8 Contour Integration

A contour integral of a complex function is nonzero only when it is not holomorphic in the region encircled by the contour. Hence. the integral is determined by the singularities of the integrand. Consequently, it is worth paying close attention to singularities. Starting with the classification of singularities, we discuss the Laurent expansion of a function around an isolated singularity. and the definition of the residue, which determines the value of integrals. Cauchy's theorem, which allows us to deform the integration path on the complex plane and the residue theorem sometimes allow us to compute definite integrals neatly. Cauchy devised complex function theory with the motivation of unifying the computation of definite integrals.

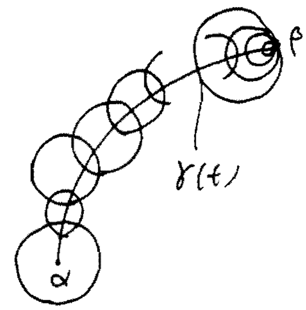
Key words: singularity. isolated singularity, branch point, degree of ramification. Laurent expansion, principal part. pole. genuine singularity. Casorati-Weierstrass' theorem. residue theorem. principal value, Plemelj formula. hyperfunction.

Summary
(1) Fate of analytic continuation around a singularity classifies the singularity (**8A.3**).
(2) Around an isolated non-branching singularity, we can expand the function into Laurent series (**8A.6**).
(3) Its coefficient of the $(z-a)^{-1}$ term is called the residue and the integral is determined by residues (**8B.2**).
(4) Deformation of the integration contour thanks to the Cauchy theorem + the residue theorem can allow many definite integrals to be evaluated exactly (**8B**). The only way to be familiar with this technique is to practice.

8.A Singularities

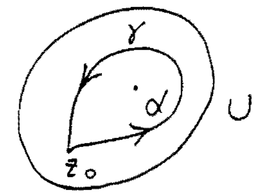
8A.1 Integral and singularities. The value of the contour integral can be nonzero only when the integrand is not holomorphic in the region encircled by the contour as can be seen from Cauchy's theorem **6.3**. The points where the function becomes nonholomorphic is called singularities. This is one good reason to pay close attention to singularities of a function. The nature and location of singularities completely specifies an analytic function as can be guessed from the theorem of identity (→**7.5**).

8A.2 Singularity. Let (P, U) be a function element (→**7.7**) defined by a power series P whose convergence disk is U centered at α. Consider a curve $\gamma : [0, 1] \to \boldsymbol{C}$ connecting α and $\beta \in \boldsymbol{C}$ such that $\gamma(0) = \alpha, \gamma(1) = \beta$ and $\gamma(t) \neq \beta$ for $t \neq 1$. If analytic continuation along γ up to $\gamma(t)$ for any $t < 1$ is possible, but not possible to β, we say the analytic function determined by the function element (P, U) has a *singularity* or a *singular point* at β.

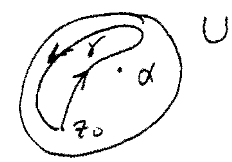


8A.3 Classification of isolated singularities. If α is a singular point of an analytic function f, and there is $r > 0$ such that there is no singularity of f in $0 < |z - \alpha| < r$, then α is called an *isolated singularity* of f.

Let α be an isolated singularity of an analytic function (→**7.10**) f. If a curve γ encircles the singularity α, there are two possible cases for the analytic continuation along the curve around α to the starting point:
(1) The result gives the same function element (P, V). In this case the singularity α is called a *non-branch point.*



(2) The result does not give the same function element. In this case α is called a *branch point.* We say we arrive at a different *branch* of the function (cf. **7.8** Discussion).



8A.4 Classification of branch point. Branch points are classified into two cases:
(2a) If the analytic continuation along γ which goes around α gives the original function element (P, U) at z_0 for the first time after going around α $m(> 1)$-times, α is called an *algebraic branch point.* $m - 1$ is called the *degree of ramification* of the branch point.
(2l) If there is no such finite m, α is called a *logarithmic branch point* (→**4.7**, see below (B)).[143]

Discussion.
(A) Read a passage from Gauss' (→**6.17**) letter to Bessel dated December 28, 1811 (not very faithfully translated as the reader sees from the modern notations. Bessel wished to introduce a new special function (→**23.5**) $\text{li}\, x = \int dx/\log x$):
"...., If a person wishes to introduce a new function in analysis, I ask first whether he confines his variable to $\boldsymbol{R}$, and regards imaginary numbers as superfluous (Überbein), or, as I accept as a principle, he allows imaginary numbers to enjoy the same right in the world of numbers. I am not discussing the practical merits. I see analysis as an independent branch of science. If those hypothetical 'imaginary numbers' were excluded from analysis, then we would suffer from enormous loss in aesthetics and flexibility, and would have to impose complicated restrictions on the truth which should be quite general. Well, when $x \in \boldsymbol{C}$ what do we mean by $\int \varphi(x)dx$? Its

[143]There are no other cases thanks to the Poincaré-Volterra theorem **7.14**.

value does not depend on the paths, if $\varphi(x) \neq \infty$ in the plane bounded by these paths. $\cdots$. By the way as is clear from the statement, the function defined by $\int \varphi(x)dx$ can be a multivalued function, because the path may not go around the point where $\varphi(x) = \infty$ or may go around it once or a few times. For example, $\cdots$" (then, Gauss discussed log defined by the integral as in (8.1) below, and continued) "However, if the function is not infinite, then the integral is univalent. For example, consider

$$\varphi(x) = \frac{e^x - 1}{x}.$$

Its integral is univalent and can be expanded into the following series

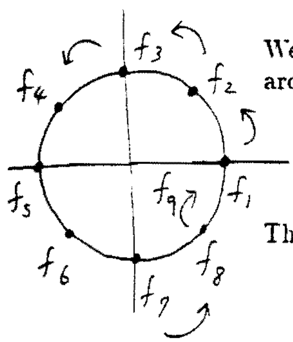
$$x + \frac{1}{2}x^2 + \frac{1}{18}x^3 + \frac{1}{96}x^4 + \cdots,$$

which is always convergent and has the unique meaning."

It is clear that Gauss had alread known by 1811 what Cauchy (→**6.11**) reached at last in 1851.[144]

(B) For $|z - 1| < 1$, we have

$$\log z = \int_1^z \frac{1}{\zeta} d\zeta = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(z-1)^k}{k} \equiv f_1(z) \tag{8.1}$$



We wish to directly analytically continue this around the unit circle. The expansion around $e^{i\pi/4}$ is obtained as

$$f_2(z) = \frac{i\pi}{4} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(z - e^{i\pi/4})^k}{k e^{ik\pi/4}}. \tag{8.2}$$

This can be obtained from

$$f_1(e^{i\pi/4}) = \int_1^{e^{i\pi/4}} \frac{1}{\zeta} d\zeta \tag{8.3}$$

and

$$f_1^{(n)}(e^{i\pi/4}) = (-1)^{n+1} e^{-in\pi/4}. \tag{8.4}$$

In this way after one rotation we obtain

$$f_9(z) = 2\pi i + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(z-1)^k}{k} \equiv f_1(z). \tag{8.5}$$

Exercise.

Compute the integral

$$\int_{-1-\sqrt{3}i}^{-1+\sqrt{3}i} z^{1/2} dz. \tag{8.6}$$

[144] based on T. Takagi, *Kinsei Sugaku Shidan* (Tales from Modern Mathematics History) (Iwanami, 1995; original from Kyoritu Publ. 1933).

8A.5 Classification of non-branch isolated singular points. Non-branch isolated singular points are classified according to the principal part (→**8A.10**) of the Laurent series (→**8A.8**) around the singularity:
(1) If there is no principal part, we can redefine appropriately the value of f at the singularity to make it holomorphic, so the singularity is called a *removable singularity.*
(2) If the principal part is a finite series, the singularity is called a *pole.* If the largest negative power of the principal part is $-m$ ($m > 0$), then the pole is called a *pole of order m.*
(3) If the principal part is an infinite series. the singularity is called a *genuine singularity* (→**8A.7**).

Exercise.
Compute the Laurent expansion around $z = a$ of

$$e^{1/(z-a)} \tag{8.7}$$

We know $z = a$ is a genuine singularity of the function.

8A.6 Branch point can be found by formal expansion. If the reader suspects α is a branch point of a function f. formally expand it around α. Then. as can be seen from the example. she can tell whether it is a branch point or not.

Consider a trivial example $\sqrt{z-1}$. $z = 1$ is an algebraic branch point with degree of ramification (→**8A.4** (2a)) 1. Let us formally expand it around $z = 1$ with the aid of the binomial theorem:

$$(z-1)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} z^n (-1)^{1/2-n}. \tag{8.8}$$

where the binomial coefficients are defined as usual:

$$\binom{1/2}{n} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)}{n(n-1)\cdots 2\cdot 1}. \tag{8.9}$$

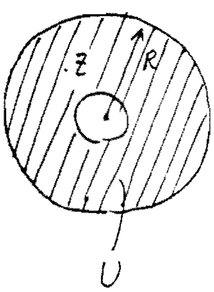
Obviously due to the double valuedness of $(-1)^{1/2}$ (→**4.9**), $z = 1$ must be a branch point.

8A.7 Casorati-Weierstrass's theorem. Let a be an isolated genuine singularity (→**8A.5**) of an analytic function $f(z)$. Then f assumes values arbitrarily close to any value in $\boldsymbol{C}$. Or, more explicitly. for any $w \in \boldsymbol{C}$ (including ∞) there is a sequence $\{z_n\}$ such that $z_n \to a$ and $f(z_n) \to w$. □

[Demo] Let $w = \infty$ and assume that there is no sequence which makes $f(z_n) \to \infty$. Then. f must be finite. so a is a removable singularity. contrary to the assumption. Hence. there must be a sequence required in the theorem for $w = \infty$. Suppose w is finite. Consider $1/(f(z)-w)$. z is still a genuine singularity of this function, but this is finite at a, a contradiction. □

Exercise.
Illustrate that the theorem is true for $e^{1/z}$.

8A.8 Laurent's theorem. Let $f(z)$ be holomorphic in $U \equiv \{z : 0 \le r < |z-\alpha| < R \le +\infty\}$ and α be not a branch point.[145] Then, f can be expanded in U in the following normally convergent series called the *Laurent series*:



$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-\alpha)^n, \tag{8.10}$$

where the coefficients are defined by

$$c_n = \frac{1}{2\pi i}\int_{\partial\{z:|z-\alpha|<s\}} (z-\alpha)^{-n-1} f(z)dz \tag{8.11}$$

with $s \in (r, R)$. □

Notice that there must be a singularity on the convergence circle for a Laurent series, since it is a kind of power series (→**7.3**).

[Demo] Choose r' and R' such that $r < r' < R' < R$. Then, f is holomorphic in $V \equiv \{z : r' < |z-\alpha| < R'\}$, so Cauchy's formula (→**6.10**) tells us

$$f(z) = \frac{1}{2\pi i}\int_{\partial\{\zeta:|\zeta-\alpha|<R'\}} \frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_{\partial\{\zeta:|\zeta-\alpha|<r'\}} \frac{f(\zeta)}{\zeta - z}d\zeta. \tag{8.12}$$

for $z \in V$. Notice that $|z-\alpha|/|\zeta-\alpha| < 1$ for the first term on RHS, and > 1 for the second term. Therefore, we may expand $1/(\zeta - z)$ to obtain

$$\begin{aligned} f(z) = & \frac{1}{2\pi i}\int_{\partial\{\zeta:|\zeta-\alpha|<R'\}} \sum_{n=0}^{\infty}\left(\frac{z-\alpha}{\zeta-\alpha}\right)^n \frac{f(\zeta)}{\zeta-\alpha}d\zeta \\ & + \frac{1}{2\pi i}\int_{\partial\{\zeta:|\zeta-\alpha|<r'\}} \sum_{n=0}^{\infty}\left(\frac{\zeta-\alpha}{z-\alpha}\right)^n \frac{f(\zeta)}{z-\alpha}d\zeta. \end{aligned} \tag{8.13}$$

The power series are uniformly convergent, so that we may exchange the order of summations and integrations. □

Exercise.
(A) Let

$$f(z) = \frac{1}{(1+z^2)(z+2)}. \tag{8.14}$$

[145] This could be a non-singular point.

Check that its Laurent expansion around $z = 0$ for
(a) $|z| < 1$ is

$$f(z) = \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{1}{2^{2n+1}} + (-1)^n 2\right) z^{2n} - \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{1}{2^{2n+2}} + (-1)^n\right) z^{2n+1}. \quad (8.15)$$

This is the Laurent series needed to classify the singularity at $z = 1$. However, as seen below, the expansion takes different forms according to the domain of z away from the singularity.
(b) $1 < |z| < 2$ is

$$f(z) = \frac{1}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}z^n + \frac{1}{5}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}} - \frac{1}{5}\sum_{n=1}^{\infty}\frac{(-1)^n 2}{z^{2n}} \quad (8.16)$$

(c) $|z| > 2$ is

$$f(z) = \frac{1}{5}\sum_{n=1}^{\infty}\frac{2^{2n} + (-1)^{n+1}}{z^{2n+1}} - \frac{1}{5}\sum_{n=1}^{\infty}\frac{2^{2n-1} + (-1)^n 2}{z^{2n}}. \quad (8.17)$$

(B) For $0 < |z| < \infty$

$$\cosh\left(z + \frac{1}{z}\right) = a_0 + \sum_{n=1}^{\infty} a_n\left(z^n + \frac{1}{z^n}\right). \quad (8.18)$$

where

$$a_n = \frac{1}{\pi}\int_0^{\pi}\cos n\theta \cosh(2\cos\theta)d\theta. \quad (8.19)$$

(C) $f(z) = (z-2)^{-1} - (z-1)^{-1}$ There are two isolated singularities 1 and 2. Hence, the Laurent expansion around $z = 0$ changes its form in three distinct regions, $|z| < 1$, $1 < |z| < 2$ and $|z| > 2$. For $|z| < 1$, this is nothing but the ordinary Taylor series. For $1 < |z| < 2$, we have

$$\begin{aligned} f(z) &= \frac{1}{2(z/2-1)} - \frac{1}{z(1-1/z)} \\ &= -\frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \cdots - \frac{1}{z} - \frac{1}{z^2} - \cdots. \end{aligned} \quad (8.20)$$

For $|z| > 2$ we have

$$f(z) = \sum_{n=2}^{\infty}\frac{2^{n-1}-1}{z^n}. \quad (8.21)$$

8A.9 Uniqueness of the Laurent expansion. This should be clear from the normal convergence of the Laurent series: Compute the coefficients of $f(z) = \sum_{-\infty}^{+\infty} b_n(z-\alpha)^n$ according to (8.11).

8A.10 Principal part. The negative power portion of the Laurent series (8.10) is called the *principal part* of the series. The rest is sometimes called the *regular part*. Taylor's theorem (→**6.15**) tells us that if a function is holomorphic at α, the Laurent series around the point does not have any principal part.

8A.11 Examples. In contrast to the Taylor series, there is no general method to compute the coefficients of the Laurent series. This is true even for the regular part, since the function is not differentiable at the expansion center. However, the uniqueness **8A.9** implies that if the reader can somehow get a Laurent series, then it is the Laurent series.

Exercise.
(A) Laurent-expand the following functions.
(1) $f(z) = 5/(z^2+1)(z+2)$ has three poles of order 1 at i, $-i$ and -2. Its Laurent expansion around $z = i$ in the range $|z-1| < 2$ is

$$f(z) = \sum_{n=0}^{\infty}(-1)^n \frac{5}{2-i}\left\{\frac{1}{(2i)^{n+1}} - \frac{1}{(2+i)^{n+1}}\right\}(z-i)^{n-1}. \tag{8.22}$$

(2) $f(z) = z/(z+1)^2(z+2)$ around $z = -1$.

$$f(z) = \frac{1}{z+1} - 2 + 2(z+1) - 2(z+1)^2 + \cdots + 2(-1)^{n+1}(z+1)^n + \cdots. \tag{8.23}$$

(3) $f(z) = z^2\exp(1/z)$ around $z = 0$.

$$f(z) = \frac{1}{2} + z + z^2 + \sum_{n=1}^{\infty}\frac{1}{(n+2)!z^n}. \tag{8.24}$$

(4) $f(z) = \exp(1/(1-z))$ around $z = 1$.

$$f(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(z-1)^n}. \tag{8.25}$$

(B) Let $f(z) = \exp[\alpha(z - z^{-1})/2]$.

$$f(z) = \sum_{n=-\infty}^{+\infty} J_n(\alpha)z^n, \tag{8.26}$$

where

$$J_n(\alpha) \equiv \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - \alpha\sin\theta)d\theta \tag{8.27}$$

is called the *Bessel function* of the first kind (→**27A.21**). f is called its *generating function* (→**27A.5**).

8A.12 Laurent expansion and Fourier expansion: Let $f(z)$ be holomorphic in the open strip $-a < \Im z < a$, and $f(z+2\pi) = f(z)$. Then,

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \tag{8.28}$$

with

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta. \tag{8.29}$$

The series converges normally in the strip (cf. **17.1**, **17.12(3)**). □

[Demo] Let $\zeta \equiv e^{iz}$. Then the strip is mapped one-to-one to the annule $e^{-a} < |\zeta| < e^{a}$. Notice that $f(z)$ is mapped to a univalent holomorphic function $g(\zeta)$ thanks to the periodicity of f. Let us Laurent expand g around $\zeta = 0$ as

$$g(\zeta) = \sum_{n=-\infty}^{\infty} c_n \zeta^n, \tag{8.30}$$

with

$$c_n = \frac{1}{2\pi i}\int_{\partial\{\zeta:|\zeta|<1\}} g(\zeta)\zeta^{-n-1}d\zeta. \tag{8.31}$$

Now, let us return to the original variable z. $f(z) = g(e^{iz})$ and (8.30) becomes (8.28) and (8.31) is translated into (8.29). □

Discussion.

Let $f(z)$ be holomorphic in the region containing the unit disk $|z| \le 1$. Let its Laurent expansion be

$$f(z) = \sum_{n=-\infty}^{\infty} A_n z^n. \tag{8.32}$$

Then

$$g(\theta) = \sum_{n=-\infty}^{\infty} A_n e^{in\theta} \tag{8.33}$$

is the Fourier expansion of $f(e^{i\theta})$.

8.B Contour Integration

8B.1 Residue: Let α be a unique singularity of an analytic function in $U \equiv \{z : 0 < |z-\alpha| < R\}$. If the singularity is not a branch point (→**8A.3**).

$$Res(\alpha; f) \equiv \frac{1}{2\pi i}\int_{\partial U} f(z)dz \tag{8.34}$$

is called the *residue* of f at α. Notice that the value does not depend on R as long as U contains only one singularity.

8B.2 Residue theorem: Let f be an analytic function holomorphic in a region D except at a finite number of points $a_1, \cdots, a_n$, and ∂D be piecewise C^1 curve. Then

$$\int_{\partial D} f(z)dz = 2\pi i \sum_j Res(a_j; f). \tag{8.35}$$

□
This is obvious from the Cauchy theorem **6.3** and the definition of residues (8.34).

8B.3 How to get residues:
(i) If we have the Laurent expansion (→**8A.8**) of f around a singularity a: $\sum c_n(z-a)^n$, then

$$Res(a; f) = c_{-1}. \tag{8.36}$$

(ii) If a is a pole of order m of f, then

$$Res(a; f) = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)]. \tag{8.37}$$

In particular, if a is a pole of order 1 (simple pole), then

$$Res(a; f) = \lim_{z \to a}(z-a)f(z). \tag{8.38}$$

Use l'Hospital's theorem to compute this.[146] This can be demonstrated easily with the aid of Cauchy's formula for derivatives (→**6.14**).

Exercise.
(A) Check that the residues of the following functions are given as follows:
(1) $z^4/(z^2-c^2)^4$. Its poles are at $\pm c$ and the order is 4. Both have the same residue $-1/(32c^3)$.
(2) $z/\sin z$. Poles are at $n\pi$ for any $n \in \mathbf{Z}$ except 0. All are order 1 and the residue is given by $(-1)^n n\pi$.
(3) $\cot \pi z/(z-a)^2$. The poles are at a and $n \in \mathbf{Z}$. The residue at $n (\neq a)$ is given by $1/\pi(n-a)^2$. If $a \notin \mathbf{Z}$, then it is second order and the residue is $-\pi/\sin^2 \pi a$. If $a \in \mathbf{Z}$, then it is a third order pole and its residue is $-\pi/3$.
(4) $z/\sin z$. Poles are at $n\pi$ for any $n \in \mathbf{Z}$ except 0. All are order 1 and the residue is given by $(-1)^n n\pi$.
(5) $\cot z/(z-1)^2$. The poles are $n \in \mathbf{Z}$. $z = 1$ is a third order pole and its residue is $-\pi/3$. (B) Find the residues (the answers are in the square brackets).
(1) $\cot z$ at $z = 0$. [1]
(2) $\log(1-z)/\sin^2 z$ at $z = 0$. [−1]

[146] L'Hospital's theorem or rule is actually due to Johann Bernoulli. L'Hospital, who got his calculus lesson from Bernoulli, wrote the first textbook of differential calculus, *Analyses des infiniment petits* (1696).

(3) $z/(\sin z - \tan z)$ at $z = 0$. [0].

~~(4) $z/\sin z$ at its poles. [$n\pi$ for $n \in \mathbb{Z} \setminus \{0\}$]~~

(5) $z^{2n}/(1+z)^n$ ($n \in N \setminus \{0\}$). [$(-1)^{n+1}(2n)!/(n-1)!(n+1)!$]

8B.4 Rational functions of sin and cos: Let $f(s,t)$ be a ratio of two polynomials of s and t.[147]

$$I = \int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta \tag{8.39}$$

can be computed with the aid of the residue theorem. Let $z = e^{i\theta}$. Then. the integral reads

$$I = \frac{1}{i}\int_{\partial(|z|<1)} f((z+z^{-1})/2.\,(z-z^{-1})/2i)\frac{dz}{z}. \tag{8.40}$$

Since f is rational. the integrand is a rational function of z. Thus the integration reduces to the problem of finding residues for the poles inside the unit disk.

Exercise.
(1)

$$I = \int_0^{2\pi} \frac{d\theta}{1+8\cos^2\theta} = \frac{2\pi}{3}. \tag{8.41}$$

Solution.

$$I = \frac{1}{i}\int_{\partial\{z||z|\le 1\}} \frac{1}{1+8\left(\frac{z+z^{-1}}{2}\right)^2}\frac{dz}{z} = \frac{1}{i}\oint \frac{z dz}{2z^4+5z^2+2} \tag{8.42}$$

The integrand has simple poles at $\pm 2i$ and $\pm i/\sqrt{2}$. Hence,

$$I = 2\pi\left[Res(i/\sqrt{2}; f) + Res(-i/\sqrt{2}; f)\right] \tag{8.43}$$

with both the residues equal to 1/6. Since the pole is simple, the residue can be obtained easily with the aid of l'Hospital's rule (→**8B.3**):

$$Res(i/\sqrt{2}; f) = \lim_{z\to i/\sqrt{2}} \frac{(z-i/\sqrt{2})z}{2z^4+5z^2+2} = \lim_{z\to i/\sqrt{2}} \frac{2z-i/\sqrt{2}}{8z^3+10z} = \frac{1}{6}. \tag{8.44}$$

(2) For $a > b > 0$

$$I = \int_0^{\pi} \frac{1}{a+b\cos\theta}d\theta = \frac{\pi}{\sqrt{a^2-b^2}}. \tag{8.45}$$

Noting that $\cos\theta$ is an even periodic function of period 2π.

$$I = \frac{1}{2}\int_0^{2\pi} \frac{1}{a+b\cos\theta}d\theta = \frac{i}{2}\int_{\partial\{z||z|\le 1\}} \frac{dz}{bz^2+2az+b}. \tag{8.46}$$

[147] Such a function is called a rational function of s and t.

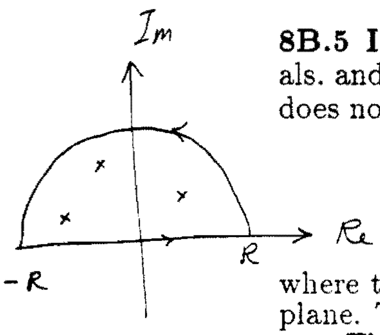
Hence.

$$I = \pi Res\left(\frac{-a+\sqrt{a^2-b^2}}{b}; f\right). \tag{8.47}$$

Its calculation is similar to (1).

(3) For $a > b > 0$

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}. \tag{8.48}$$



8B.5 Integral of rational functions: Let P and Q be polynomials. and the order of Q be not smaller than 2 plus the order of P. If Q does not have any zero point on the real axis, then

8B.3 ii can be used

$$I = \int_{-\infty}^{\infty} \frac{P(z)}{Q(z)} dz = 2\pi i \sum_j Res(a_j; P/Q). \tag{8.49}$$

where the sum is over all the poles of the integrand on the upper half plane. This can be demonstrated with the aid of the contour in the figure. The contribution from the large semicircle vanishes in the $R \to \infty$ limit (the condition on Q is required to guarantee this).

Exercise.

(1)

$$I = \int_0^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}. \tag{8.50}$$

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{1}{2}2\pi i\left[Res\left(e^{i\pi/4}; f\right) + Res\left(e^{i3\pi/4}; f\right)\right]. \tag{8.51}$$

The residues can be computed with the aid of l'Hospital's rule as

$$Res\left(e^{i\pi/4}; f\right) = \frac{1}{4e^{3\pi i/4}}. \tag{8.52}$$

We know the integral must be real. so we do not need the real part of the residues. $\Im Res(e^{i\pi/4}; f) = -\sqrt{2}/8$.

(2)

$$\int_0^{\infty} \frac{dx}{1+x^6} = \frac{\pi}{3}. \tag{8.53}$$

In this case perhaps a better way to compute is **8B.8**. See Exercise (3) there.

(3)

$$I = \int_{-\infty}^{\infty} \frac{x^2}{x^4+10x^2+9} dx = \frac{5}{8}\pi. \tag{8.54}$$

1/4

The integrand has four simple poles at $\pm 3i$ and $\pm i$.

8B.6 Integral of rational function times sin **or** cos: Let R be a rational function without any pole on the real axis and the order of the denominator is no less than the order of the numerator +1. The definite integral

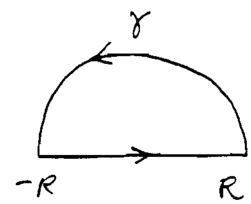
$$\int_{-\infty}^{\infty} R(x)\{\cos \alpha x \text{ or } \sin \alpha x\} dx \tag{8.55}$$

($\alpha > 0$) is computed as the real or imaginary part of

$$\int R(z) e^{i\alpha z} dz = 2\pi i \sum_j Res(a_j; R(z)e^{i\alpha z}), \tag{8.56}$$

where the sum is over all the poles of R in the upper half plane. We can use the same contour as in **8B.5** to demonstrate the formula with the aid of the following lemma:

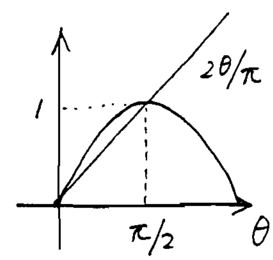
8B.7 Jordan's lemma: Let f be rational.[148] and for $\theta \in [0.\pi]$ $f(Re^{i\theta}) \to 0$ uniformly in the $R \to \infty$ limit. Then. for the semicircle γ in the figure



$$\lim_{R\to\infty} \int_\gamma e^{i\alpha z} f(z) dz = 0 \tag{8.57}$$

for any $\alpha > 0$. (For $\alpha < 0$ we have a similar lemma using the semicircle in the lower half plane.) □

[Demo] Let $M(R)$ be the maximum value of $|f|$ on γ. Then



$$\left|\int_\gamma e^{i\alpha z} f(z)dz\right| \le RM(R)\int_0^\pi e^{\alpha R\sin\theta}d\theta = 2RM(R)\int_0^{\pi/2} e^{-\alpha R\sin\theta}d\theta. \tag{8.58}$$

Notice that $2\theta/\pi \le \sin\theta$ for $\theta \in [0.\pi/2]$. so we can have the following upper bound of the above formula

$$2RM(R)\int_0^{\pi/2} e^{2\alpha R\theta/\pi}d\theta = \frac{\pi M(R)}{\alpha}(1 - e^{-\alpha R}) < \frac{\pi}{\alpha}M(R). \tag{8.59}$$

We know $M(R) \to 0$ as $R \to \infty$. □

Exercise.

(1)

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{1 + x + x^2} dx = \frac{2\pi}{\sqrt{3}}\left(\cos\frac{1}{2}\right)e^{-\sqrt{3}/2}. \tag{8.60}$$

I is the real part of the following integral

$$J = \int_{-\infty}^{\infty} \frac{e^{iz}}{1 + z + z^2} dz = 2\pi i Res\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i; f\right) = 2\pi i \frac{1}{\sqrt{3}i} e^{-i/2-\sqrt{3}/2}. \tag{8.61}$$

[148]Actually. f may be meromorphic (i.e.. there are only poles as singularities) on the upper half plane including the real axis.

(2) For $a > 0$

$$I = \int_{-\infty}^{\infty} \frac{\cos ax}{x^2 + a^2} dx = \frac{\pi}{2a} e^{-aa}. \tag{8.62}$$

(3) For $a > 0$

$$I = \int_{0}^{\infty} \frac{x \sin ax}{a^2 + x^2} dx = \frac{\pi}{2} e^{-aa}. \tag{8.63}$$

This is the imaginary part of

$$J = \frac{1}{2} \int_{-\infty}^{\infty} \frac{z e^{iaz}}{a^2 + z^2} dz. \tag{8.64}$$

The residue can be calculated just as the examples above.

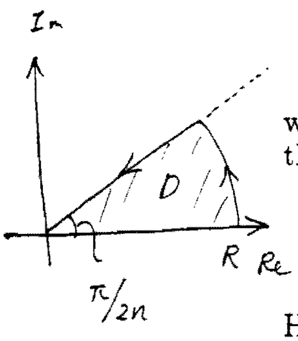
(4)

$$\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}. \tag{8.65}$$

This is obtained from

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{z} dz = \pi i. \tag{8.66}$$

8B.8 Integral of $\cos z^n$ or $\sin z^n$: Consider



$$\int_{\partial D} e^{iz^n} dz = 0. \tag{8.67}$$

where D is given in the figure. The contribution from the portion of the circle goes to zero as $R \to \infty$, so we get

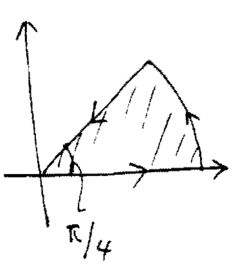
$$\int_0^\infty e^{ix^n} dx = \int_0^\infty e^{-x^n} \left(\cos \frac{\pi}{2n} + i \sin \frac{\pi}{2n} \right) dx. \tag{8.68}$$

Hence, the integral is reduced to the calculation of

$$\int_0^\infty e^{-x^n} dx. \tag{8.69}$$

This can be reduced further to the calculation of the Γ-function ($\to$**9.1**) by changing the variable $x \to t = x^n$.

This is an example of reducing the integral to other known integrals.



Exercise.

(1) Fresnel integral:

$$\int_0^\infty \sin(x^2) dx = \frac{\sqrt{\pi}}{2\sqrt{2}}. \tag{8.70}$$

The integration path is the boundary of the fan shape of angle $\pi/4$. Without computing the actual value, can you show that the integral is indeed well-defined?
(2)

$$\int_0^\infty \cos(x^3)dx = \frac{1}{2\sqrt{3}}\Gamma\left(\frac{1}{3}\right), \quad \int_0^\infty \sin(x^3)dx = \frac{1}{6}\Gamma\left(\frac{1}{3}\right). \tag{8.71}$$

For Γ, see **9**. This is obtained from

$$\int_0^\infty e^{ix^3}dx = \int_0^\infty e^{-x^3}dx\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right). \tag{8.72}$$

Then, we use

$$\int_0^\infty e^{-x^3}dx = \frac{1}{3}\int_0^\infty e^{-t}t^{-2/3} = \frac{1}{3}\Gamma\left(\frac{1}{3}\right). \tag{8.73}$$

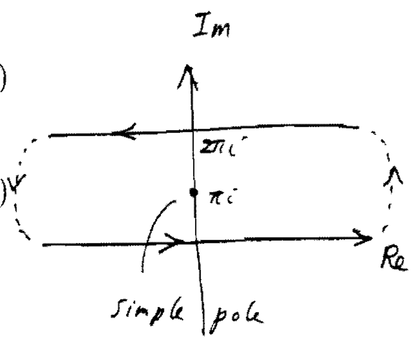
(3) Show

$$\int_0^\infty \frac{1}{1+x^n}dx = \frac{\pi/n}{\sin(\pi/n)}. \tag{8.74}$$

[Hint. A similar contour works.]

8B.9 Use of periodicity of e^x. The periodicity of e^x (period $2\pi i$) can be used as in the following example. Compute

$$I = \int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}dx \tag{8.75}$$



for $a \in (0,1)$. The path is shown in the figure. See **9.8** as well.

8B.10 Use of the residue to compute series: If we have a holomorphic function which has poles at all the points of $\boldsymbol{N}$ or $\boldsymbol{Z}$, we can use it to convert the sum over integers to contour integrals. $z^{-2}\cot\pi z$ is such a function. Let S_N be the square whose vertices are at $(\pm 1 \pm i)(n+1/2)$ for any positive $n \in \boldsymbol{N}$. If $f(Re^{i\theta})$ behaves as $o[R]$ in the $R \to \infty$ limit, then

$$\sum_{n=1}^{N}\frac{f(n)}{n^2} = \frac{1}{4i}\int_{\partial S_N} f(z)z^2\cot\pi z dz - Res(0; f(z)\cot\pi z/\pi z^2), \tag{8.76}$$

but the integral vanishes in the large square limit.

Exercise.
Compute the following sum

$$\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}. \tag{8.77}$$

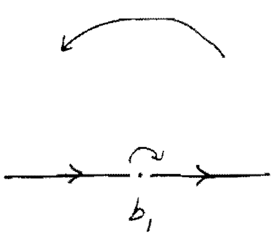
8B.11 Cauchy principal value of integral. Let f be a real function defined on the real axis, and $f(x) \to \infty$ at c. In this case the integral of f over an interval (a, b) containing c may not exist, but even in such cases the following limit may exist

$$P\int_a^b f(x)dx \equiv \lim_{\delta\to 0}\left[\int_a^{c-\delta} f(x)dx + \int_{c+\delta}^b f(x)dx\right], \quad \delta > 0. \tag{8.78}$$

If this is well defined, it is called the *(Cauchy) principal value* of $\int_a^b f dx$ (→**14.17**, **8B.13**).

Exercise.
Check

$$P\int_{-1}^{1}\frac{1}{x}dx = 0. \tag{8.79}$$



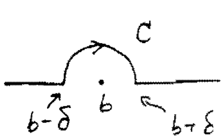
8B.12 Simple poles on the real axis. Let Q be a function for which $z^2Q(z)$ vanishes in the $z \to \infty$ limit for $\Re z > 0$. Suppose there are finite number of simple poles $\{b_j\}$ on the real axis. Then, if the real integral near the singularities is interpreted as Cauchy's principal values, we have

$$P\int_{-\infty}^{\infty} Q(x)dx = 2\pi i\left[\sum_j Res(a_j; Q(z)) + \frac{1}{2}\sum_k Res(b_k, Q(z))\right]. \tag{8.80}$$

where the sum over j is for all the singularities of Q on the upper half plane, and the sum over k is for all the simple poles on the real axis. □
[Demo] We choose the region D with *indentations* around the poles on the real axis. We can apply the residue theorem **8B.2** to D to get

$$\int_{\partial D} d\zeta f(\zeta) = 2\pi i\sum_j Res(a_j; Q(z)). \tag{8.81}$$

Now we magnify the neighborhood of a simple pole b on the real axis. Then, the integral along the path ℓ can be written symbolically as



$$\int_\ell f(\zeta)d\zeta = \int^{b-\delta} + \int_{b+\delta} + \int_C, \tag{8.82}$$

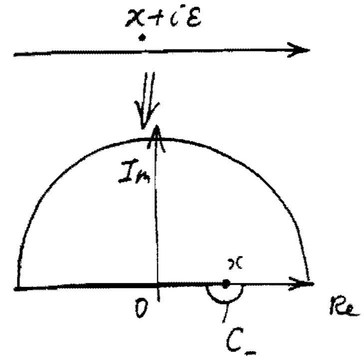
where C is the contour around the small semicircle of radius $\delta(>0)$ in the clockwise direction. The first two terms give the principal value of $\int f dx$ near the singularity b. Using the Laurent expansion (→**8A.8**) of f around b

$$f(z) = \frac{Res(b; f)}{z-b} + \cdots, \tag{8.83}$$

and the parametrization $\tau = b + \delta e^{i\theta}$, we have in the $\delta \to 0$ limit

$$\int_C f(\zeta)d\zeta = -\pi i Res(b; f) + O[\delta]. \tag{8.84}$$

8B.13 Plemelj formula: Let f be a function holomorphic on the upper half plane and $|f| \sim |z|^{-\epsilon}$ as $z \to \infty$ on the upper half plane for some positive ϵ. Furthermore, if f is holomorphic in some nbh of x, then

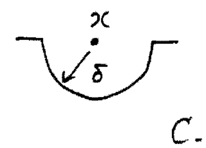


$$\lim_{\epsilon \to 0+} \int_{-\infty}^{\infty} \frac{f(y)}{y - x \pm i\epsilon} dy = P \int_{-\infty}^{\infty} \frac{f(y)}{y - x} dy \mp i\pi f(x). \tag{8.85}$$

[Demo] Cauchy's formula **6.10** applied to the upper large semidisk tells us

$$2\pi i f(x \mp i\epsilon) = \int_{-\infty}^{\infty} \frac{f(y)}{y - x \pm i\epsilon} dy \tag{8.86}$$

$$= \int_{-\infty}^{x-\delta} + \int_{x+\delta}^{\infty} + \int_{C_-}. \tag{8.87}$$



The first two integrals combine to the principal value integral. In the third integral, the holomorphy of f near x allows us to deform C_- into a lower semicircle of radius δ. The exchange of the two limiting procedures $\epsilon \to 0$ and $\delta \to 0$ is allowed, because (8.87) does not depend on δ at all: we may let $\epsilon \to 0$ for each δ. We can explicitly calculate the third integral just as in **8B.12**. □
Usually, we write (8.85) as (→**3.8**)

$$\frac{1}{y - x \mp i\epsilon} = P\frac{1}{y - x} \pm \pi i \delta(y - x). \tag{8.88}$$

This is called the *Plemelj formula*. From this we obtain

$$\delta(x - a) = \frac{1}{2\pi i}\left(\frac{1}{x - a - i0} - \frac{1}{x - a + i0}\right). \tag{8.89}$$

Here $i0$ implies $i\epsilon$ for infinitesimal $\epsilon(> 0)$. See **32C.13**, **8B.16**.

8B.14 Kramers-Kronig relation. Let $G(t)$ be the response of a linear system to a perturbation F. Since G must be a linear functional of F, it has the following general form:

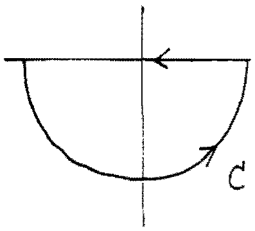
$$G(t) = \int ds \phi(t - s) F(s), \tag{8.90}$$

where ϕ is called the *response function*. Its Fourier transform

$$\chi(\omega) = \int dt \phi(t) e^{-i\omega t} \tag{8.91}$$

is called the *admittance*. Due to *causality*, $\phi(t) = 0$ for $t < 0$. Hence,

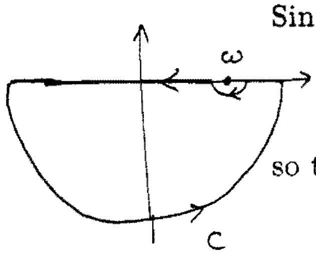
$$\int d\omega \phi(\omega) e^{i\omega t} = 0. \tag{8.92}$$



if $t < 0$. To compute this by a contour integration $e^{i\omega t} \to 0$ for $t < 0$ is needed for large $|\omega|$. Therfore, $Im\,(\omega t) > 0$ implies $Im\,\omega < 0$. Hence, causality implies

$$\oint_C \chi(\omega) e^{i\omega t} d\omega = 0. \tag{8.93}$$

Since $e^{i\omega t}$ is analytic in ω, this implies (see Morera's theorem →**6.16**) that $\chi(\omega)$ is holomorphic in the lower half plane. Conversely, if χ is holomorphic in the lower half plane, then the causality is satisfied.[149] Since χ is holomorphic in the lower half plane.



$$0 = \oint_C \frac{\chi(z)}{z - \omega} dz, \tag{8.94}$$

so that

$$\chi(\omega) = \frac{i}{\pi} P \int_{-\infty}^{\infty} \frac{\chi(z)}{z - \omega} dz. \tag{8.95}$$

Splitting the real and imaginary parts as $\chi = \chi' - i\chi''$, the formula can be written as

$$\chi'(\omega) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\chi''(z)}{z - \omega} dz. \tag{8.96}$$

$$\chi''(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\chi'(z)}{z - \omega} dz. \tag{8.97}$$

These are called the *Kramers-Kronig relation*,[150] but had been known to mathematicians like Hilbert long before. χ'' describes the dissipation part of the response, so it is more easily measured experimentally. Therefore, the relation becomes useful to reconstruct the whole admittance.

8B.15 Hilbert transformation. If

$$u(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v(\zeta)}{\zeta - z} d\zeta. \tag{8.98}$$

$$v(z) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(\zeta)}{\zeta - z} d\zeta, \tag{8.99}$$

we say u and v are *Hilbert transforms* of each other. This can be demonstrated with the aid of

$$\frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{dz}{(x - z)(z - y)} = \delta(x - y). \tag{8.100}$$

[149] First pointed out by H. Takahashi in 1942. See Butsuri **40**, 188 (1985) for an interview [Japanese].

[150] It is a kind of dispersion relations extensively used in high energy physics.

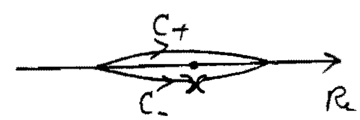
This is formally demonstrable by the Plemelj formula and the Hilbert transformation.
(1) For this pair.

$$\int_{-\infty}^{\infty} |u(x)|^2 dx = \int_{-\infty}^{\infty} |v(x)|^2 dx. \qquad (8.101)$$

This is called the *Parseval relation.*
(2) Let f and g be a Hilbert transform pair. Then

$$\int_{-\infty}^{\infty} f(t)g(t)dt = 0. \qquad (8.102)$$

8B.16 Delta function as Sato hyperfunction. (8.89) can be obtained more directly as follows: Let f be real analytic. Take integration paths $C_\pm$ very close to the real axis surrounding $x \in \boldsymbol{R}$. Then. Cauchy's formula **6.10** gives



$$\int_{C_\mp} \frac{f(z)}{z-x} dz = P\int f(z)dz \mp i\pi f(x). \qquad (8.103)$$

or

$$2\pi i f(x) = -\int_{C_+} \frac{f(z)}{z-x} dz + \int_{C_-} \frac{f(z)}{z-x} dz. \qquad (8.104)$$

Flattening the paths toward the real axis. we could formally obtain

$$2\pi i f(x) = \int_a^b dx f(y)\left(\frac{1}{y-x-i0} - \frac{1}{y-x+i0}\right). \qquad (8.105)$$

Thus we have arrived at (8.89). To define such a singular object as the difference of boundary values $F(x+i0) - F(x-i0)$ of analytic functions has been proposed by Sato (in 1955). and is called the *Sato hyperfunction.*[151]

[151] For the history. see N. Kaneko. "The history of Sato hyperfunction. from a private point of view." Surikagaku. March. p15. April. p22. May p29 (1986).