To prove (2) note that u and v cannot have a common zero, since $W \neq 0$. Let a_1 and a_2 ($> a_1$) be two adjacent zeros of u, and assume that v does not vanish in the interval $J = (a_1, a_2)$. Then u/v is well defined in J, and is differentiable:

$$\frac{d(u/v)}{dx} = \frac{W(x)}{v^2}. \tag{24.12}$$

This cannot vanish. However, $u/v = 0$ at the both ends of J, so Rolle's theorem asserts that (24.12) must vanish in J, a contradiction. We can exchange u and v to complete the proof.

Exercise.
Consider the following 1-Schrödinger problem

$$(-\Delta + V)\psi = E\psi, \tag{24.13}$$

where V vanishes at infinity. If this equation has a bound state, it cannot be degenerate. In particular, the lowest energy bound state (ground state) cannot be degenerate. Prove this showing or answering the following:
(1) Degeneracy implies that there are two independent solutions for a given energy. What must be their Wronskian?
(2) The Wronskian for localized state is zero.

24A.13 Making a partner. Suppose we have found one solution v to (24.8). We wish to make u (a partner of v) so that $\{u, v\}$ becomes a fundamental system of solutions (→**24A.11**). We use (24.12). To compute the Wronskian W we can use (24.5) (→**24A.8**) with $Tr\,A = -P$ for (24.8). W can be solved as

$$W = W_0 \exp\left(-\int^x P(y)dy\right). \tag{24.14}$$

From (24.12) we obtain

$$u = v\left[\int^z d\xi v^{-2} e^{-\int^\xi P(\xi')d\xi'} + c\right], \tag{24.15}$$

where c is a constant.

Exercise.
One solution of

$$\frac{d^2y}{dx^2} - \left(\frac{1}{x} + 1\right)\frac{dy}{dx} + \frac{1}{x}y = 0 \tag{24.16}$$

is e^x. Find its partner.

24.B Frobenius' theory

24B.1 Analiticity of solutions. **24A.3** implies that if P and Q are analytic in a region D, then the solution to (24.8) is unique and analytic in D. Hence, a local solution can be assumed to be in the power series form around a point where P and Q are holomorphic.

24B.2 Singular points. If P or Q becomes singular (→**8A.2**) at

a point a, a is called a *singular point* of the ODE (24.8).
(1) At a singular point a, if the singularity of P is at worst a pole of order one, and that of Q is at worst a pole of order two (→**8A.5(4)**(ii)), then a is called a *regular singular point* of the ODE.
(2) Otherwise, a is called an *irregular singular point* of the ODE.

Discussion.[343]
In general (in more standard pure math literatures) the definition of a regular singular point is as follows. Let u be any solution of (24.8).
Definition. z_0 is a regular singular point of (24.8), if there is a positive number ρ such that for any of its solution u satisfies

$$\lim_{z \to z_0} (z - z_0)^\rho u(z) = 0 \tag{24.17}$$

That is, if the singularity of the solution (remember **24A.3**, **24B.1**) is at worst algebraic at z_0, we say z_0 is a regular singular point.
Theorem [Fuchs]. A necessary and sufficient condition for z_0 to be a regular singular point of (24.8) is that z_0 is a regular singular point in the sense of **24B.2**. □
Its proof is not very simple (elementary but lengthy). An intuitive understanding is the 'balance condition' of the singularities (divergences) around z_0 in (24.8). Consider only the most singular terms in (24.8) near z_0. If the 'aggravation' by differentiation of the singularity in the solution is balanced by the singularities in the coefficients, then we say the singularity is regular.

24B.3 Expansion around regular singular point. Frobenius showed that power series expansion can give a local solution around a regular singular point as well. Around a regular singular point a, which we may set to be 0 without any loss of generality, we expect the following form

$$u(z) = z^\mu \sum_{k=0}^{\infty} a_k z^k, \tag{24.18}$$

where μ is an appropriate complex constant. We may expand P and Q as (Laurent expansion →**8A.8**)

$$zP(z) = \sum_{k=0}^{\infty} p_k z^k, \tag{24.19}$$

$$z^2 Q(z) = \sum_{k=0}^{\infty} q_k z^k. \tag{24.20}$$

Formally substituting these expansions into the differential equation (24.8), we get conditions for the equation to be satisfied identically:

$$a_0 \phi(\mu) = 0, \tag{24.21}$$

$$a_1 \phi(\mu + 1) + a_0 \theta_1(\mu) = 0 \tag{24.22}$$

[343] Yosida p86

and generally for $n = 1, 2, \cdots$

$$\phi(\mu + n)a_n + \sum_{k=1}^{n} a_{n-k}\theta_k(\mu + n - k) = 0, \tag{24.23}$$

where

$$\begin{aligned} \phi(\mu) &= \mu^2 + (p_0 - 1)\mu + q_0, & (24.24)\\ \theta_i(\mu) &= \mu p_i + q_i. & (24.25) \end{aligned}$$

24B.4 Indicial equation. We may assume $a_0 = 1$ without any loss of generality. However, if (24.23) couples only even coefficients $\{a_{2n}$ with each other (or only odd coefficients), then even and odd coefficients are decoupled. Therefore, the choice $a_0 = 1$, $a_1 = 0$ and that $a_0 = 0$, $a_1 = 1$ both give different solutions (cf. **24C.2**). (24.21) or $\phi(\mu) = 0$ is called the *indicial equation.* It determines two (possibly identical) values of μ, μ_1 and μ_2 (Henceforth, we assume $Re\mu_1 \geq Re\mu_2$).

Exercise.
Find the indicial equation for

$$\frac{d}{dz}\left\{(1 - z^2)\frac{d}{dz}u\right\} + \left\{l(l+1) - \frac{m^2}{1 - z^2}\right\}u = 0. \tag{24.26}$$

24B.5 Use of symbol manipulation programs. Expanding and regrouping expanded terms is performed by symbol manipulating programs very efficiently. In practice, Frobenius' method will not be used often, but if needed, the best way is to use computers to compute the series.

24B.6 Theorem. Assume that $z = 0$ is a regular singular point ($\rightarrow$**24B.2**(1)) of (24.8) and μ_1, μ_2 are the roots of the indicial equation $\phi(\mu) = 0$ (cf.(24.24)). Then
[1] If $\mu_1 - \mu_2 \notin \boldsymbol{N}$, there is a fundamental system of solutions ($\rightarrow$**24A.11**) in the form of (24.18) converging in some neighborhood of 0.
[2] If $\mu_1 - \mu_2 \in \boldsymbol{N}$, generally only one solution in the form of (24.18) is uniquely determined by the expansion method. See **24B.7** for further classification. □

[Demo] Choose $\mu = \mu_1$. Then, $\phi(\mu + n)$ cannot be zero for any $n = 1, 2, \cdots$, so that a_n can be uniquely determined from (24.23). The resultant series is convergent in some small neighborhood of $z = 0$. This can be demonstrated by constructing a majorizing series.[344] If $\mu_1 - \mu_2$ is not in $\boldsymbol{N}$, then $\mu = \mu_2$ also allows us to determine a_n uniquely, and the resultant solution is distinct from the one obtained for

[344] See, for example, H. S. Wilf, *Mathematics for the Physical Sciences* (Dover, 1962), or E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge UP, 1927), Sect. 10.31 for an explicit demonstration.

μ_1. However, if $\mu_1 - \mu_2 \in \boldsymbol{N}$, then there is $m \in \boldsymbol{N}$ such that $\mu_2 + m = \mu_1$ or $\phi(\mu_2 + m) = 0$. Therefore, we may not generally determine a_m for this μ_2.

24B.7 Theorem [For $\mu_1 - \mu_2 \in \boldsymbol{N}$]. In case **[2]** of Theorem **24B.6**.
[21] If $\mu_1 = \mu_2$, then any partner u (to make a fundamental system) of the solution v constructed for μ_1 in the form (24.18) must contain a logarithmic term and has the following general form

$$u(z) = Av(z)\ln z + z^{\mu_1}\psi(z), \tag{24.27}$$

where A is a nonzero constant, and ψ is analytic around $z = 0$. This function can be determined by substituting the series expansion form of (24.27) into (24.8).
[22] If $\mu_1 - \mu_2 \in \boldsymbol{N} \setminus \{0\}$, then a partner u of the solution v constructed for μ_1 in the form (24.18) has the following general form

$$u(z) = Av(z)\ln z + z^{\mu_2}\psi(z), \tag{24.28}$$

where v is again the solution constructed for μ_1 in the form (24.18), A is a constant (can be zero), and ψ is analytic around $z = 0$. This function can be determined by substituting the series expansion form of (24.28) into (24.8).
□
[Demo] According to (24.15) (→**24A.13**) the ratio $q(z) = u/v$ of v and its partner u is given by (c_1 and c_0 are integration constants)

$$\begin{aligned} q(z) &= c_1 + c_0 \int^z d\zeta v(\zeta)^{-2} \exp\left[-\int^\zeta P(\zeta')d\zeta'\right] \\ &= c_1 + c_0 \int^z d\zeta \frac{1}{[\zeta^{\mu_1}(1 + a_1\zeta + \cdots)]^2} \exp\left[-\int^\zeta \left(\frac{p_0}{\zeta'} + p_1 + \cdots\right) d\zeta'\right] \\ &= c_1 + c_0 \int^z \zeta^{-(p_0 + 2\mu_1)} h(\zeta) d\zeta, \end{aligned} \tag{24.29}$$

where $h(z)$ is analytic around $z = 0$ as can be seen from

$$h(z) = \exp\left[-\int^z d\zeta(p_1 - p_2\zeta + \cdots)\right] / (1 + a_1\zeta + \cdots)^2. \tag{24.30}$$

Since from the indicial equation (→**24B.4**) or $\phi(\mu) = 0$ (cf.(24.24)) $-p_0 + 1 = \mu_1 + \mu_2$, we know $p_0 + 2\mu_1 = 1 + \mu_1 - \mu_2 \in \boldsymbol{N} \setminus \{0\}$. Therefore, (24.29) has the following form

$$q(z) = A\ln z + z^{\mu_2 - \mu_1}\varphi(z), \tag{24.31}$$

where A is a constant and φ is a function analytic around $z = 0$. Hence, u must have the form (24.28). For $\mu_1 = \mu_2$ A cannot be zero to make u functionally independent of v.□.

24B.8 Practical Frobenius.
(0) Check the expansion center is at worst regularly singular (→**24B.2**).
(1) Compute the indices μ_1 and μ_2 according to **24B.4**.
(2) Choose the index with the larger real part μ_1 and construct the series solution following Frobenius (**24B.3**).
(3) If μ_2 is not equal to μ_1, try to construct the second solution just as before. If the obtained solution is different (functionally independent[345] from the first one, we are done.
(4) If we obtain the same solution or $\mu_1 = \mu_2$, assume the form with logarithm as in **24B.7**, and determine v in a power series form.

Exercise.[346]
(1) Show that a fundamental system of solutions of the equation

$$\frac{d^2u}{dx^2} + xu = 0 \tag{24.32}$$

consists of

$$u_1 = x - \frac{1}{12}x^4 + \cdots, \tag{24.33}$$

$$u_2 = 1 - \frac{1}{6}x^3 + \cdots. \tag{24.34}$$

(2) Show that a fundamental system of solutions of the equation

$$\frac{d^2u}{dx^2} + \frac{1}{4x^2}(1 - x^2)u = 0 \tag{24.35}$$

consists of

$$u_1 = x^{1/2}\left\{1 + \frac{1}{16}x^2 + \frac{1}{1024}x^4 + \cdots\right\}, \tag{24.36}$$

$$u_2 = u_1(x)\log x - \frac{1}{16}x^{3/2} + \cdots. \tag{24.37}$$

24B.9 Construction of the second solution by differentiation.
Let us write the solution obtained by Frobenius' method with the index λ as $u(x;\lambda)$. If $u(x,\lambda_1)$ and $u(x,\lambda_2)$ are functionally independent, then we can use $u(x,\lambda_1)$ and a linear combination of the two as a fundamental system of solutions. Consider

$$\frac{(\lambda_1 - \lambda_2)u(x,\lambda_1) - m\,u(x,\lambda_2)}{\lambda_1 - \lambda_2 - m}. \tag{24.38}$$

[345] That is, their Wronskian (→**24A.6**) is not identically zero. Often, without checking the Wronskian, we can recognize the independence by inspection.
[346] Watson-Whittaker p209.

For the case **[21]**, we choose $m = 0$ and compute the limit of $\lambda_1 \to \lambda_2$ with the aid of l'Hospital's rule. That is, we compute

$$\left.\frac{\partial}{\partial\lambda}u(x;\lambda)\right|_{\lambda=\lambda_1}. \tag{24.39}$$

Computing this explicitly, we obtain the general form given in **24B.7[21]**. When $\lambda_1 - \lambda_2 = m \in \boldsymbol{N}$, we perform a similar calculation:

$$\left.\frac{\partial}{\partial\lambda}[(\lambda-\lambda_2)u(x;\lambda)]\right|_{\lambda=\lambda_1}. \tag{24.40}$$

Again we recover the form asserted in **24B.7**.

24B.10 Examples.[347]
(1) Case **[1]**: $\mu_1 - \mu_2 \neq \boldsymbol{N}$.

$$x^2y'' + \left(x^2 + \frac{5}{36}\right)y = 0 \tag{24.41}$$

with

$$v = x^{5/6}\left(1 - \frac{3}{16}x^2 + \frac{9}{896}x^4 + \cdots\right), \tag{24.42}$$

$$u = x^{1/6}\left(1 - \frac{3}{8}x^2 + \cdots\right). \tag{24.43}$$

(2) Case **[21]**: $\mu_1 = \mu_2$

$$x(x-1)y'' + (3x-1)y' + y = 0 \tag{24.44}$$

with

$$v = 1/(1-x),\ u = \ln x/(1-x). \tag{24.45}$$

(3) Case **[22]**: $\mu_1 - \mu_2 \in \boldsymbol{N} \setminus \{0\}$ with a logarithmic term.

$$(x^2-1)x^2y'' - (x^2+1)xy' + (x^2+1)y = 0 \tag{24.46}$$

with

$$v = x, \quad u = x\ln x + 1/2x. \tag{24.47}$$

(4) Case **[22]** $\mu_1 - \mu_2 \in \boldsymbol{N} \setminus \{0\}$ without any logarithmic term (cf. **27A.19**, **27A.25**).

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{24.48}$$

[347] They are taken from E. Kreyszig, *Advanced Engineering Mathematics* (Wiley, 1983 Fifth edition) p163.

with

$$v = \sin x/\sqrt{x}, \quad u = \cos x/\sqrt{x}. \tag{24.49}$$

See **27A.2** also, for example.

24B.11 Singularity at infinity. To study the singularity of the equation (24.8) at infinity, we introduce $\zeta = z^{-1}$ as usual in complex function theory. The equation reads in terms of ζ

$$\frac{d^2u}{d\zeta^2} + \left[\frac{2}{\zeta} - \frac{1}{\zeta^2}P(\zeta^{-1})\right]\frac{du}{d\zeta} + \frac{1}{\zeta^4}Q(\zeta^{-1})u = 0. \tag{24.50}$$

Therefore (→**24B.2**),
(1) If $2z - z^2P(z)$ and $z^4Q(z)$ is regular at ∞, $z = \infty$ is a non-singular point.
(2) If $zP(z)$ and $z^2Q(z)$ are regular at ∞, then $z = \infty$ is a regular singular point.
(3) Otherwise, $z = \infty$ is an irregular singular point.

24B.12 How to solve inhomogeneous problem. To solve the inhomogeneous version of (24.8)

$$\frac{d^2u}{dx^2} + P(x)\frac{du}{dx} + Q(x)u = f(x), \tag{24.51}$$

where f is a piecewise continuous function, we have only to find one special solution to this inhomogeneous equation; the general solution is the sum of that for (24.8) and this special solution. If one cannot get it by inspection, then perhaps the most systematic way is to use Lagrange's method of variation of constants described in **11B.13**.

24.C Representative Examples

24C.1 Legendre equation. If the method of separation of variables is used in the spherical coordinates for the Laplace equation (→**2D.10**), the angular part can further be split into the parts $\Theta(\theta)$ and $\Phi(\varphi)$ as (cf. **26A.2**)

$$\frac{d^2\Phi}{d\varphi^2} + m^2\Phi = 0, \tag{24.52}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right)\Theta = 0. \tag{24.53}$$

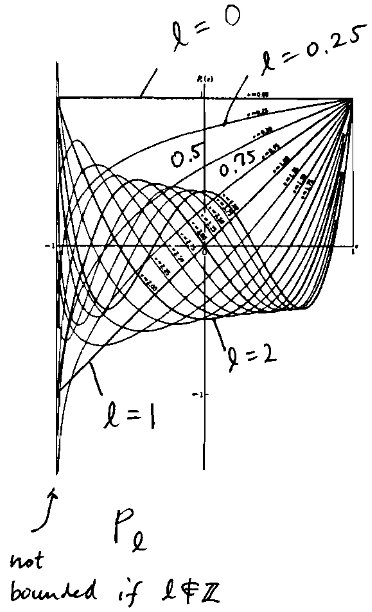
If there is no φ dependence, then $m = 0$, and (24.53) simplifies to (→**26B.6**)

$$\frac{d^2P}{dz^2} - \frac{2z}{1-z^2}\frac{dP}{dz} + \frac{\ell(\ell+1)}{1-z^2}P = 0, \tag{24.54}$$

where $z = \cos\theta$ and $P(z) = \Theta(\theta)$. Or, we get

$$\frac{d}{dz}(1-z^2)\frac{d}{dz}P + \ell(\ell+1)P = 0, \tag{24.55}$$

which is called the *Legendre equation.* $z = \pm 1$ are regular singular points (→**24B.2**) of (24.54). ($z = \infty$ is also a regular singular point. See **24B.11**.)



24C.2 Series expansion method applied to Legendre's equation; around $z = 0$. Since $z = 0$ is a regular point, solutions can be obtained in the series form $P(z) = \sum_{k=0}^{\infty} a_k z^k$ with the radius of convergence at least unity (→**24B.1**, **7.3**).
(1) Introducing this into (24.55), we get

$$(n+1)(n+2)a_{n+2} + (\ell - n)(\ell + n + 1)a_n = 0. \tag{24.56}$$

(2) This implies that a_n can be expressed in terms of a_0 and a_1. The choice $a_0 = 1$, $a_1 = 0$ gives an even power series

$$P_{even} = 1 - \frac{\ell(\ell+1)}{2!}z^2 + \frac{\ell(\ell+1)(\ell-2)(\ell+3)}{4!} - \cdots, \tag{24.57}$$

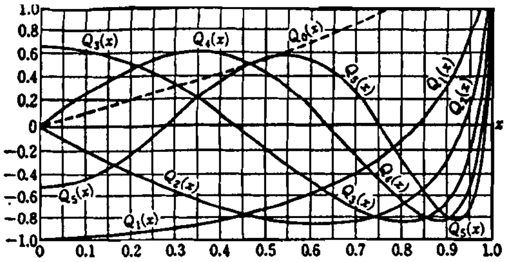
and $a_0 = 0$, $a_1 = 1$ gives an odd power series

$$P_{odd} = z - \frac{(\ell-1)(\ell+2)}{3!}z^3 + \frac{(\ell-1)(\ell+2)(\ell-3)(\ell+4)}{5!}z^5 - \cdots. \tag{24.58}$$

(3) Notice that these two solutions make a fundamental system of solutions (→**24A.11**). If $\ell = n \in N \setminus \{0\}$, then they become polynomials called the *Legendre polynomials* $P_n(z)$ (→**21B.2**).

24C.3 Series expansion method applied to Legendre's equation; around $z = 1$. The indicial equation (24.24) is $\phi(\mu) = \mu^2 = 0$, so this is the case [21] of Theorem **24B.7**. One solution in the series form is

$$P_\ell(z) = \sum_{k=0}^{\infty} \frac{(\ell+1)(\ell+2)\cdots(\ell+k)(-\ell)(-\ell+1)\cdots(-\ell+k-1)}{k!^2}\left(\frac{1-z}{2}\right)^k. \tag{24.59}$$



↑ not bounded

This is called the *Legendre function of degree ℓ of the first kind.* Its partner in the fundamental system is obtained in the form of (24.28) (→**24B.7**). For a positive integer $\ell = n$

$$Q_n(z) = \frac{1}{2}P_n(z)\ln\frac{1+z}{1-z} - \sum_{k=1}^{n}\frac{2n-4k+3}{(2k-1)(n-k+1)}P_{n-2k+1}(z). \quad (24.60)$$

This is called the *Legendre function of degree ℓ of the second kind.* Since P_n and Q_n make a fundamental system of solutions (→**24A.11**), their zeros separate each other (→**24A.12**(2)).

24C.4 Gauss' hypergeometric equation. The following equation is called *Gauss' hypergeometric equation*

$$z(1-z)\frac{d^2u}{dz^2} + [\gamma - (\alpha+\beta+1)z]\frac{du}{dz} - \alpha\beta u = 0, \quad (24.61)$$

where α, β and γ are constants. $z = 0, 1$ and ∞ are the regular singular points (→**24B.2**(1)). The indicial equation (→**24B.4**) around $z = 0$ is

$$\phi(\mu) = \mu(\mu - 1 + \gamma) = 0. \quad (24.62)$$

For $\mu = 0$ we can get (→**24B.6**) for $-\gamma \notin \boldsymbol{N}$

$$F(\alpha,\beta,\gamma;z) \equiv \sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k}z^k, \quad (24.63)$$

where

$$(\lambda)_k = \lambda(\lambda+1)\cdots(\lambda+k-1). \quad (24.64)$$

F is called the *hypergeometric function.* For $\mu = 1-\gamma$, if $\gamma - 2 \notin \boldsymbol{N}$, we get a partner of the above solution as

$$z^{1-\gamma}F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma; z). \quad (24.65)$$

Notice that from (24.59)

$$P_\nu(z) = F(\nu+1, -\nu, 1; (1-z)/2). \quad (24.66)$$

Discussion.
If we scale z as kz in Gauss's equation, we obtain the equation of the following form:

$$z(1-kz)u'' + (c-bz)u' - au = 0. \quad (24.67)$$

Its regular singular points are at 0, $1/k$ and ∞. There is no other singularities. Take the $k \to 0$ limit to make $1/k$ confluent to ∞. Then, we obtain

$$zu'' + (c-bz)u' - au = 0. \quad (24.68)$$

If we set $b = 0$, the equation is Bessel's equation (→**27A.1**). Indeed, replacing az with $-t^2/4$, $c = \nu + 1$, and $v = t^\nu u$, then we have

$$t^2 v'' + tv' + (t^2 - \nu^2)v = 0. \tag{24.69}$$

It is obvious that ∞ is its irregular singularity (→**24B.2**).

24C.5 Associate Legendre functions. Consider the case with $m \neq 0$ for (24.53) (→**24C.1**). Using the same transformation of the variable $z = \cos\theta$, (24.53) becomes

$$\frac{d}{dz}\left((1 - z^2)\frac{d\Theta}{dz}\right) + \left(\ell(\ell + 1) - \frac{m^2}{1 - z^2}\right)\Theta = 0. \tag{24.70}$$

$z = \pm 1$ are regular singular points (→**24B.2**). Instead of solving this with the aid of the series expansion, introduce Z as

$$\Theta = (1 - z^2)^{m/2} Z(z). \tag{24.71}$$

Then, we have

$$(1 - z^2)\frac{d^2 Z}{dz^2} - 2(m + 1)z\frac{dZ}{dz} + (\ell - m)(\ell + m + 1)Z = 0. \tag{24.72}$$

Differentiate Legendre's equation (24.55) m times, we get

$$(1 - z^2)\frac{d^{2+m}u}{dz^{2+m}} - 2(m+1)z\frac{d^{m+1}u}{dz^{m+1}} + (\ell - m)(\ell + m + 1)\frac{d^m u}{dz^m} = 0. \tag{24.73}$$

Therefore, in terms of Legendre functions P_ℓ and Q_ℓ (→**24C.3**)

$$P_\ell^m(z) = (1 - z^2)^{m/2}\frac{d^m}{dz^m}P_\ell(z), \quad Q_\ell^m(z) = (1 - z^2)^{m/2}\frac{d^m}{dz^m}Q_\ell(z) \tag{24.74}$$

are the fundamental system of solutions (→**24A.11**) of (24.70), and are called *associate Legendre functions* (→**26A.5-6**). Notice that P_ℓ^m is not a polynomial, if m is odd. Also

$$P_\ell^m(\pm 1) = 0 \quad \text{for } m \geq 1. \tag{24.75}$$

24C.6 Confluent hypergeometric equation. Replace z in the hypergeometric equation (24.61) with z/β and let $\beta \to \infty$. We get

$$z\frac{d^2 u}{dz^2} + (\gamma - z)\frac{du}{dz} - \alpha u = 0 \tag{24.76}$$

This is called the *confluent hypergeometric equation* or *Kummer's equation.* $z = 0$ is a regular singular point (→**24B.2**), but $z = \infty$ is an irregular singular point (→**24B.2**), which is created by the confluence of two regular singular points 1 (which is scaled to β by the variable change) and ∞ of the hypergeometric equation. The indicial equation (24.24) is $\phi(\mu) = \mu(\mu - 1) + \gamma\mu = 0$. The series solution method gives

$$u_1 = F(\alpha, \gamma; z), \quad u_2 = z^{1-\gamma} F(\alpha - \gamma + 1, 2 - \gamma; z), \tag{24.77}$$

where

$$F(\alpha, \gamma; z) \equiv \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!(\gamma)_k} z^k, \quad \gamma \neq 0, 1, 2, \cdots. \tag{24.78}$$

This function is called the *confluent hypergeometric function.*

Exercise.
Show that
(1) $(1 + z)^n = F(-n, \beta, \beta, z)$,
(2) $(1/z)\log(1 + z) = F(1, 1, 2, -z)$.

APPENDIX a24 Floquet Theory

a24.1 We consider (24.1) with periodic $A(x)$, that is, there is $\omega > 0$ such that

$$A(x+\omega) = A(x). \tag{24.79}$$

a24.2 Theorem [Floquet]. If A in (24.1) is periodic, then there is a fundamental matrix such that

$$\Phi(x) = F(x)e^{x\Lambda}, \tag{24.80}$$

where F is a $n \times n$ matrix with period ω, and Λ is a constant $n \times n$ matrix. □
[Demo] Let $\Phi(x)$ be a fundamental matrix (→**24A.5**) for (24.1). Then $\Phi(x+\omega)$ is also a fundamental matrix. Therefore, Theorem **24A.9** tells us that there is a constant non-singular matrix M such that $\Phi(x+\omega) = \Phi(x)M$. Since M is non-singular, its logarithm $\ln M = N$ is well defined. Define $\Lambda = N/\omega$, and set

$$F(x) = \Phi(x)e^{-x\Lambda}. \tag{24.81}$$

We get with the aid of $\Phi(x+\omega) = \Phi(x)M$

$$\Phi(x+\omega) = F(x+\omega)e^{(x+\omega)\Lambda} = F(x+\omega)e^{x\Lambda}M = F(x)e^{x\Lambda}M. \tag{24.82}$$

Hence,

$$F(x+\omega) = F(x). \tag{24.83}$$

In other words,

a24.3 Theorem. A linear ordinary differential equation (24.1) with a periodic matrix A can be converted into a constant coefficient ordinary differential equation

$$\frac{d\boldsymbol{v}(x)}{dx} = \Lambda \boldsymbol{v}(x) \tag{24.84}$$

with $\boldsymbol{u} = F(x)\boldsymbol{v}$, where F is defined by (24.81).□

a24.4 Characteristic exponents. The eigenvalues of Λ in (24.81) are called the *characteristic exponents.* There is no systematic way to obtain these exponents.