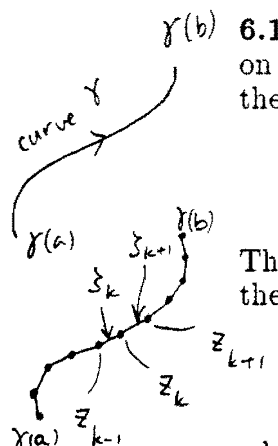
6 Integration on Complex Plane

Contour integrals on C are defined, and Cauchy's theorem is demonstrated. Derivatives of holomorphic functions can be written in terms of integration. and this leads to a remarkable property: analyticity = holomorphy. This is of course untrue for real functions. Complex functions enjoy this equivalence thanks to the strong nature of its differentiability.

Key words: contour integral. Cauchy's theorem, orientation. Cauchy's formula. Morera's theorem.

Summary

(1) An integral of a holomorphic function along a closed contour vanishes – Cauchy's theorem (**6.3**). The essence of the theorem is the local linearizability of holomorphic functions (**6.7**).
(2) The converse of Cauchy's theorem holds (Morera's theorem **6.16**).
(3) Holomorphic functions are infinite times differentiable (**6.12**). Derivatives of any order can be written in terms of contour integrals (**6.14**).
(4) Holomorphic functions are Taylor-expandable (analytic) (**6.15**).
(5) Pay due attention to the orientation of the boundary curve (or the right-hand rule) (**6.4**).



6.1 Integration along contour. Let $\gamma(t)$ $(t \in [a.b])$ be a C^1-curve[121] on the complex plane. The integration of a function $f = u + iv$ along the curve (contour) is defined by

$$\int_\gamma f(z)dz \equiv \int_a^b u(\gamma(t))\gamma'(t)dt + i\int_a^b v(\gamma(t))\gamma'(t)dt. \qquad (6.1)$$

This definition is equivalent to the following definition with the aid of the Riemann sum:

$$\int_\gamma f(z)dz \equiv \lim_{\|\delta\|\to 0}\sum_n f(\zeta_n)(z_n - z_{n-1}), \qquad (6.2)$$

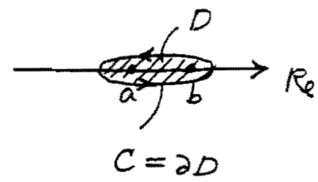
where $\|\delta\| = \max\{|z_k - z_{k-1}|\}$ and ζ_k is a point on the curve between z_k and z_{k-1}.

Exercise.

[121] We should say more correctly a Jordan curve. Smoothness of piecewise C^1 is enough.

(1) Let f be holomorphic in a region D which contains a real closed interval $[a, b]$. Then,

$$\int_a^b f(x)dx = \frac{1}{2\pi i}\int_C f(z)\mathrm{Log}\frac{z-a}{z-b}dz, \tag{6.3}$$



where the contour C is the boundary of D (cf. Discussion in **4.7**).
(2) Show

$$\left|\int_C f(z)dz\right| \le \int_C |f(z)||dz|. \tag{6.4}$$

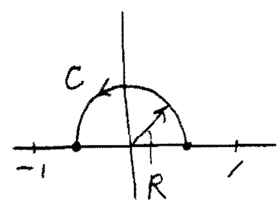
Here integration over $|dz|$ is understood as the limit of an analogue of (6.2).
(3) Parametrizing the integral path as $z = a + re^{i\theta}$, demonstrate ($n \in \boldsymbol{Z}$)

$$\oint_{|z-a|=r}(z-a)^n dz = 2\pi i\sigma(n). \tag{6.5}$$

Here $\sigma(n) = 1$ if $n = -1$, and zero, otherwise.
(4)

dz

$$\int_C \frac{1}{1+z^2} = -2\arctan R, \tag{6.6}$$



where C is the semicircle of radius R from R to $-R$ for $R < 1$ in the upper half plane.

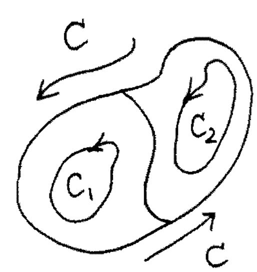
6.2 Bilinear nature of integration $\int$. The integration operator $\int$ can be regarded as a linear operator ($\to$**1.4**) from the set of integrable functions on $\boldsymbol{C}$ to $\boldsymbol{C}$, and also as a linear operator from the set of oriented curves on the complex plane to $\boldsymbol{C}$: for integrable functions f, g and oriented piecewise C^1-curves $\gamma, \gamma_1, \gamma_2$

$$\int_\gamma(\alpha f + \beta g)dz = \alpha\int_\gamma fdz + \beta\int_\gamma gdz. \tag{6.7}$$

$$\int_{a\gamma_1+b\gamma_2} fdz = a\int_{\gamma_1} fdz + b\int_{\gamma_2} fdz. \tag{6.8}$$

Here $\alpha, \beta \in \boldsymbol{C}$, and $a, b \in \boldsymbol{N}$. $\gamma_1 + \gamma_2$ is interpreted as the successive trip along γ_1 and γ_2 in their specified directions. $a\gamma$ is interpreted as a-time repeated trip along γ. $-\gamma$ is interpreted as the trip along γ in the opposite direction specified by the definition of γ. For example, if two cycles share a portion on which the two cycles have opposite orientations, then we can make a single larger cycle. Consequently,

$$\oint_{c_1} f(z)dz + \oint_{c_2} f(z)dz = \oint_c f(z)dz. \tag{6.9}$$



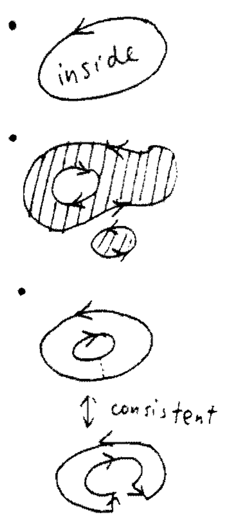
6.3 Cauchy's Theorem. Let $f(z)$ be a holomorphic function ($\to$**5.4**) on a closed region D[122] whose boundary consists of finite number of piecewise C^1-curves. Then.

$$\int_{\partial D} f(z)dz = 0. \tag{6.10}$$

□

Pay attention to the definition of the boundary in **6.4**. The proof is given by the combination of the following **6.5** and **6.6**.

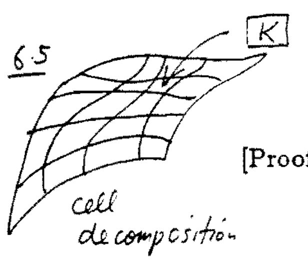
Exercise.
(1) Integrate $\exp(-3z)$ from $1 - \pi i$ to $2 + \pi i$.
(2) Integrate $\cos 3z$ from $\pi - \pi i$ to $2\pi i$.



6.4 Orientation of the boundary. To specify the contour, we must specify the direction in which we integrate along the contour. This is called the *orientation* of a curve. The orientation of the boundary of a set E is chosen so that the reader sees the inside of E always on her left-hand side. when she travels along the boundary in the positive direction (i.e.. the specified direction) of the contour. This is consistent with the right-hand rule.

6.5 Cell decomposition of a bounded closed region: Let D be a closed bounded region whose boundary consists of finite number of piecewise C^1-curves.[123] Then. D can be decomposed into the 'cells' D_i which are the images of a rectangle K $(0.1) \times (0.1)$ by C^1-maps. where no two cells share their internal points. (Although intuitively obvious. the proof of this theorem is not at all simple.)

6.6 Cauchy's theorem for a cell. Let $f(z)$ be a holomorphic function on a region E. and γ is the boundary of a cell $D(\subset E)$. Then



$$\int_\gamma f(z)dz = 0. \tag{6.11}$$

[Proof by Goursat] It is easy to explicitly demonstrate (→**6.1** Exercise (3))

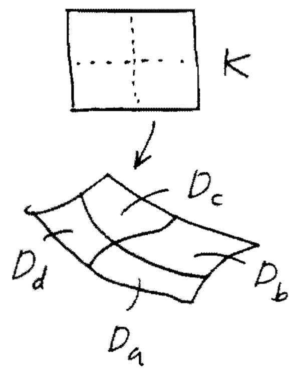
$$\int_\gamma (\alpha z + \beta)dz = 0. \tag{6.12}$$

[122] Closedness is required to get rid of any singularity from the boundary of D as well as from the inside of D. However, actually a stronger form of the theorem assuming only the continuity on the boundary of D holds.

[123] Henceforth a 'C^1-curve' can always be replaced by a curve with length. A necessary and sufficient condition for a curve $\gamma(t)$ to have length is that γ is of bounded variation.

where α, β are constants. The cell D is the image of K. Decompose K into four identical squares, and their images $D_a, \cdots, D_d$ divide D into four cells. Then due to the bilinearity of integral operation ($\rightarrow$**6.2**)

$$\int_{\partial D} f dz = \sum_{i=a}^{d} \int_{\partial D_i} f dz. \tag{6.13}$$



Hence,

$$\frac{1}{4}\left|\int_{\partial D} f dz\right| \le \left|\int_{\partial D_1} f dz\right|. \tag{6.14}$$

Here D_1 is the cell which gives the largest modulus of the integral among $D_a, \cdots, D_d$. Repeat the procedure m-times, and we get

$$\frac{1}{4^m}\left|\int_{\partial D} f dz\right| \le \left|\int_{\partial D_m} f dz\right|, \tag{6.15}$$

where D_m is the image of the square whose edge size is $1/2^m$ and gives the largest modulus of the integral among the four pieces in D_{m-1}. Since the size of D_m is sufficiently small and f is holomorphic, we may write in D_m

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + o[z - z_0]. \tag{6.16}$$

For any ϵ there is a positive integer N such that if $m > N$, then $|o[z - z_0]| < \epsilon\delta(D_m)$ in D_m, where $\delta(D_m)$ is the radius of D_m. With the aid of (6.12)

$$\left|\int_{\partial D_m} f dz\right| = \left|\int_{\partial D_m} o[z - z_0] dz\right| \le \frac{\epsilon}{2^m}\delta(D_m) \times const. \le \frac{\epsilon}{4^m} \times const., \tag{6.17}$$

where the constant is independent of m. From (6.15) and (6.17), we have

$$\left|\int_{\partial D} f dz\right| \le \epsilon \times const., \tag{6.18}$$

where $const.$ is independent of m. We know ϵ is arbitrary, so the left-hand side must be zero. □

As can be seen from the proof, we cannot claim the theorem, if the boundary of D does not have length.

6.7 The essence of Cauchy's theorem. Holomorphic functions are locally linearizable ($\rightarrow$**5.1**), and any contour integral can be decomposed into the sum of contour integrals around tiny cells. Hence, the essence of Cauchy's theorem is that contour integral of a constant and z are both zero as used in **6.6**.

6.8 Cauchy's theorem from Green's formula. The contour integral of $f = u + iv$ on the complex plane can be written as

$$\int_{\partial D} f(z) dz = \int_{\partial D}(udx - vdy) + i\int_{\partial D}(udy + vdx). \tag{6.19}$$

if the boundary has
a length, OK

Apply Green's theorem (→**2C.13**(3)) to this and use the Cauchy-Riemann equation (→**5.3**). We will immediately see that the RHS vanishes.

Exercise.
Let D be a bounded region whose boundary is piecewise smooth. Show that

$$\int_{\partial D} \bar{z}dz = 2i \times \text{the area of } D \tag{6.20}$$

Or, equivalently, let C be a smooth simple closed curve on the complex plane. Demonstrate that

$$\int_C xdz = iS, \int_C ydz = -S. \tag{6.21}$$

where S is the area encircled by C. Notice that S has a sign depending on the orientation of the curve C. See **6.4**.

6.9 Indefinite integral theorem.. Let f be a holomorphic function on a region D and γ be a C^1-curve in D connecting $z \in D$ and an arbitrary fixed point $\alpha \in D$. Then

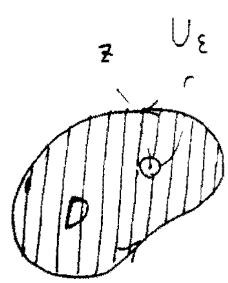
$$F(z) = \int_\gamma f(z)dz \tag{6.22}$$

is holomorphic in D and $F' = f$. [Obvious.]

6.10 Cauchy's formula. Let f be holomorphic on a closed region D.[124] Then

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - z}d\zeta. \tag{6.23}$$

□



[Demo] Choose a sufficiently small positive number ϵ. Let U_ϵ be the disk of radius ϵ with its center at z. Since $f(\zeta)/(\zeta - z)$ is holomorphic on $D \setminus U_\epsilon$, Cauchy's theorem implies

$$\int_{\partial(D\setminus U_\epsilon)} \frac{f(\zeta)}{\zeta - z}d\zeta = 0. \tag{6.24}$$

That is, (→**6.2**)

$$\int_{\partial D} \frac{f(\zeta)}{\zeta - z}d\zeta = \int_{\partial U_\epsilon} \frac{f(\zeta)}{\zeta - z}d\zeta. \tag{6.25}$$

Parametrize ∂U_ϵ as $\gamma(\theta) = z + \epsilon e^{i\theta}$. Then

$$\frac{1}{2\pi i}\int_{\partial U_\epsilon} \frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi}\int_0^{2\pi} f(z + \epsilon e^{i\theta})d\theta. \tag{6.26}$$

[124] As is in the footnote of **6.3** we need not require that f is holomorphic on the boundary of D: we have only to require the continuity of f on ∂D. Furthermore, D need not be singly connected.

Since f is holomorphic. this implies

$$\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\partial U_\epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta = f(z). \tag{6.27}$$

□

Discussion.
Let f be holomorphic in the unit disk D given by $|z| \le 1$. Show

$$\frac{1}{2\pi i} \int_{\partial D} \frac{\overline{f(\zeta)}}{\zeta - z} d\zeta = \overline{f(0)}, \tag{6.28}$$

if $|z| < 1$. (Hint: Change the integration variable to $\bar{\zeta}$.)

What happens if $|z| > 1$? [Note: This is NOT for general region D. but only for the unit disc.]

6.11 Who was Cauchy?[125] Augustin-Louis cauchy was born in Paris in the year the Revolution began (Aug. 21. 1789). His father, a barrister and police lieutenant escaped the Reign of Terror (1793-4) to live in Arceuil. as the neighbors of Laplace (→**33.3**) and Berthollet. Lagrange (→**3.5**) reportedly forecast the scientific genius of the boy. Cauchy thought pure mathematics was over. and the remaining task was applied mathematics. He worked as a military engineer at Cherbourg for two years from 1811, but resigned due to ill-health. Lagrange and Laplace persuaded him to leave engineering and to turn exclusively to mathematics in 1813.

Cauchy was a politically ultraconservative royalist. and after Restoration in 1814. he was appointed a member of the Paris Academy after Monge and 'regicide' Carnot (father of Sadi Carnot) were expelled.

Spurred by Fourier's work on Fourier series (→**17.18**). Cauchy tried to rationalize analysis. His results were published in Cour d'Analyse (1821) and Résumé des Lecons sur le Calcul infinitésimal (1823). In the former. he introduced the concept of functions as maps. He proved for the first time that continuous functions have primitive functions. The proof itself is important. but the recognition that a proof is needed was novel and more important. His course is almost the same as we teach now in the introductory calculus courses (for example, ϵ-δ).[126]

He tried to unify methods to calculate definite integrals in 1825 (14 years after Gauss's letter to Bessel revealing Gauss' full knowledge of complex analysis →**8A.4**. Also see **6.17**). Even in the proof of the

[125] See also B. Belhost. *Augustin-Louis Cauchy. a biography* (Springer, 1991).

[126] It is a famous story that Lagrange hurried home, and checked his celestial mechanics book. when Cauchy published his work on convergence in 1820, which he started around 1814. but after 1818 when he knew Fourier's work (→**1.7**) he was convinced that his program to rationalize calculus was meaningful.

residue theorem. Cauchy did not denote complex numbers with single letters. but always wrote in the two real number form, $x + iy$. Although this paper of 1825 is now regarded as the historic starting point of complex function theory, Cauchy did not recognize so at least for a very long time. because his main purpose was to unify and streamline the methods of calculating definite integrals ($\rightarrow$**8**)with the aid of the changing of the order of double integration used extensively by Laplace, Legendre. and others.

His life was quiet until the July Revolution of 1830. He refused to take the oath of allegiance to the new king who replaced a Bourbon king. and went into a self-imposed exile of 8 years. In 1832 he realized the relation between complex analysis and power series ($\rightarrow$**7.1**, **7.10**). Especially. he realized the relation between the radius of convergence and the singularity (published in 1837) ($\rightarrow$**7.3**). Now, there was a chance to relate his integration theory and the Taylor expansion theory. but it took for him for about 20 years to clearly recognize as a mathematical object 'analytic function.'

He returned to Paris in 1838 to resume his work at the Academy. In 1851. he introduced the concept of differentiability (**5.1** strong differentiability in our terminology $\rightarrow$**2A.8**). which was Riemann's starting point in his thesis (1851) ($\rightarrow$**7.15**).

Devoutly catholic. he was a social worker in the town of Sceaux (his house is still there on the corner next to Mary-Curie High School), and occasionally criticized scientists for research that he considered dangerous to religion – he was absolutely correct in this respect. because institutionalized religions and science cannot be compatible in a conscientious and at the same time intelligent person. Cauchy published 789 papers. and died in 1857.

Cauchy provided the first phase of rigorous foundation of calculus. He also gave an important contribution to group theory.

6.12 Infinite differentiability of holomorphic functions. If f is holomorphic on a closed region D. then f is infinite times differentiable inside of D. and the derivatives are holomorphic there.

The proof is given in **6.13-6.14**. As the reader will see the kernel is that differentiation can be written in terms of integrals. Generally speaking. differentiation 'magnifies' small scale features, making the function less differentiable (recall that differentiation maps C^n-class into C^{n-1}-class). In contrast. integration 'coarse-grains' a function. Hence. it is crucial that differentiation is expressible with the aid of integration.

6.13 Derivative can be computed through integration. If f

is holomorphic on a closed region D. then for $\forall z \in D^\circ$[127]

$$f'(z) = \frac{1}{2\pi i}\int_{\partial V} d\zeta \frac{f(\zeta)}{(\zeta - z)^2}, \tag{6.29}$$

where V ($\subset D^\circ$) is any singly connected open set containing z whose boundary ∂V is a C^1 curve. □

Notice that the result can be obtained by formally exchanging the order of integration and differentiation.

[Demo] We have only to compute the derivative with the aid of Cauchy's formula **6.10**. Let z and $z+h$ be in V; this is always possible, since V is an open set and h can be very small. We have

$$\frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi i h}\int_{\partial V} d\zeta f(\zeta)\left(\frac{1}{\zeta - z - h} - \frac{1}{\zeta - z}\right) \tag{6.30}$$

$$= \frac{1}{2\pi i}\int_{\partial V} d\zeta f(\zeta)\frac{1}{(\zeta - z)^2} + J(h), \tag{6.31}$$

where

$$J(h) \equiv \frac{h}{2\pi i}\int_{\partial V} d\zeta \frac{f(\zeta)}{(\zeta - z)^2(\zeta - z - h)}. \tag{6.32}$$

Let δ be the distance between ∂V and $\{z, z+h\}$.[128] Since there is a positive number M such that $|f| < M$, we have a bound

$$|J(h)| \le \frac{h}{2\pi}\frac{M}{\delta^3}|\partial V|, \tag{6.33}$$

where $|\partial V|$ is the circumference of V. Hence, in the $h \to 0$ limit, J vanishes. □

6.14 General formula for derivative. Let f and V be the same as in **6.13**. Then

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial V} d\zeta \frac{f(\zeta)}{(\zeta - z)^{n+1}}. \tag{6.34}$$

□

Again notice that the result can be obtained by formally differentiating the integrand by exchanging the order of integration and differentiation.

[Demo] For $n = 0$ (→**6.10**) and 1 (→**6.13**) we know the formula is correct. Let us assume that the **formula** for $n = k - 1$ is correct. Then

$$\frac{f^{(k-1)}(z+h) - f^{(k-1)}(z)}{h} = \frac{(k-1)!}{2\pi i h}\int_{\partial V} d\zeta f(\zeta)\left[\frac{1}{(\zeta - z - h)^k} - \frac{1}{(\zeta - z)^k}\right]. \tag{6.35}$$

[127] This denotes the open kernel of D. That is, the largest open set in D.

[128] The distance between two sets A and B is the infimum of the distance between the point in A and that in B, i.e., distance $(A, B) \equiv \inf\{\rho(a, b) : a \in A, b \in B\}$.

Since

$$\frac{1}{(\zeta - z - h)^k} = \frac{1}{(\zeta - z)^k} + \frac{kh}{(\zeta - z)^{k+1}} + O[h^2], \tag{6.36}$$

$$\frac{f^{(k-1)}(z+h) - f^{(k-1)}(z)}{h} = \frac{k!}{2\pi i}\int_{\partial V} d\zeta f(\zeta)\frac{1}{(\zeta - z)^{k+1}} + O[h]. \tag{6.37}$$

□

Exercise.
(1) Let D be a region containing the origin. Show

$$\frac{1}{2\pi i}\int_{\partial D} \frac{z^n e^{z\zeta}}{n!\zeta^{n+1}} d\zeta = \left(\frac{z^n}{n!}\right)^2. \tag{6.38}$$

Using this relation, demonstrate

$$\sum_{n=0}^{\infty}\left(\frac{z^n}{n!}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2z\cos\theta} d\theta. \tag{6.39}$$

(2) Show for a holomorphic function f in the region containing the unit disk that

$$\frac{2}{\pi}\int_0^{2\pi} f(e^{i\theta})\cos^2\frac{\theta}{2} d\theta = 2f(0) + f'(0). \tag{6.40}$$

6.15 Holomorphic functions are Taylor expandable.[129] Note that for some $c \in \boldsymbol{C}$[130]

$$\frac{1}{\zeta - z} = \sum_{n=0}^{\infty} \frac{(z-c)^n}{(\zeta - c)^{n+1}} \tag{6.41}$$

is uniformly convergent (→**A5.11**), so that we can put this formula into Cauchy's formula **6.10** and integrate termwisely (→**A5.10**). The result is the following Taylor series expansion of f around c thanks to the integral expressions of derivatives in **6.14**:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(z-c)^n. \tag{6.42}$$

We have arrived at a very remarkable conclusion: if a complex function is differentiable in a region, it can be Taylor-expanded in it; that

[129] Brooks Taylor, 1685-1731.

[130] The reader should remember

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots + + z^n + \cdots$$

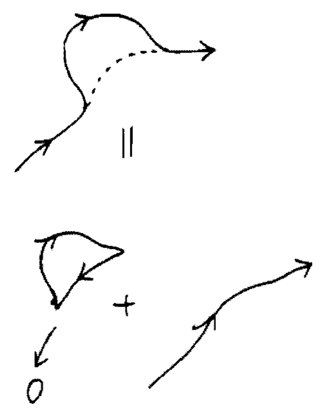
for $|z| < 1$.

is. it can be written as a convergent power series around any point in the region. This strong result is solely due to the strong definition of differentiability ($\rightarrow$**2A.4**).

6.16 Morera's theorem – converse of Cauchy's theorem.[131] Let D be a singly connected region. f be continuous on ∂D and

$$\int_{\partial U} dz f(z) = 0 \tag{6.43}$$

for any connected region $U \subset D$ such that ∂U is a C^1 curve. Then, f is holomorphic in D.[132]□



[Demo] If we can show that there is a holomorphic function F such that $F' = f$, we are done ($\rightarrow$**6.1-2**). Consider

$$F(z) = \int_\gamma f(\zeta)d\zeta. \tag{6.44}$$

where γ is an oriented C^1 curve in D from $a \in D$ to $z \in D$. The condition (6.43) and the linearity of integration with respect to the 'algebra' of oriented curves ($\rightarrow$**6.2**) implies that F does not depend on the choice of the path connecting a and z. Hence, F is holomorphic. □

6.17 Who was Gauss?[133] Carl Friedrich Gauss was born on April 30. 1777 in Braunschweig. Although he studied at University of Göttingen from 1795 to 98. he was already the first rate mathematician, and completed his number theory masterpiece (*Disquisitiones Arithmeticae*) when he was 20 (the printing of this famous book. which Dirichlet carried wherever he went. started in April. 1798. but was interrupted several times. and was published only in 1801).

He obtained his PhD in 1799 from University of Helmstedt with the thesis on the existence of the roots of algebraic equations. This was his favorite topic. which he proved several times with different methods in his life. The thesis avoided the use of imaginary numbers. because he was afraid that he might not get PhD due to conventional professors. Therefore. the statement was that any algebraic equation can be factorized into first or second order factors.

After his PhD. from 1799 to 1807. he was fully supported by Prince Ferdinand of Braunschweig. and could concentrate on mathematics until he was 30. Almost all his great accomplishments started during this

[131] Giacinto Morera. 1856-1909.

[132] Morera's theorem still holds even if we restrict U in (6.43) to be triangles or rectangles in D.

[133] heavily relying on T. Takagi. ***Kinsei Suugaku Shidan*** (Tales from Modern Mathematics History) (Kyoritu. 1933). See also W. K. Bühler. *Gauss. a biographical sketch* (Springer. 1981). All of his offsprings seem to be in the US.

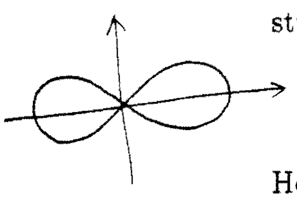
'happiest time of my life' (according to old Gauss in his 70s, "for mathematics unhindered and uninterrupted time is mandatory").

After 1807, he was a professor and the chief astronomer at Göttingen, and 'could not have any time to do big work.' "When my head is completely occupied by the effort to grasp a shadow of the spirit floating in the air comes the time to give a lecture. I must jump up and switch my attention to a completely different world. The pain is beyond any expression ⋯." In his memoir mixed with the calculations on elliptic functions one finds, "Der Tod ist mir lieber als ein solches Leben."

From 1816 he participated in the field work to make the map of Hannover. It is a famous story that he was attempted to check the flatness of the space. In 1828 Gauss invited Wilhelm Weber to Göttingen, and for a few tens of years they collaborated on the study of electromagnetism. He died on May 22. 1855 in Göttingen. His monument carries 'Mathematicorum princeps.'

As we see in Discussion in **8A.4**, Gauss had already known the main part of complex function theory by 1811. but he never published it. He should have known elliptic function theory. but he did not publish it. Later Abel and Jacobi constructed the theory. expecting that Gauss should have known most results. Gauss knew non-Euclidean geometry. but he did not publish it. He avoided debate and argument with reactionary conservatives (recall what he did in his thesis). His seal had one tree with a couple of fruits with the motto 'pauca sed matura.'[134]

Gauss wrote in his diary on January 8. 1797 that he started to study lemniscate in conjunction to



lemniscate

$$u = \int_0^x \frac{dx}{\sqrt{1-x^4}}. \tag{6.45}$$

He was trying to generalize trigonometric functions for some time using the analogy

$$\text{arc } \sin x = \int_0^x \frac{dx}{\sqrt{1-x^2}}. \tag{6.46}$$

This was the starting point of his study of elliptic functions. In this study (and in many others) he did a lot of experimental mathematics using numerical studies. He loved numbers; for example, when one of his acquaintances died. he computed the life span of the deceased in days on the back of the notice. In one of his note he gave $e^{-\pi}$ up to 50 decimal places. His computations were extremely elegant and clever, often exploiting number theory. His mathematics was inductive; he was an explorer of the universe of numbers.

He developed fast Fourier transform (→**32B.12**), one of the best

[134] cf.. paucity. maturity.

numerical integration schemes (1814 →**22A** needed to perform perturbation calculation), the least square approximation method (1821-3 in order to study the motion of planetoids), etc. His theory of curved surfaces (1827) was during his map making activity, and his potential theory (1839-40) was related to his electromagnetism study.

His pure and applied mathematics were inseparably intertwined, that is, his applied mathematics was the true applied mathematics; we saw such examples recently in Kolmogorov.