# 7 Analytic Continuation

If a Taylor expansion of an analytic function around some point is given, this series completely and globally determines the analytic function. With a review of power series, analytic continuation is introduced to construct analytic functions.

**Key words**: power series, convergence disk, convergence circle, convergence radius, theorem of identity, function element, analytic continuation, analytic completion, analytic function, Riemann surface.

**Summary**
(1) If two analytic functions agree on infinitely many points whose accumulation point is inside the domains of both functions, then they are actually identical (Theorem of identity **7.5**).
(2) This theorem is the key to analytic continuation and completion (**7.7-9**).
(3) Analytic function is completely determined by its property on an arbitrarily small open set in its domain (**7.10**).

**7.1 Motivation to study series and sequence, analyticity**. We have learned a remarkable fact that holomorphic functions are Taylor-expandable (Taylor's theorem **6.15**). Taylor-expandable functions are called *analytic functions*. Or, more precisely, a function $f : D \to \boldsymbol{C}$ is said to be *analytic* in a region $D$, if it is Taylor-expandable there.

**7.2 Review of power series**. A *(formal) power series* around $z_0 \in \boldsymbol{C}$ is a series of the form $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. [Without loss of generality, we may consider the case with $z_0 = 0$ to study its convergence.]
For the series $\sum_{n=0}^{\infty} a_n z^n$, the number $R$ defined by

$$\lim_{n\to\infty} \sup |a_n|^{1/n} = 1/R \quad \text{(Cauchy-Hadamard's formula)} \qquad (7.1)$$

is called the *convergence radius* of the series.[135]

The disk of radius $R$ (the convergence radius) around the expansion center $z_0$ of a power series $\sum a_n(z-z_0)^n$ is called its *convergence*

[135] $\lim_{N\to\infty} \sup a_N$ means $\lim_{N\to\infty}\{\sup_{n\geq N} a_n\}$. That is, find $\sup_{m\geq n} a_m = b_n$, then make $\lim_{n\to\infty} b_n$. This limit is always well defined (may be $+\infty$, but in such a case the limit is definitely larger than any positive number) because $\{b_n\}$ is monotonically non-increasing.

*disk.* Its boundary is called the *convergence circle.*
(1) $\sum_{n=0}^{\infty} \alpha_n z^n$ converges absolutely and uniformly to a holomorphic function $f(z)$ such that $\alpha_n = f^{(n)}(0)/n!$ for $|z| < R$, where $R$ is the radius of convergence of the series.
(2) The power series is termwisely differentiable, and the resultant power series converges to $f'(z)$ in the open disk $D$.
(3) Let $\sum_{n=0}^{\infty} \alpha_n z^n$ be a power series converging to $f(z)$. Then, for $z$ such that $|z| < R$

$$\int_0^z d\zeta f(\zeta) = \sum_{n=0}^{\infty} \frac{\alpha_n}{n+1} z^{n+1}. \tag{7.2}$$

□

**Discussion.** Use of *generating function* to compute the sum of series.
This is a discrete version of integral transformations like the Laplace transformation (→**33**).
To compute

$$S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \tag{7.3}$$

we use

$$G(z) = \sum_{n=1}^{\infty} z^{n-1} = (1-z)^{-1}. \tag{7.4}$$

Integrating this twice from zero to $z$, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} z^{n+1} = (1-z)\ln(1-z) + z. \tag{7.5}$$

Hence the desired sum is 1.[136] Such a formal calculation is often useful. In this case the calculation is not only formal but mathematically respectable. Can you justify all the formal steps?

Newton already did this

**Exercise.**
(A) Let $|z| < 1$. Show[137] that for any complex number $\alpha$

$$(1+z)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n, \tag{7.6}$$

[136] Of course, in this case, a clever way is to realize

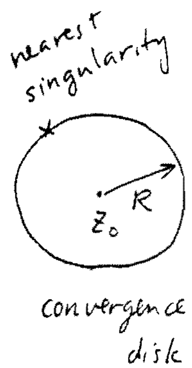
$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

[137] Formally. Return to this problem after learning singularities to justify the formal justification.

where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}, \tag{7.7}$$

the binomial coefficient (for $n = 0$ this is defined to be 1).
(B) Expand the following function in powers of $z - 1$.
(1) $z/(z+1)$.
(2) $\log z$.




**7.3 Convergence circle must contain a non-holomorphic point.** Let $f$ be a holomorphic function in a region $D$ which contains the origin. The convergence radius $R$ of its Taylor series around $z = 0$ is the minimum distance between $z = 0$ and the points where $f(z)$ is not holomorphic (Cauchy realized this. and was a breakthrough. →**6.11**). □

[Demo] Suppose the radius of convergence $R$ is larger than the minimum distance between the origin and the nearest non-holomorphic point of $f(z)$. Then. the series defines a function which is holomorphic in $|z| < R$ (see Taylor's theorem **6.15**), but this means that the function does not have any non-holomorphic point within the disk contrary to the assumption. □

**Discussion.**
**Vivanti's theorem**. The singularity of the positive real coefficient convergent series is on the real positive axis.

**7.4 Convergence radius is a continuous function of expansion center**: The radius of convergence of the Taylor series of a holomorphic function depends continuously on the position of its expansion center. (This should be obvious from **7.3**).

**Discussion**. It is a good occasion to review the radius of convergence.
Consider the power series $\sum a_n z^n$.
(1) If $a_n \neq 0$. and if

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \tag{7.8}$$

converges. then it is the radius of convergence $\rho$. This formula seems to be due to d'Alembert.

For example. $1 + z + 2z^2 + \cdots + nz^n + \cdots$ has $\rho = 1$. The radius of convergence of $1 + z + 2!z^2 + \cdots + n!z^n + \cdots$ is zero.
(2) [Cauchy-Hadamard's formula] (discussed in **7.2**)

$$\frac{1}{\rho} = \limsup_{n\to\infty} |a_n|^{1/n}. \tag{7.9}$$

For example. if $a_n = (2 + (-1)^n)^n$. then (1) does not work, but this formula gives $\rho = 1/3$. (For lim sup see **7.2** or **A1.10**)
(3) Let $a$ and $b$ be positive reals. Discuss the convergence radius of

$$1 + az + (bz)^2 + (az)^3 + (bz)^4 + \cdots + (az)^{2n-1} + (bz)^{2n} + \cdots. \tag{7.10}$$

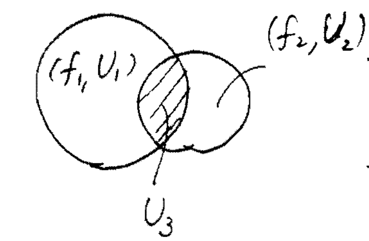
**7.5 Theorem of Identity**: Let $f$ and $g$ be holomorphic on a region $D$, and $\{a_n\}$ be a sequence whose accumulation point $a$ is in $D$. If $f(a_n) = g(a_n)$ for all $n$, then $f \equiv g$ on $D$. □

That $a \in D$ is crucial. For example, $\sin(1/z)$ is not identically zero, and its zeros accumulate at the origin, but there the function is not holomorphic.

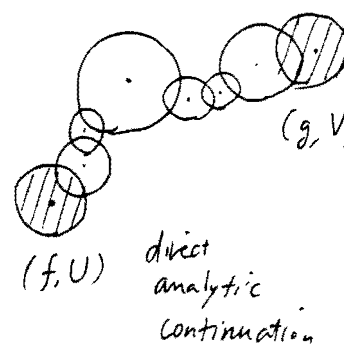
This follows from the fact that the zeros of holomorphic functions ($\not\equiv 0$) are all isolated and no accumulation point exists in its domain.

**7.6 Principle of invariance of functional relations**: **7.5** implies that if $f = g$ holds on an infinite set and $f = g$ is also true on its accumulation point, then actually $f = g$.

**7.7 Function element**. Let $U$ be a region, and $f$ be a holomorphic function on $U$. $(f, U)$ is called a *function element*. Two function elements $(f_1, U_1)$ and $(f_2, U_2)$ are said to be *equivalent*, if $f_1 \equiv f_2$ on $U_3 \equiv U_1 \cap U_2$. This definition of equivalence is sensible because of the identity theorem **7.5**.




**7.8 Direct analytic continuation, indirect analytic continuation**. Let $(f, U)$ and $(g, V)$ be function elements. If $f \equiv g$ on $U \cap V$ and $V \not\subset U$, $(g, V)$ is called the *direct analytic continuation* of $(f, U)$. If we can find a finite chain of direct analytic continuations between $(f, U)$ and $(g, V)$, we say $(g, V)$ is called the *indirect analytic continuation* of $(f, U)$. Both continuations are collectively called *analytic continuations*.




Expand around e^{i \theta}

on.

Along the unit circle centered at the origin, directly analytically continue the Taylor expansion of $\sqrt{z}$ around $z = 1$:

$$P(z; 1) = \sum_{k=0}^{\infty} (-1)^k \binom{1/2}{k} (1 - z)^k \tag{7.11}$$

to $z = 1$ again. Show that the result is $-P(z; 1)$ (cf. **8A.3**).

**7.9 Uniqueness of direct analytic continuation**: Let $(g, V)$ and $(h, W)$ be both direct analytic continuations of $(f, U)$. $U \cap V \cap W \neq \emptyset$ and $V \cap W$ be connected. Then $(h, W)$ is a direct analytic continuation of $(g, V)$. □

[Demo] $X \equiv U \cap V \cap W$ is an open set in $V \cap W$, and $g \equiv h$ on $X$. Since $g$ and $h$ are holomorphic on $V \cap W$, the theorem of identity **7.5** tells us that $h \equiv g$ there. □

**7.10 Analytic function**. The totality of the function elements which can be analytically continued from a function element $(f, U)$ is called the *analytic function* determined by $(f, U)$. To make this totality is

Take 1/(1-x) as an example and do all explained here.

called *analytic completion*. The procedure uniquely defines an analytic function from a single function element. Notice that 'analytic function' is a global concept in contrast to 'holomorphic function' (→**5.4**). In a certain sense an analytic function is a function whose total information resides in any small piece of its domain like 'holography.'

In practice, analytic functions are functions which can be expanded into Taylor series. Clearly recognize that the equivalence of holomorphy and analyticity is almost a miracle. Needless to say, this is untrue for real-valued functions on $\boldsymbol{R}$.

**Exercise.**
(1) Gauss' hypergeometric function is obtained by the analytic completion of the following series (→**24C.4**):

$$F(\alpha, \beta, \gamma, z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{k! (\gamma)_k} z^k. \tag{7.12}$$

where $(\lambda)_k = \lambda(\lambda+1)\cdots(\lambda+k-1)$ and $(\lambda)_0 = 1$. What are the following functions (they are elementary functions →**4.2** Discussion)?
(i) $F(-p, \beta, \beta, z)$.
(ii) $zF(1, 1, 2, z)$.
(iii) $F(n/2, -n/2, 1/2, \sin^2 x)$.
(iv) $\sin^{-1} x = xF(1/2, 1/2, 3/2, x^2)$.
(2) Show that

$$\frac{d}{dz} F(\alpha, \beta, \gamma, z) = \frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1, \gamma+1, z). \tag{7.13}$$

More generally, (For $\Gamma$ see **9**, but the reader needs only $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ here.)

$$\frac{d^n}{dz^n} F(\alpha, \beta, \gamma, z) = \frac{\Gamma(\alpha+n)\Gamma(\beta+n)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma+n)} F(\alpha+n, \beta+n, \gamma+n, z). \tag{7.14}$$

**7.11 Natural boundary.** Analytic continuation cannot always allow a function element (→**7.7**) to be extended to the whole complex plane. The boundary beyond which analytic continuation is impossible is called the *natural boundary* of the analytic function. For example,

$$\sum_{n=0}^{\infty} z^{n!} \tag{7.15}$$

(such a series with very sparse distribution of the exponents is called a *lacunary series*) defines an analytic function whose natural boundary is the unit circle centered at the origin. This can be seen from the fact that the series diverges when $z \to \exp(2\pi n/m!)$ along the radius ($m$ is a positive integer, and $p \in \{0, 1, \cdots, m!-1\}$). Actually, *Hadamard's* *gap theorem* says the following:

**Theorem [Hadamard].**

Let

$$f(z) = \sum_{\nu=0}^{\infty} c_{n_\nu} z^{n_\nu}. \tag{7.16}$$

If

$$\liminf_{n\to\infty} \frac{n_{\nu+1}}{n_\nu} > 1, \tag{7.17}$$

then the convergence circle of $f$ is its natural boundary.[138]

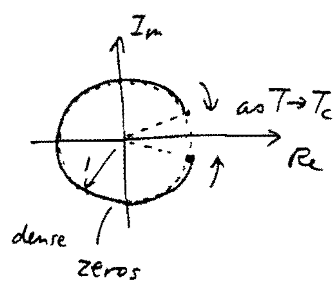
**Discussion [Lee-Yang circle theorem].**[139]

Let $-1 \le a_{ij} \le 1$ and $\{i, j\} \subset \{1, 2, \cdots, n\}$. Make the following polynomial

$$P(z) = \sum_{X \subset \{1,\cdots,n\}} z^{|X|} \prod_{i \in X} \prod_{j \in X} a_{ij}. \tag{7.18}$$

where $|X|$ is the cardinality of $X$ (the number of elements in $X$), and the terms of order 0 and $n$ are 1 and $z^n$, respectively. Then, all the zeros of $P$ is on the unit circle $|z| = 1$.□

In the original statistical mechanical context (T. D. Lee and C. N. Yang, Phys. Rev. **87**, 410 (1952)), $P$ is the grand canonical partition function of the Ising model, and $z$ is the (complex) fugacity. In the thermodynamic limit, we must take the $n \to \infty$ limit. Then, $P(z)$ becomes an entire function (→**4.2**) and below the critical temperature $T_c$ the zeros are dense on the unit circle everywhere. Hence, $\ln P$ (the thermodynamic potential) consists of two analytic functions inside and outside the unit circle. Generally speaking, phase transition is associated with the nonanalyticity of thermodynamic functions on the complex plane (as a function of complex temperature, complex fugacity, etc.).




The Lee-Yang gap.

**7.12 $C^\infty$ but not $C^\omega$ functions.** Suppose we have two real analytic functions[140] $f$ and $g$ crossing at $x = a$. Then, grafting $f$ and $g$ smoothly near $a$, we can make a function which is not analytic but infinite times differentiable. Thus, infinite times differentiability is not enough to guarantee the analyticity. $e^{-1/x(1-x)}$ is a famous example of such a function.

This is not for complex function; only grafting real valued functions

**7.13 Values of analytic functions, Riemann surface.** Even if we start with a single function element $(f, U)$, its analytic continuation to a nbh of some point $\alpha$, even if it exists, may not be unique.

[138] lim inf is defined analogously as lim sup (→**7.2**) with sup replaced with inf.

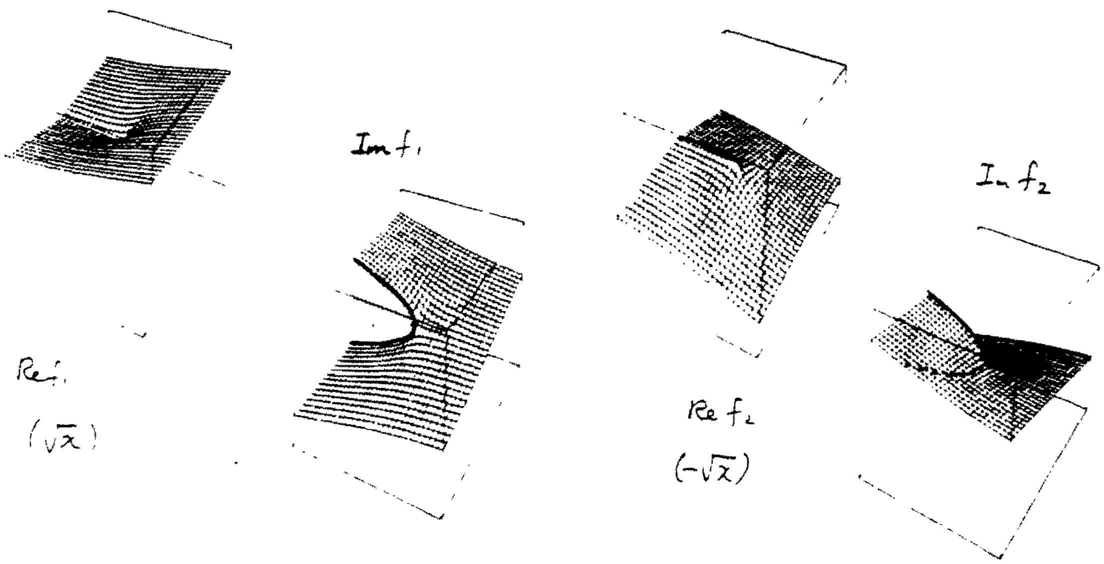
[139] See D. Ruelle, "Is our mathematics natural? The case of equilibrium statistical mechanics," Bull. Amer. Math. Soc. **19**, 259 (1988) for the shortest proof of the theorem.

[140] Analytic functions which are real on the real axis is called real analytic functions.

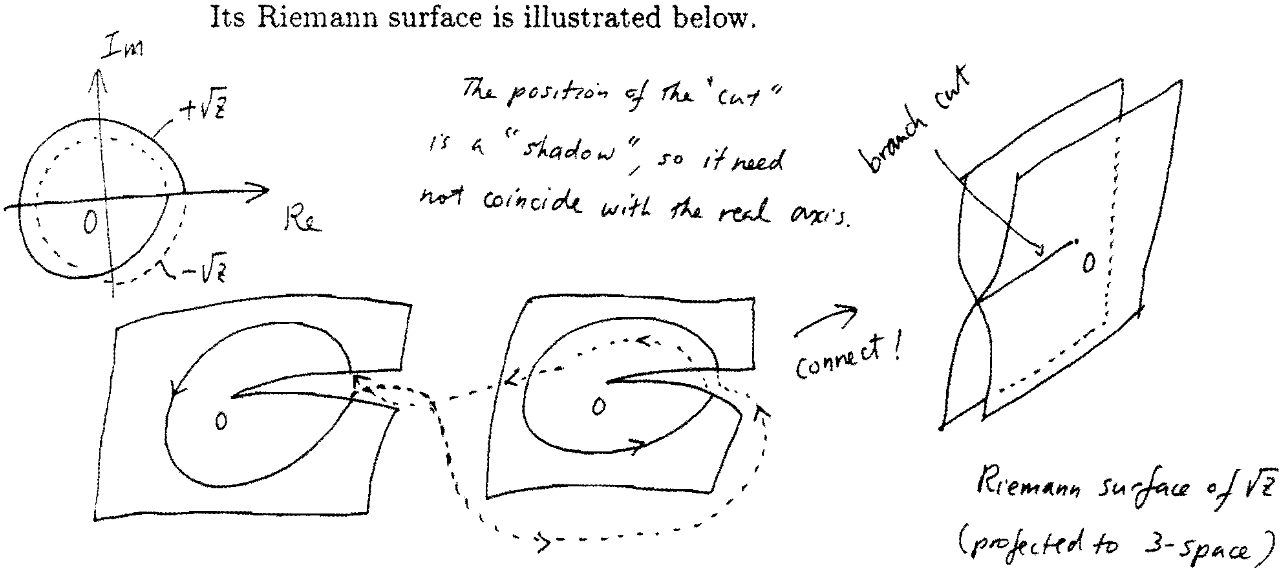
Illustrate the Riemann surface of log z

Hence, in general, an analytic function takes multiple values at a given point on its domain. Thus, 'analytic function' is not a function defined on a region of complex plane in the usual sense of the word 'function' (as a map). However, it cannot assume uncountably (→**A1.16**) many values (see the next theorem). We can conceive a space on which a given analytic function becomes a function in the usual sense of the word 'function' as a map. The space is called the *Riemann surface* for the given function. This is a two-dimensional object which may not be realizable in 3-space just as Klein's bottle.

See the real and imaginary parts of $z^{1/2}$. Two branches (see also **8A.4** Discussion (B)) $f_1$ and $f_2$ are plotted.




Its Riemann surface is illustrated below.



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**7.14 Poincaré-Vivanti's theorem.** The multiplicity of an analytic function must be countable (→**A1.16**, **17.18(4)**). □

Intuitively, this can be seen from the smoothness of analytic functions and the fact that direct analytic continuation can be done along a piecewise linear curve. There are at most countably many (→**A1.16**) such curves on the complex plane.

**7.15 Who was Riemann?**[141] Georg Friedrich Bernhard Riemann was born on September 17, 1826 in a small village on the Elbe near Lüneburg. He was the second of six children of a poor pastor. He was educated by his father before he entered the gymnasium. When he was fourteen, he lived with his grandmother in Hanover and entered the third grade of the gymnasium there. After his grandmother died, he transferred to the second grade of a gymnasium in Lüneburg in April, 1842. The principal of the school recognized his mathematical genius and lent his math books. Riemann always returned the books within a couple of days, so the principal was surprised but found that Rieman understood them. He became familiar with Euler's work in those days.

He entered University of Göttingen in April, 1846 as a Linguistics and Theology major to get a job as quickly as possible to support his parents and siblings. He also attended Gauss' (→**6.17**) lectures on the least square method. His desire to study mathematics became irrepressible, and he finally asked for his father's permission to switch his major. In those days Gauss was about 70, and gave only a few applied mathematics courses, so he was disappointed and moved to the University of Berlin in 1847.

In Berlin, Jacobi (algebra and analytical mechanics), Dirichlet (number theory, integration theory, PDE), Steiner, and other professors gave lectures on their new results. Dirichlet aimed at logical rigor and avoided calculations as much as possible. This style met Riemann's taste.

In the spring of 1849, he returned to Göttingen, and was attracted to Weber's experimental physics course. Weber recognized his genius and became his patron. Riemann did not get any direct instruction from Gauss, but was strongly influenced by the atmosphere created by the great mathematician. For example, Riemann accepted the idea of 'ether' which Gauss also had.

In November 1851, he submitted his thesis entitled, The foundation of general theory of functions of one complex variable. He defined holomorphic functions in terms of the Cauchy-Riemann equation (→**5.3**). The idea of conformal maps (→**10**) was also conceived. He also introduced Riemann surfaces (→**7.13**). Gauss praised the thesis: Mr.

[141] Mainly based on K. Kobori, *Great Mathematicians of the 19th Century* (Kobundon, 1940).

Riemann's thesis clearly tells us that his study is thorough, that he has a sharp brain, and that he has a magnificent and rich creativity. From every point, the thesis is a precious accompishment and far surpasses the standard of doctoral theses. When Riemann visited Gauss after the exam, Gauss told him that he had similar thoughts (→**6.17**, **8A.4** Discussion (A)), and that he had a similar aim.

He next started preparation for the Habilitation paper. He chose to study Fourier series (→**17**), but this was not an easy task. Fortunately. Dirichlet visited Göttingen, who checked Riemann's manuscript together, and "Professor Dirichlet gave me detailed suggestions with kindness I could not imagine when I took into account the difference of our social statuses. I pray Professor will remember me forever."(from a letter to his father). He submitted his paper, The expressibility of functions by trigonometric series. in December 1853. The Riemann integration appeared for the first time in this paper (→**17.18**(3)). In those days he was an assistant of Weber.

The famous Habilitations exam was held on June 10, 1854 (→**2**). He introduced (1) the concept of manifold, (2) a new definition of distance through the quadratic form, and (3) the concept of curvature.[142]

He became a lecturer in 1854. His first lecture was on PDE and its applications to physics. He had eight students ("I am glad that I have so many students."(from a letter to his father)). In 1855, Dirichlet succeeded Gauss. Dirichlet made effort to make Riemann an associate professor, but failed. He finished his study of elliptic functions which was started in ca. 1851. His lecture on elliptic functions attracted only three participants including Dedekind. He became an associate professor on January 9, 1857.

In 1857 he completed "On the number of prime numbers less than a given number." He introduced the zeta function (→**9.9**)

$$\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{n^s}. \tag{7.19}$$

and conjectured that all the zeros in the strip $0 < Re\, s < 1$ are on $Re\, s = 1/2$ (the Riemann conjecture). With Dedekind, he is the founder of analytic number theory. Dirichlet died on March 9, 1859. Riemann became a full professor on July 30, 1859. He got married on June 3, 1863 with his sister's friend Elise Koch, but this was his last happy period. He became ill in August. Weber persuaded the government to support his stay in Italy to recover his health. He had a wonderful time in Italy, befriending Italian mathematicians, Betti, Beltrami, and others.

[142] This is a generalization of Gauss's curvature, but the new aspect was to write it in terms of the metric tensor.

His health never recovered fully, and in June 15, 1866, he went on his third Italian trip to rest at Selasca on Lake Maggiore. He died there in July, 1866.