5 Differentiation of Complex Functions

Differentiation of complex functions is defined as a strong differentiation. This requires a special relation between the real and imaginary components of derivatives (the Cauchy-Riemann equation).

Key words: holomorphy, Cauchy-Riemann equation, conjugate harmonic function

Summary
(1) Differentiability or holomorphy is defined as strong differentiability (**5.1**).
(2) Consequently, we have the Cauchy-Riemann equation (**5.3**).
(3) Real and imaginary parts of holomorphic functions are harmonic (**5.6**).

5.1 Differentiation. We have already discussed the meaning of differentiability of complex functions (→**2A.8**). Let us repeat the definition (Review **2A.4** and **2A.8**). If the following limit exists[117]

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}, \tag{5.1}$$

we say f is *differentiable* at z. Note that this is a strong derivative (→**2A.4**). The limit is written as $f'(z)$ or df/dz and is called the *derivative* of f at z. As usual, differentiability is local linearizability (→**2A.1**).

5.2 Differentiation rules are the same as in elementary real analysis: Differentiation of sums and products of complex functions can be computed with the aid of the ordinary rules used in the real analysis.

5.3 Cauchy-Riemann equation: Differentiability of $f(z) = u(x,y) + iv(x,y)$ w.r.t. $z = x+iy$,[118] where u, v, x and y are real, is very different from its differentiability as a function of x and y. Write (5.1) in the following form $f(z+h) - f(z) = f'(z)h + o[h]$, or

$$u(x+s,y+t)-u(x,y)+i[v(x+s,y+t)-v(x,y)] = (P+iQ)(s+it)+o[s,t], \tag{5.2}$$

[117] This means that the limit <u>does not</u> depend on how z is approached.

[118] Henceforth, if f is written as $u + iv$, and z is written as $x + iy$ without any comments, u, v, x, y are understood as real quantities throughout this lecture notes.

where $f' = P + iQ$ and $h = s + it$ with P, Q, s and t being all real. The differentiability **5.1** implies that P and Q does not depend on the choice of s and t, so that (5.2) must be an identity w.r.t. s and t. Hence, we get $u_x = P$, $u_y = -Q$, $v_x = Q$ and $v_y = P$, i.e.,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{5.3}$$

This set of equations is called the *Cauchy-Riemann equation.* central defining feature

Cauchy 1827, Riemann 1851

Exercise
(1) Demonstrate that

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}, \quad \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}. \tag{5.4}$$

(2) Let $f(r, \theta) = Re^{i\Theta}$ (the polar expression). If f is holomorphic, then

$$r\frac{\partial R}{\partial r} = R\frac{\partial\Theta}{\partial\theta}, \quad \frac{\partial R}{\partial\theta} = -rR\frac{\partial\Theta}{\partial r}. \tag{5.5}$$

Discussion.
(A) The cleverest way to derive the Cauchy-Riemann equation is to use the fact that the derivative df/dz is a strong derivative. Consequently, $\partial f/\partial x = \partial f/\partial iy$.
(B) Even if the Cauchy-Riemann equation holds, it does not guarantee the holomorphy. Consider $f(z) = e^{-1/z^4}$ at $z = 0$. At $z = 0$ the CR equation indeed holds as $0 = 0$, but if $z = re^{i\pi/4}$, then obviously f diverges in the $r \to +0$ limit.

5.4 Holomorphic function: If $f(z)$ is differentiable at each point of a region D, f is called a *holomorphic function* on D.[119] [When D is not open, f is called a holomorphic function on D, if it is holomorphic on an open set containing D].

This is not about a single point; Think about z + f (\theta) z^2

Exercise
(A) Are the following functions holomorphic?
(1) $x^2 + iy$.
(2) $(x^2 - y^2 + 3x) + i(2xy + 3y)$.
(B)
(1) Demonstrate that if $f(z)$ is holomorphic and its argument ($arg(f(z))$) is constant, then f must be constant.
(2) Demonstrate that if $f(z)$ is holomorphic, then $\overline{f(\bar{z})}$ is also holomorphic.
(C) Let f and g be holomorphic for all $\mathbf{C}$, and $f' = g$, $g' = -f$, $f(0) = 0$ and $g(0) = 1$. Find f and g. [$f(z) = \sin z$.]

[119] 'Analytic function' (→**7**) is a distinct concept: f is analytic, if it can be Taylor-expanded. For a function on a real axis differentiability does not guarantee the analyticity (**7.11**). Miraculously, for a function on a complex plane differentiability implies Taylor-expandability as we will see later (→**6.15**).

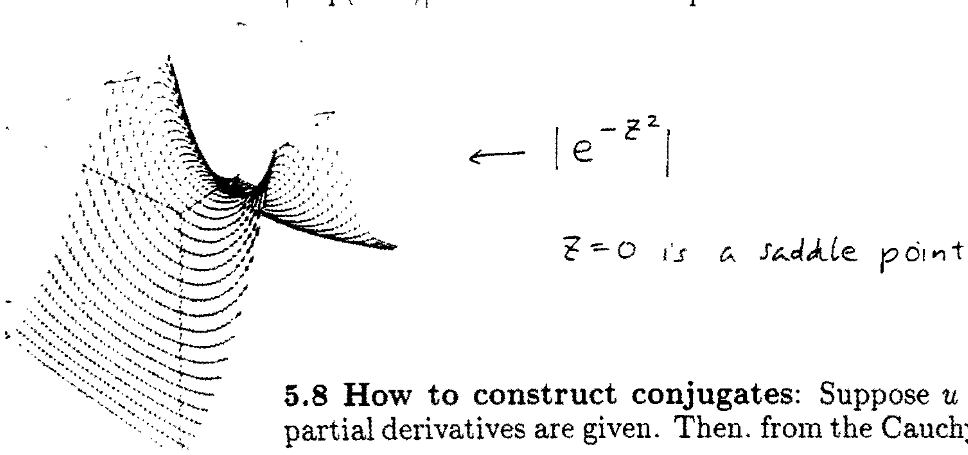
5.5 Examples:
(1) Show that z is holomorphic. but $\bar{z}$ is not. Cf. **10.15** Exercise (3).
(2) Show that $f(z) = x^2 + y^2 + 2ixy$ is not holomorphic anywhere on C (check that the Cauchy-Riemann equation **5.3** does not hold).

5.6 Re and Im of holomorphic function are harmonic: Let $f = u+iv$. u. v being real. be holomorphic on a region D. Then, we have a Cauchy-Riemann equation (5.3). If u and v are twice differentiable,[120] then it is easy to see u and v are harmonic(→**2C.11**) on D. v is called the *conjugate harmonic function* of u.

Exercise.
(1) Is there any holomorphic function whose real part is $e^{x/y}$? See **5.8** also.
(2) Demonstrate

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4|f'(z)|^2. \tag{5.6}$$

5.7 $\log|f|$ **is harmonic** : Let f be holomorphic on a region D without zero in the region. Then. $\log|f(z)|$ is harmonic as a function of x and y on D. □
This is obvious from $\log|f| = \Re \log f$ (→**4.7**).
The harmonicity of $\log|f|$ implies that $|f|$ cannot have any local maximum or minimum. This could be guessed from the nature of the Laplacian **1.13**. or rigorously from **29.4-29.6**. See the illustration of $|\exp(-z^2)|$. $z = 0$ is a saddle point.



5.8 How to construct conjugates: Suppose u is harmonic and its partial derivatives are given. Then. from the Cauchy-Riemann equation

[120] We will later prove that the real and imaginary parts of a holomorphic function are infinite times differentiable. so these assumptions are actually not needed (→**6.12**).

(→**5.3**) we have

$$v(x,y) = \int v_x(x,y)dx = -\int u_y(x,y)dx + C_1(y), \tag{5.7}$$

$$v(x,y) = \int v_y(x,y)dy = \int u_x(x,y)dy + C_2(x), \tag{5.8}$$

where C_1 and C_2 are unknown functions. Compare these two expressions, and we can actually fix v up to an additive constant (the method seems to work invariably at least for physicists' problems). This is also a method to reconstruct a holomorphic function from its real or imaginary part.

Exercise.
(A) Let

$$u(x,y) = (x-y)(x^2+4xy+y^2). \tag{5.9}$$

(1) Demonstrate that u is harmonic (→**2C.11**) on the (x,y) plane.
(2) Find its conjugate harmonic function.
(3) Write down the holomorphic function $f(z)$ whose real part is given by u.
(B) Find the holomorphic function whose real part is given by (additive constants are ignored)
(1)

$$u = \frac{x}{x^2+y^2-2y+1} \Rightarrow f(z) = \frac{1}{z-i}.$$

f(x0 is obtained by setting x= z and y=0

(2)

$$u = \frac{\sin x}{\cos x + \cosh y} \Rightarrow f(z) = \tan\frac{z}{2}.$$