9 Γ-functions

The gamma function was introduced by Euler through his integral: its analytic completion defines an analytic function called the Gamma function. $\Gamma(m+1) = m!$ for $m \in \boldsymbol{N}$ makes this function very useful in theoretical physics. Elementary results are collected here.

Key words Gamma function, Euler's integral, beta function, Schwinger-Feynman's parameter formula, Stirling's formula.

Summary
(1) Remember the definition of Gamma function in terms of Euler's integral (**9.1**). This is practically important in calculating definite integrals (**9.10**).
(2) $N!$ is roughly equal to $(N/e)^N$ for large N (Stirling's formula **9.11**).
(3) Half integer values of Γ can be evaluated exactly (**9.6**).

9.1 Euler's integral. For $\mathbb{R}z > 0$.

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt. \tag{9.1}$$

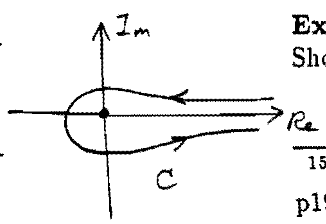
This is called *Euler's integral.* This integral is defined only for $\mathbb{R}z > 0$, but the *Gamma function* is defined by the analytic completion (→**7.10**) of (9.1). A rough idea is as follows. Note that

$$\int_0^m \left(1 - \frac{t}{m}\right)^m t^{z-1} dt = \frac{m! m^z}{z(z+1)\cdots(z+m)}. \tag{9.2}$$

This can be shown by repeated integration by parts. Hence, (9.1) can be written as

$$\Gamma(z) = \lim_{m\to\infty} \frac{m! m^z}{z(z+1)\cdots(z+m)}. \tag{9.3}$$

The RHS is well defined for all $z \notin -\boldsymbol{N}$.[152]



Exercise.
Show that

$$H(z) = \int_C (-\zeta)^{z-1} e^{-\zeta} d\zeta = -2i \sin \pi z \Gamma(z). \tag{9.4}$$

[152] A rigorous version may be found in J. W. Dettman, ***Applied Complex Analysis***, p194 (Dover, 1965), or E.T. Whittaker and G.N. Watson, ***A Course of Modern Analysis***, Chapter XII, which is a convenient reference source of the Γ-function.

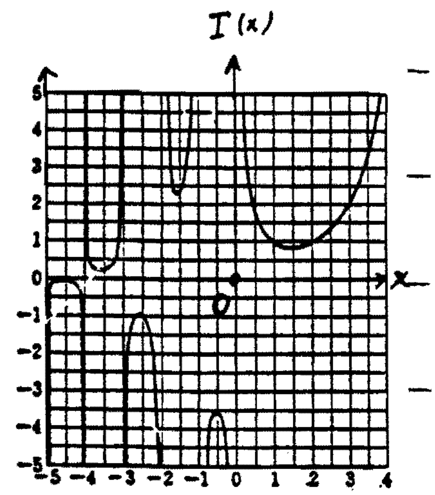
From this we obtain the following Hankel's formula

$$\frac{1}{\Gamma(z)} = \frac{i}{2\pi}\int_C (-\zeta)^z e^{-\zeta} d\zeta. \tag{9.5}$$

(2) Draw the graph of $1/\Gamma$ for the interval $[-2, 4]$.

9.2 $\Gamma(z+1) = z\Gamma(z)$. This is for $z \neq 0, -1, -2, \cdots$.
[Demo] From (9.3)

$$z\Gamma(z) = \lim_{m\to\infty} \frac{m! m^{z+1}}{(z+1)(z+2)\cdots(z+1+m)} \frac{z+1+m}{m} = \Gamma(z+1). \tag{9.6}$$



We can compute the Laurent expansion (→**8A.8**) of the Gamma function around negative integers s

$$\Gamma(z) = \frac{(-1)^n}{n!}\frac{1}{z+n} + \cdots. \tag{9.7}$$

Exercise.
Laurent-expand $\Gamma(-2+\epsilon)$ around $\epsilon = 0$ and find its principal part. You may use the Taylor expansion formula (→**9.9**), if needed.

9.3 Factorial. Obviously from **9.2**, we have

$$\Gamma(m+1) = m! \tag{9.8}$$

for $m \in \boldsymbol{N}$. $0! = 1$ as usual.

9.4 9.2 directly from Euler's integral. From (9.1) we get with the aid of integration by parts

$$\Gamma(z+1) = -\int_0^\infty (e^{-t})' t^z dt = z\int_0^\infty (e^{-t})' t^{z-1} dt. \tag{9.9}$$

This is **9.2**, which is demonstrated here for $\mathbb{R}z > 0$, but the principle of invariance of functional relations **7.6** can be invoked to demonstrate **9.2** for all $z \notin -\boldsymbol{N}$.

However, notice that the functional relation **9.2** combined with $\Gamma(1) = 1$ is <u>not</u> enough to characterize the Γ-function.[153]

[153] However, there is a
Theorem [Wielandt] Let $F(z)$ be a holomorphic function in the right half plane having the following two properties:
(i) $F(z+1) = zF(z)$ on the right half plane.
(ii) $F(z)$ is bounded in the strip $\{1 \leq \mathbb{R}z < 2\}$.
Then, $F(z)$ is proportional to $\Gamma(z)$.
See R. Remmert, "Wielandt's theorem about the Γ-function," Am. Math. Month. p214-220, March 1996.

Discussion.
Thus

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} \tag{9.10}$$

is true for z on the right half plane. However, the RHS is meaningful for $Re\, z > -1$ except $z = 0$. Continue this argument to show that $\Gamma(z)$ is analytic except negative integer values of z.

9.5 Formula of complementary arguments: For $z \notin \boldsymbol{Z}$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{9.11}$$

[Demo] We note $\Gamma(1-z) = -z\Gamma(-z)$ from **9.1**.

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = z\prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2}\right). \tag{9.12}$$

The RHS is an entire function (let us call it $o(z)$) with simple zeros at all $\boldsymbol{Z}$, and $o(z)/z$ at $z = 0$ is 1. Actually, the product is $\sin \pi z/\pi z$. An easier demonstration will be given in **9.8** below. □
Analogously, we have

$$\Gamma(z+1/2)\Gamma(z-1/2) = \pi/\cos \pi z. \tag{9.13}$$

Exercise.
Show that

$$\Gamma(z)\Gamma(-z) = \frac{\pi}{z \sin \pi z}. \tag{9.14}$$

Using this, demonstrate

$$|\Gamma(iy)|^2 = \frac{\pi}{y \sinh \pi y}. \tag{9.15}$$

9.6 Γ for half integers: The formula of complementary arguments allows us to compute $\Gamma(1/2)$.[154] Since this is positive as seen from the definition (9.3).

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{9.16}$$

With **9.1** we get

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi} = \frac{(2n)!}{2^{2n}n!}\sqrt{\pi}, \tag{9.17}$$

[154] This can be computed directly with the aid of the Gaussian integral as (1) In Exercise.

and

$$\Gamma\left(-n+\frac{1}{2}\right) = \frac{(-1)^n 2^n}{(2n-1)!!}\sqrt{\pi} = \frac{(-4)^n n!}{(2n)!}\sqrt{\pi}. \tag{9.18}$$

Exercise.
(1) $\Gamma(1/2)$ can be computed directly as follows:

$$\Gamma(1/2) = \int_0^\infty e^{-t}\frac{1}{\sqrt{t}}dt = \int_{-\infty}^\infty e^{-x^2}dx. \tag{9.19}$$

Hence. we have only to compute the Gaussian integral. The best method to compute this integral is the following trick:

$$\left\{\int_{-\infty}^\infty e^{-x^2}dx\right\}^2 = \int_{\boldsymbol{R}^2} dxdy e^{-(x^2+y^2)} = 2\pi\int_0^\infty e^{-r^2}rdr. \tag{9.20}$$

Complete the calculation.
(2) Compute $\Gamma(7.5)$ and $\Gamma(-1.5)$.
(3) How fast does $\Gamma(-n+1/2)$ converges to 0 in the $n \to \infty$ limit?
(4) Show

$$\lim_{n\to\infty}\frac{(2n-1)!!}{(2n)!!}n^{1/2} = \pi^{-1/2}. \tag{9.21}$$

9.7 Beta function. The *beta function* $B(p.q)$ is an analytic function of two variables obtained by the analytic completion ($\to$**7.10**) of the following integral

$$B(p.q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt. \tag{9.22}$$

$$= 2\int_0^{\pi/2} d\theta\cos^{2p-1}\theta\sin^{2q-1}\theta. \tag{9.23}$$

$$= \int_0^\infty dx\, x^{p-1}(1+x)^{-(p+q)}. \tag{9.24}$$

where $\Re p$ and $\Re q$ must be positive. The second line can be obtained by setting $t = \sin^2\theta$. and the third line by $t = x/(1+x)$. Assume $p, q \in \boldsymbol{R}$ and positive. We get ($t = x^2$ or y^2 in (9.1))

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty dx\, e^{-x^2}x^{2u-1}\int_0^\infty dy\, e^{-y^2}y^{2u-1} \tag{9.25}$$

$$= 4\int_0^\infty rdr\, e^{-r^2}r^{2(p+q-1)}\int_0^{\pi/2} d\theta\cos^{2p-1}\theta\sin^{2q-1}\theta.$$

$$= \Gamma(p+q)B(p.q). \tag{9.26}$$

Hence. we have

$$B(p.q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{9.27}$$

The RHS is meaningful for all p, q except for negative integers, so that we may define the beta function by this formula.

Exercise.
(1) Because

$$B(p,q) = B(q,p) = \int_0^\infty \frac{x^{q-1}}{(1+x)^{p+q}}dx, \tag{9.28}$$

we obtain

$$B(p,q) = \frac{1}{2}\int_0^\infty \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}dx \tag{9.29}$$

and

$$\int_0^\infty \frac{x^{p-1}-x^{q-1}}{(1+x)^{p+q}}dx = 0. \tag{9.30}$$

(2)

$$I = \int_0^{\pi/2} \sin^p\theta\cos^q\theta d\theta = \frac{1}{2}\int_0^1 x^{(p+1)/2-1}(1-x)^{(q+1)/2-1}dx = \frac{1}{2}B\left(\frac{p+1}{2},\frac{q+1}{2}\right). \tag{9.31}$$

For example,

$$\int_0^{\pi/2} \sin^p\theta d\theta = \frac{\sqrt{\pi}}{2}\Gamma((p+1)/2)/\Gamma(p/2+1). \tag{9.32}$$

This is called *Wallis' formula*, if p is a positive integer.
(3) Computing

$$\int_{-1}^1 (1-x^2)^{z-1}dx \tag{9.33}$$

with two different change of variables ($t = x^2$ and $t = (x+1)/2$), show

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right). \tag{9.34}$$

More generally, it is known that

$$\Gamma(nz) = (2\pi)^{(1-n)/2}n^{nz-1/2}\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\cdots\Gamma\left(z+\frac{n_1}{n}\right). \tag{9.35}$$

9.8 Proof of 9.5: From (9.27) and (9.24) we get for $0 < \Re z < 1$

$$\Gamma(z)\Gamma(1-z) = \Gamma(1)B(z,1-z) = \int_0^\infty dx\frac{x^{z-1}}{1+x} = \frac{\pi}{\sin\pi z}. \tag{9.36}$$

We can apply the principle of invariance of functional relation **7.6** to complete the proof of **9.5**.

Exercise.

To compute the integral in (9.36) we can also use the transformation $x = e^y$ to convert the integral to

$$\int_{-\infty}^{\infty} \frac{e^{zy}}{1+e^y} dy. \tag{9.37}$$

This is the same problem in **8B.9**.

9.9 Taylor expansion:

$$\Gamma(1+z) = 1 - \gamma z + \frac{1}{2}\left(\gamma^2 + \frac{\pi^2}{6}\right) z^2 + \cdots. \tag{9.38}$$

Here γ is called *Euler's constant* defined by

$$\gamma \equiv \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right) \tag{9.39}$$

and $\gamma = 0.577215664\cdots$.[155] □
[Demo] Calculate the logarithmic derivative of (9.3) (Uniform convergence allows termwise operations)

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+z}\right), \tag{9.40}$$

so that

$$\Gamma'(1) = -\gamma - 1 + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = -\gamma. \tag{9.41}$$

Differentiating (9.40) once more, we get

$$\frac{d^2}{dz^2}\log\Gamma(z) = \sum_{k=0}^{\infty}\frac{1}{(z+k)^2}. \tag{9.42}$$

Hence,

$$\Gamma''(1) - \Gamma'(1)^2 = \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}.^{156} \tag{9.45}$$

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[155] Whether γ is irrational or not is not known: it is known that if it is rational, both the denominator and the numerator must have at least 30,000 digits.
[156] To compute this sum or the *zeta function* (→**7.15**)

$$\zeta(z) \equiv \sum_{k=1}^{\infty}\frac{1}{k^z}, \tag{9.44}$$

we use

$$\zeta(z)\Gamma(z) = \int_0^{\infty}\frac{t^{z-1}}{e^t - 1} dt. \tag{9.45}$$

See T. M. Apostol, *Math. Intelligencer*, **5**(3), 59-60 (1983) "A proof the Euler missed: evaluation of $\zeta(2)$ the easy way." See **4.4** Discussion (1).

This gives the desired second derivative. □

Exercise.
(1) Demonstrate that the Γ-function is a convex function (→**2A.1** Discussion) for $x > 0$.
(2) Using the fact

$$\frac{\Gamma'(1)}{\Gamma(1)} = -\gamma. \tag{9.46}$$

demonstrate

$$\gamma = -\int_0^\infty e^{-t}\log t\,dt. \tag{9.47}$$

9.10 Use in perturbative field theories:[157]

(1) When we compute (bare) perturbation series, we have to compute integrals of the following type:

$$I \equiv \int d\boldsymbol{q}\frac{1}{(q^2+2\boldsymbol{k}\cdot\boldsymbol{q}+m^2)^\alpha} = \pi^{d/2}\frac{\Gamma(\alpha-d/2)}{\Gamma(\alpha)}(m^2-k^2)^{d/2-\alpha}. \tag{9.48}$$

Here the integral may not exist even when the RHS exists. In such cases the integral is <u>defined</u> by the RHS (analytic continuation). This formula can be demonstrated as follows: First we exponentiate the denominator with the aid of Euler's integral (9.1)

$$\frac{1}{a^\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty dt\, t^{\alpha-1}e^{-at} \tag{9.49}$$

as

$$I = \int d\boldsymbol{q}\int_0^\infty dt\, t^{\alpha-1}\exp[-t(q^2+2\boldsymbol{k}\cdot\boldsymbol{q}+m^2)]. \tag{9.50}$$

This is a standard trick. We can legitimately exchange the order of the two integrations (Fubini's theorem→**19.14**). and perform the d-dimensional Gaussian integral[158](→**19.19**) to get

$$I = \frac{1}{\Gamma(\alpha)}\int_0^\infty dt\, t^{\alpha-1}\left(\frac{\pi}{t}\right)^{d/2} e^{-(m^2-k^2)t}. \tag{9.51}$$

[157] J. Zinn-Justin. *Quantum Field Theory and Critical Phenomena*, (Oxford, 1989); D. J. Amit. *Field Theory, the Renormalization Group, and Critical Phenomena* (World Scientific, original from McGraw-Hill 1978).

[158]
$$\int_{-\infty}^{\infty} dq^d e^{-a^2q^2} = \left(\frac{\sqrt{\pi}}{a}\right)^d.$$

This gives the desired result. See also (D) below.
(2) We often need the integral of the product of the factors $1/(q^2+m^2)$. In this case the q-integral (momentum integral) can be reduced to (1) by the so-called *Schwinger-Feynman parameter formula*:

$$\begin{aligned}
&\frac{1}{a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n}} \\
= &\frac{\Gamma(\alpha_1+\cdots+\alpha_n)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(a_n)} \int_V dt_1 dt_2 \cdots dt_{n-1} \frac{t_1^{\alpha_1-1}\cdots t_n^{\alpha_n-1}}{(t_1 a_1+\cdots+t_n a_n)^{\alpha_1+\cdots+\alpha_n}},
\end{aligned} \tag{9.52}$$

where

$$V = \{(t_1,\cdots,t_{n-1}) : t_i \in [0,1], t_1+t_2+\cdots+t_{n-1} \le 1\}. \tag{9.53}$$

To demonstrate this we start with (9.49). We have, using Fubini's theorem

$$\prod_{i=1}^n a_i^{-\alpha_i} = \int_0^\infty dx_1 dx_2 \cdots dx_n \frac{\prod_{i=1}^n x_i^{\alpha_i-1} e^{-\sum a_i x_i}}{\prod_{i=1}^n \Gamma(\alpha_i)}. \tag{9.54}$$

Now, introduce new variables $(t_1, t_2, \cdots, t_{n-1}, y)$ as

$$x_i = t_i y, \quad (i = 1, 2, \cdots, n-1), \tag{9.55}$$

$$x_n = (1-t_1-t_2-\cdots-t_{n-1})y. \tag{9.56}$$

The Jacobian for this transformation is y^{n-1}, so that

$$\prod_{i=1}^n a_i^{-\alpha_i} = \int_0^\infty dy \int_V dt_1 \cdots dt_{n-1} y^{n-1} y^{\sum \alpha_i - n} \frac{e^{-y\sum a_i t_i}}{\prod_{i=1}^n \Gamma(\alpha_i)}. \tag{9.57}$$

This leads to the desired result. □

Exercise.
(A) Demonstrate

$$I = \int\int f(t_1+t_2) t_1^{a_1-1} t_2^{a_2-1} dt_1 dt_2 = \frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(a_1+a_2)} \int_0^1 f(t) t^{a_1+a_2-1} dt, \tag{9.58}$$

where the integration range is $t_1+t_2 \le 1$ and $t_1 > 0$, $t_2 > 0$.

(B) From a similar calculation as (A), we get the following formula:

$$\begin{aligned}
&\int\int\cdots\int f(t_1+t_2+\cdots+t_n) t_1^{a_1-1} t_2^{a_2-1} \cdots t_n^{a_n-1} dt_1 dt_2 \cdots dt_n \\
&\qquad = \frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_n)}{\Gamma(a_1+a_2+\cdots+a_n)} \int_0^1 f(t) t^{a_1+a_2+\cdots+a_n-1} dt
\end{aligned} \tag{9.59}$$

where the integration range is $t_i > 0$ and $t_1 + \cdots + t_n < 1$ as in (A). You need not demonstrate this formula (if you feel it is correct). Using this formula, demonstrate that the volume V of the n-ball of radius r is given by

$$V = \frac{2r^n\pi^{n/2}}{n\Gamma(n/2)}. \tag{9.60}$$

Compute the surface area of the n-ball (i.e., the volume of the $(n-1)$-sphere).

Estimate the ratio of the volume of d-ball and that of its thin skin of thickness $\epsilon \ll 1$ for very large d.[159] [Hint. Actually, dimensional analysis is enough. Look at the ratio of the volume of n-sphere of radius r and that of radius $r - \epsilon$.]

(C) This formula can be generalized to the following. Let D be a domain in n-space defined by

$$\left(\frac{x_1}{a_1}\right)^{b_1} + \cdots + \left(\frac{x_n}{a_n}\right)^{b_n} \le 1 \tag{9.61}$$

and $x_1 \ge 0, \cdots, x_n \ge 0$.

$$\int \cdots \int_D dx_1 dx_2 \cdots dx_n x_1^{l_1-1} \cdots x_n^{l_n-1} = \frac{a_1^{l_1}\cdots a_n^{l_n}}{b_1\cdots b_n}\frac{\Gamma\left(\frac{l_1}{b_1}\right)\cdots\Gamma\left(\frac{l_n}{b_n}\right)}{\Gamma\left(\frac{l_1}{b_1}+\cdots+\frac{l_n}{b_n}+1\right)}. \tag{9.62}$$

(D) Demonstrate (9.48). That is,

$$\int dq \frac{1}{(q^2+2kq+m^2)^{\alpha}} = \frac{1}{2}S_{d-1}\frac{\Gamma(d/2)\Gamma(\alpha-d/2)}{\Gamma(\alpha)}(m^2-k^2)^{d/2-\alpha}. \tag{9.63}$$

9.11 Stirling's formula.[160] Uniformly in $|\arg z| \le \pi - \delta$ for any small positive δ,

$$\Gamma(z) \sim \sqrt{2\pi}e^{-z}z^{z-1/2}\left[1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51849z^3} + \cdots\right]. \tag{9.64}$$

Here $\sim$ implies that the expansion is asymptotic (→**25.3**, **25.14**).□

A practical way to remember the salient feature is

$$n! \sim (n/e)^n. \tag{9.65}$$

[Demo][161] We only demonstrate

$$\frac{\Gamma(n)e^n\sqrt{n}}{n^n} \to \sqrt{2\pi} \quad \text{as} \quad n \to \infty. \tag{9.66}$$

[159] In very high dimensional spaces, almost all the volume is always very close to the skin. This is a very important fact for statistical mechanics, and coding theory.

[160] James Stirling, 1692-1770.

[161] J. M. Patin, *Am. Math. Month.* **96**, 41-42 (1989).

In Euler's integral (9.1) set $x = \sqrt{t} - \sqrt{n}$ to find

$$\frac{\Gamma(n)e^n\sqrt{n}}{n^n} = 2\int_{-\sqrt{n}}^{\infty}\left(1+\frac{x}{\sqrt{n}}\right)^{2n+1} e^{-2\sqrt{n}x}e^{-x^2}dx \tag{9.67}$$

Now, the integrand is uniformly bounded in n by the integrable function e^{-x^2-1}, because

$$\left(1+\frac{x}{\sqrt{n}}\right)^{2n-1} e^{-2\sqrt{n}x} \le \exp\left\{\frac{x}{\sqrt{n}}(2n-1)\right\}\exp(-2\sqrt{n}x) = e^{-x/\sqrt{n}} \le e. \tag{9.68}$$

Since for each x

$$\lim_{n\to\infty} \log\left\{\left(1+\frac{x}{\sqrt{n}}\right)^{2n-1} e^{-2\sqrt{n}x}\right\} = -x^2. \tag{9.69}$$

the dominated convergence theorem[162] tells us

$$\frac{\Gamma(n)e^n\sqrt{n}}{n^n} \to 2\int_{-\infty}^{\infty} e^{-2x^2}dx = \sqrt{2\pi}. \tag{9.70}$$

□

Discussion.[163] The above proof does not tell us why the ratio (9.66) must be considered. Let us give a more 'constructive' proof.

(1) Notice that for $\mathbb{R}z > 0$

$$\Gamma'(z) = \int_0^{\infty} e^{-t}t^{z-1}\ln t\, dt. \tag{9.71}$$

(2) To rewrite $\ln t$ let us show that integration of

$$\frac{1}{t} = \int_0^{\infty} e^{-xt}dx \tag{9.72}$$

implies for $\mathbb{R}t > 0$

$$\ln t = \int_0^{\infty} \frac{e^{-x} - e^{-xt}}{x}dx. \tag{9.73}$$

This integral is called Frullani's integral.[164]

(3) Combining the above results, we obtain

$$\Gamma'(z) = \int_0^{\infty}\frac{dx}{x}\left[e^{-x}\Gamma(z) - \int_0^{\infty} e^{-t(x+1)}t^{z-1}dt\right]. \tag{9.74}$$

[162] Again, this is a rudimentary theorem of Lebesgue integral (→**19.11**).

[163] This is adapted from Lebedev.

[164] To justify the changing the order of integrations, we may rely on Fubini's theorem (→**19.14**). The same is true for the exchange in (3).

(4) From this we obtain

$$\frac{d}{dz}\ln\Gamma(z+1) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{e^t-1}\right)dt, \tag{9.75}$$

$$= \int_0^\infty \frac{e^{-t}-e^{-tz}}{t}dt + \frac{1}{2}\int_0^\infty e^{-tz}dt - \int_0^\infty \left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)e^{-tz}dt, \tag{9.76}$$

$$= \ln z + \frac{1}{2z} - \int_0^\infty \left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)e^{-tz}dt. \tag{9.77}$$

(5) Integrating this with z from 0 to z, we obtain

$$\ln\Gamma(z) = \left(z-\frac{1}{2}\right)\ln z - z + 1 + \int_0^\infty \left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-t}-e^{-tz}}{t}dt. \tag{9.78}$$

(6) This can be rewritten as

$$\ln\Gamma(z) = \left(z-\frac{1}{2}\right)\ln z - z + \omega(z) - \omega(1). \tag{9.79}$$

where

$$\omega(z) = \int_0^\infty f(t)e^{-tz}dt \tag{9.80}$$

with

$$f(t) = \left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{1}{t}. \tag{9.81}$$

To compute $\omega(1)$. notice that

$$\omega(1/2) - \omega(1) = \int_0^\infty \left(\frac{e^{-t/2}}{t} - \frac{1}{e^t-1}\right)dt \tag{9.82}$$

but this can be obtained from the result of (5) with $z = 1/2$ ($\to$**9.6**) as

$$\omega(1/2) - \omega(1) = \frac{1}{2}\ln\pi - \frac{1}{2}. \tag{9.83}$$

On the other hand. we can compute $\omega(1/2)$ directly as

$$\omega(1/2) = \frac{1}{2} + \frac{1}{2}\ln\frac{1}{2}. \tag{9.84}$$

Hence. $\omega(1) = -(1/2)\ln 2\pi$

(7) For large $\mathbb{R}z > 0$ we can expect that ω is small. Actually it is of order $1/z$.