Introduction to Applicable Analysis: Part I (1997 Spring version)

Y. Oono
Beckman Institute and Department of Physics
405 N. Mathews Av.
University of Illinois at Urbana-Champaign
Urbana, IL 61801

January 6, 1997

Read This First

These are the lecture notes for 'innovated' Physics 411/412. Traditionally. the course was regarded as a remedial one for those who did not have sufficient mathematics background to pass the qualification examination. Now, the boundary-value problem has been removed from the qual. Therefore. the main obstacle is gone that hindered making the course useful in the life beyond the qual (i.e.. the major part of our life).

The main purpose of the lecture notes is to give an overview of mathematical tools and theories relevant and/or related to understanding extended systems – space-time phenomena. Needless to say, most interesting natural phenomena are space-time phenomena.

The main difficulty of the course is that it is likely to be the first not-so-elementary mathematics-related course. and. simultaneously. likely to be the last such course for many physics students. Hence. the lecturer must start from a rather basic level. However. the lecturer must push the future researchers up to the level from which they can take off for their own adventures. Consequently. perhaps a two-year course may have to be pushed into one semester. Therefore. the reader should not try to understand everything. Rather. she should regard the course as an overview or perspective course for her future self-study. If the notes could lower the 'activation energy' to learn. e.g.. distribution. Hilbert space. Lebesgue integration. Fourier. Laplace. Radon. etc.. transformations. then the course is a success.

The main difference between the Fall 1996 version and this version is that examples and exercises are extensively incorporated in this version. The final result may be about 200 pages thicker.

The lecturer is still looking for a coauthor mathematician who has some respect of physics.

Study Guide

To build analytical muscles is NOT the purpose of these notes; to this end the reader must practice for a sufficiently long time. However, if she tries to solve many exercises in these notes.[1] the lecturer believes that the notes can give a reasonably extensive training. If she wishes to study some topics further. perhaps, using the introduced references, the aim of the lecture notes is accomplished.

1 Structure of the notes. The notes consist of the main body (+ miscellaneous appendices) and one large Appendix. The Appendix (which will come with Part II) is a summary of elementary calculus at the freshman level (outside the US). The reader should use them as check lists to confirm her knowledge. The notes are made of entries that are numbered to facilitate cross referencing. An entry may occupy more than one page. but its core is always bite sized. A slightly advanced student should be able to read these notes starting from any section or even any entry in any section with the help of extensive references denoted by $\rightarrow$.

2 Summary. Each section begins with an introductory remark with key words. **Summary** provides the basic check list for the reader's knowledge and also contains some instructions. Immediately after reading a section. she must not have any difficulty in any of the entries in the summary. That is. she must be able to give a reasonable explanation of each entry in the summary to her friends.[2]

3 Active learning. There is only one mode of meaningful learning: self-study. The instructor's role is to show a map. It is the reader who explores. Only the exploration gives her the real experience and training. Alway check whether the reader can explain each entry to her smart (but perhaps ignorant) friends.

4 Discussion and Exercise. Many entries contain **Discussion** and/or **Exercise**. **Discussion** contains related topics to each entry and may require more advanced knowledge or patience. **Exercise** is less sophisticated. or more directly related to the entry than Discussion. but not

[1] Do not hesitate to ask for hints. etc. yoshi@kolmogorov.pt.uiuc.edu.

[2] However. in actual researches often a rough idea of what we can do is crucial. The reader can learn necessary details very quickly if she konws what to learn and where she can get the relevant material. Therefore. the reader need not store many things in her long term memory.

necessarily simpler. It is strongly recommended that the reader at least think how to solve Exercise problems.

5 Recording the activity. If the reader wishes to read these notes or any textbook systematically, it is not a bad habit to keep an accompanying notebook (a logbook) to record all the calculations she does (with the corresponding pages of the book) just as any reasonable experimentalists and field workers do. When the reader reads the same book again for some particular knowledge later, the notebook will significantly shorten the necessary time (and sometimes she will realize how smart she was).

1 Numbering of items. To facilitate frequent cross referencing. and to shorten the notes, all the items are numbered and short titles are attached as **28.10 Short-time modification of diffusion equation** or **14.22 Convolution**. The idea is that each entry can be understood with the aid of a few cross references therein by the reader who knows analysis reasonably well (that is, who has taken good introductory courses given by mathematicians, or who does not have any difficulty with Appendix A). The entry number starting with **a** denotes the smaller appendices interspersed throughout the notes. Thus **15.3** means that the entry is the 3rd in Section 15. **8A.7** means that the entry is in subsection 8A. **A5.14** means the 14th entry in Section 5 of the Appendix A at the end of the notes. **a33.2** means the second entry in the Appendix of Section 33. etc.

2 Proofs and the level of mathematical rigor. Proofs of advanced statements are often omitted. but usually notes are given as to where they can be found. These lecture notes are like an extended menu with some food samples. so the lecturer believes that the style is permitted. However. it is unethical to use the theorem without understanding how it can be demonstrated. When (an outline of) a proof is given. constructive proofs are preferred to merely elegant ones. Tedious but mechanical proofs are relegated to references (if it is conceptually important. an outline is given in the notes).

3 Figures. Mostly due to the technical reason (no proof reader. no typist. in short. no departmental help at all). many figures are still missing from this version.

Ideal practical course of applicable analysis

From the lecturer's point of view. an ideal course for physicists of mathematical tools (mainly to study extended systems) should contain the topics mentioned in the following. Unfortunately, we will be able to cover its tiny subset in our course.

What we should learn. First. we must be able to capture the physics in a form mathematics can be applied. Hence, we need to know
(1) Modeling methods.
Traditionally. partial differential equations (PDE) are almost the sole means. However. after the advent of efficient numerical tools, a more flexible attitude is often rewarding. Still. PDE are the most suitable to perform analytical studies. and we should know the classical topics and tools:
(2) Study of linear PDE.
The traditional (obsolete) mathematical physics courses were how-to courses to solve linear PDE without much logic (like a cookbook). We should know the traditional topics such as separation of variables. etc. However. these traditional methods work only in very special cases (e.g.. the domain shape must be very special). Therefore. we must understand the nature of PDE and their close correspondence to the physical phenomena described by them. Hence. we should know
(3) General theory of (linear) PDE.
This should cover at least the mathematical theories developed by 1960. As mentioned above. even linear PDE cannot be solved analytically. Therefore we need
(4) Qualitative tools.
and
(5) Numerical tools.
We need qualitative understanding of PDE (i.e., the study of PDE as dynamical systems) to build sound intuition. This is also very important for us not to be fooled by unreliable computational or uncontrolled approximation results. Needless to say. computational means is the most important practical method to solve PDE. Hence, to know the best available algorithms is advantageous with a sound basic knowledge of numerical analysis. Even with supercomputers solving a PDE is costly. so it is desirable that we can simplify the problem. preserving

the main features we are interested in. Hence, the study of
(6) System reduction and approximation tools
are important. This is also theoretically very important, because reduced systems may give us deeper insight about the original complicated systems.

Book Guide

This is a biased book guide of the books the lecturer has found useful. Most books are reserved in physics library.

1. General Reference Books.

The following dictionary is an authoritative dictionary for mathematicians:

Encyclopedic Dictionary of Mathematics (MIT Press, Third Edition. 1990?)

Physics Library has the second edition. This is a translation of a dictionary compiled by the Mathematical Society of Japan. The lecturer consulted this dictionary to avoid gross oversight. To use this dictionary the reader need a fairly good knowledge of mathematics.

2. Tables, Handbooks.

E Jahnke and F Emde, Tables of Functions (or Funktionentafeln) (Dover. 1945) is still a classic. which contains not only tables but graphs and reliefs of special functions. **M Abramowitz and I A Stegun (ed.), Handbook of Mathematical Functions** (Dover. 1964) is a popular book of tables. Its large part is an American counterpart of Funktionentafeln. but contains much more. e.g.. necessary tables and methods for numerical analysis. I S Gradshteyn and M Ryzhik. Table of Integrals. Series. and Products (Academic Press) [the lecturer has not used the latest version] is essentially a table of integrals. Now a large part of this book can be ignored with the advent of Maxima. Maple. Mathematica. etc. For example. the PC version of Maxima called μ-math does an excellent job. and is easier to use than books of this sort for may integrals. Besides. the lecturer cannot stop feeling that the table was not compiled by working scientists. The table is not as intelligently organized as the three compact volume set of 'Mathematical Formulas' from Iwanami (alas. in Japanese. but formulas are not in Japanese. of course).

3. General Introduction to Mathematics (or Mathematical mode of thinking).

S Mac Lane, Mathematics : Form and Function (Springer. 1986) readably explains basic concepts. The lecturer is sure that the reader will find topics she does not know. Its explanation of category theory may be the best introduction to the topic (by one of the founders of the theory: what is it for physicists? The lecturer does not know, but if the reader really wishes to think carefully about 'complex systems,' then she should not avoid the topic.) Every 'mathophile' should enjoy this book. and every scientist serious about mathematics should read it. **L. Gårding, Encounter with Mathematics** (Springer, 1977) is a tour of mathematics guided by a leading researcher of partial differential equations. Each chapter explains representative concepts and theorems of a chosen field. and is concluded with a document section. which quotes original references (translated into modern English).

4. Introductory books.

[Real Analysis]

American college textbooks are too wordy and often unstructured. Some of the college outline series may be used as a check list (Appendix A of the notes should be usable for this purpose). For a good thorough re-introduction to elementary calculus J D DePree and C W Swartz. Introduction to Real Analysis (Wiley, 1988) may be recommended (but may be too mathematical for physicists).

[Complex Analysis]

There are numerous books on the topic. but **H A Priestley, Introduction to Complex Analysis** (Oxford. 1990) is a cute (thin and elegant) book. For a crush course. the reader could use my course note (1992) for Phys413. which condenses elementary facts into 100 pages, so it could be used as a check list. If she needs a deeper knowledge of the topic. for physicists probably to read classic textbooks as E T Copson. An Introduction to the Theory of Functions of Complex Variables (Oxford UP. 1935) or E C Titchmarsh. The Theory of Functions is good. **M Rao and H Stetker, Complex Analysis, an invitation** (World Scientific. 1991) is a nice modern book. **E T Whittaker and G N Watson, A Course of Modern Analysis** (Cambridge. 1927.

fourth edition) contains excellent chapters.

[Differential Topology, Geometry

No book can beat **I M Singer and J A Thorpe, Lecture Notes on Elementary Topology and Geometry** (Scott, Foresman, and Company. Glenview, IL 1967). The book explains well elementary topology needed in real analysis in the first two chapters. Chapter 3 introduces the fundamental group. which is important in classifying singularities and defects in fields and condensed matters. Although it is a thick book. Y G Borisovich. N M Bliznyakov. T N Fomenko and Y A Izrailevich. Introduction to Differential and Algebraic Topology (Kluwer. 1995. original 1980) is enjoyable.

[Functional Analysis]

I recommend **A N Kolmogorov and S V Fomin, Introductory Real Analysis** (Dover 1970) which is the first 2/3 of the original Russian textbook designed for the course Analysis III for engineers and scientists of Moscow State University. **W Rudin, Functional Analysis** (McGraw-Hill. 1973) is also a good introductory textbook.[3]

[Fourier Analysis]

T W Körner, Fourier Amalysis (Cambridge. 1988) is an excellent textbook for those who have time to read 600 pages. The lecturer likes the style of mixing mathematics and other cultural aspects in a unified fashion. The lecturer wishes to write a set of lecture notes in this style. if our course is at least a one year course. The book could be introduced under the entry 3 above. The book has an accompanying problem book. Exercises for Fourier Analysis (Cambridge. 1993). This does not contain solutions. but most problems are step-by-step problems.

[Ordinary Differential Equations]

For the conceptual aspects. I recommend **V I Arnold, Ordinary Differential Equations** (Springer). This tells the reader how to think

[3] **J P Keener, Principles of Applied Mathematics** (Addison-Wesley. 1988) is a new generation of applied mathematics textbook trying to add functional flavor. but PDEs are not discussed in depth.

qualitatively or geometrically.

[Partial Differential Equations]
G L Lamb, Jr. Introductory Applications of Partial Differential Equations (Wiley. 1995) is a very elementary and problem-solving-oriented introduction to PDE. The book should be good for those who have never seen PDE. For a more theoretical side. **F John, Partial Differential Equations** (Springer, 1982) seems to be a classic now, but the latest book: **E DiBenedetto, Partial Differential Equations** (Birkhäuser. 1995) is excellent, although the first introductory chapter may be steep. **G Barton, Elements of Green's Functions and Propagation** is a friendly book of how to use Green's functions. J Smoller. Shock Waves and Reaction-Diffusion Equations (Springer. 1994. second version) is not an elementary introduction to PDE. but explains most basic theories needed to understand reaction-diffusion equations in a self-contained fashion.

[Special Functions]
Needless to say. the classic is **E T Whittaker and G N Watson, A Course of Modern Analysis** (Cambridge. 1927. fourth edition). The lecturer recommends **N N Lebedev, Special Functions & Their applications** (Dover. 1972) and its accompanying workbook **N N Lebedev, I P Skalskaya and Y S Uflyand, Worked Problems in Applied Mathematics** (Dover. 1965). If the reader cannot find a problem similar to the one she wishes to solve in the book. it is not unwise to go to numerical methods.

[Numerical Analysis]
G Hämmerlin and K-H Hoffmann. Numerical Mathematics (Springer. 1991) may be a good introduction. To learn the basic of numerical analysis of PDE. the recent book **K W Morton and D F Mayers, Numerical Solution of Partial Differential Equations** (Cambridge, 1994) is very friendly and readable (and thin). Although 'Numerical Recipes' may be quoted. the book does not explain the state-of-the art methods especially in case of PDE.

[Approximation]

Perturbation techniques are useful analytical means. **E J Hinch, Perturbation Methods** (Cambridge UP, 1991) is a thin and nice introduction. Notice that many such techniques can be unified from the renormalization group point of view.

[Mathematics Typing]
Everyone should be familiar with TeX or LaTeX. To learn this, the best way is to get a one or two page exanple from someone (or ask the instructor for one). and start writing your own. The lecturer finds **Diller, LaTex line by line** (Wiley.) very useful as a practical reference book.