4 Complex Functions

The shortest path between two truths in the real domain passes through the complex domain. J. Hadamard

Elementary complex functions are reviewed. Remember that logarithm is an infinitely many-valued function. and, consequently. some care is needed to interpret α^β. For example. $\sqrt{1} + \sqrt{1}$ is not equal to $2\sqrt{1}$.

Key words: complex function. exponential function. entire function. logarithm. many-valued function. power.

Summary
(1) Some functions are multi-valued and need extra care (**4.7-10**).
(2) Except for this. we may treat complex functions just as real functions.

4.1 Preliminary. We already discussed complex-valued functions on $\boldsymbol{C}$ in **2A.8**. A map from a region in $\boldsymbol{C}$ to $\boldsymbol{C}$ is called a *complex function.*
Remark [Why complex numbers?] i is called the imaginary unit. because it was. as a number. long thought to be fictitious. However. there are many reasons to regard $\boldsymbol{C}$ as the most natural number system for 'applied mathematics': for example. a general n-th order polynomial $P(x)$ has n roots only on the complex plane (that is. an equation $P(x) = 0$ is solvable only when x is allowed to be a complex number →**6.17**): A general normal square matrix T (i.e.. $T^*T = TT^*$ holds) is diagonalizable by a unitary transformation only when the matrix is considered on $\boldsymbol{C}$.
Historical remark.
The name 'imaginary number' (nombre imaginaire) is due to Descartes. The name 'complex number' (komplex Zahl) is due to Gauss (→**6.17**). The existence of complex numbers was accepted first in conjunction to the third order equation $x^3 = 15x + 4$. If Cardano's method is applied. the root is give by

$$x = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}. \tag{4.1}$$

but the equation clearly has a root $x = 4$. Bombelli used

$$2 \pm \sqrt{-11} = (2 \pm i)^3 \tag{4.2}$$

It is impossible to make the formula for the roots of the tertiary equations without using complex numbers 上野代数2 p236

and showed that the above formula indeed expresses 4. That is, he for the first time demonstrated that 'actual numbers' (real numbers) could be obtained through the use of imaginary numbers.[109]

4.2 Exponential function. For $z = x + iy \in \boldsymbol{C}$, where $x, y \in \boldsymbol{R}$, its *exponential function* is defined by

$$e^z \equiv e^x(\cos y + i \sin y). \tag{4.3}$$

We can explicitly check, using the properties of (real) trigonometric functions, $e^{iy_1}e^{iy_2} = e^{i(y_1+y_2)}$, so that for $z_1, z_2 \in \boldsymbol{C}$, $e^{z_1}e^{z_2} = e^{z_1+z_2}$ (addition theorem). Using this, we can check the differentiability of e^z:

$$e^{z+h} = e^z(1 + \Re h)(1 + i\Im h) + o[h] = e^z(1 + h) + o[h]. \tag{4.4}$$

Hence, e^z is holomorphic (→**2A.8**, **5.4**) for any $z \in \boldsymbol{C}$. A function holomorphic everywhere in $\boldsymbol{C}$ is called an *entire function*. e^z is an entire function. It is easy to demonstrate:
(i) $\overline{e^z} = e^{\bar{z}}$.
(ii) $de^z/dz = e^z$.
(iii) $e^z \neq 0$ for any $z \in \boldsymbol{C}$. This is obvious from $e^z e^{-z} = 1$.
(iv) $|e^{x+iy}| = e^x$, or $|e^{iy}| = 1$ for any real y.
(v) $e^{i2n\pi} = 1$ for any integer n. The *primitive period* of e^z is said to be $2\pi i$.

Discussion [Elementary functions]. A function $w(z)$ determined by an irreducible complex polynomial $P(w, z) = 0$ is called an *algebraic function*. Therefore, the ordinary polynomials (called *rational integral functions*) and the ratio of polynomials (called *rational functions*) are algebraic functions. Algebraic functions, exponential functions, logarithmic functions, trigonometric functions, their inverse functions, and the functions made from these functions through finite number of compositions are called *elementary functions*. Elementary functions which are not algebraic functions are called *elementary transcendental functions*.

4.3 How Euler arrived at Euler's formula. With the aid of the addition theorems for trigonometric functions, he first noted that

$$(\cos x \pm i \sin x)(\cos y \pm i \sin y) = \cos(x + y) \pm i \sin(x + y). \tag{4.5}$$

Hence, he realized that for any positive integer n

$$(\cos z \pm i \sin z)^n = \cos nz \pm i \sin nz, \tag{4.6}$$

[109] R. Nagaoka, Sugaku Seminar, 1986 July, p36.

that is,

$$\cos nz = \frac{1}{2}\{(\cos z + i\sin z)^n + (\cos z - i\sin z)^n\} \tag{4.7}$$

$$\sin nz = \frac{1}{2i}\{(\cos z + i\sin z)^n - (\cos z - i\sin z)^n\}. \tag{4.8}$$

Then. he took an 'infinitely small' z and 'infinitely large' number n such that $nz = v$ in these formulas.[110] Since $\cos z = 1$ and $\sin z = z$. we get. noting that $(1+v/n)^n = e^v$, for example, $\cos v = (e^{iv} + e^{-iv})/2$. In this way he arrived at the famous formula.

Discussion [Non-standard analysis].
The reader should have realized that in the above. infinitesimals are treated as such (not as limits). It is a very productive way of using infinitesimals just as physicists like. However. it was hard to put the concept of infinitesimal on a firm ground. This became only possible after the introduction of 'non-standard models of real numbers' by A Robinson.[111] The analysis based on the nonstandard model of reals is called the *non-standard analysis.* With its aid. what Euler did can be justified. To understad what really 'non-standard' means. you need rudimentary knowledge of formal logic.

Exercise.
Let $z = x + iy$. Demonstrate Euler's formula $e^z = e^x(\cos y + i\sin y)$ by showing the following formulas:

$$\log\left|\left(1+\frac{z}{n}\right)^n\right| \to x \tag{4.9}$$

$$arg\left(1+\frac{z}{n}\right)^n \to y \bmod 2\pi \tag{4.10}$$

as $n \to \infty$.

4.4 Who was Euler?[112] Euler's analysis textbook *Introductio in analysin infinitorum* (1748) was extremely influential beyond 1800, so that his notations such as sin. cos. e. π, i. $\sum$, etc.. became conventional. He. not Newton. wrote down the so-called Newton's equation of motion for the first time. and laid the foundations of continuum mechanics including fluid dynamics (but see d'alembert **2B.7**). F

Leonhardt Euler was born in Basel. Switzerland, on April 15, 1707. He revealed a photographic memory by reciting *Aeneid* page by page by heart. In 1720 he enrolled at the University of Basel and graduated with first honors two years later. His master's thesis in 1724 compared the natural philosophies of Descartes and Newton. Euler convinced Johann Bernoulli to tutor him in mathematics and natural philosophy for

[110] Here the words between ' and ' are his.

[111] A. Robinson. *Non-standard analysis* (North-Holland. 1966).

[112] This entry is mainly based on p486– of R. Calinger. *Classics of Mathematics* (Prentice-Hall. 1995).

one hour on Saturday afternoons. Bernoulli quickly recognized Euler's genius and helped convince his father to allow his son to concentrate on mathematics. After failing to get a physics position at Basel, he joined the St. Petersburg Academy of Sciences in 1727, boarding at Daniel Bernoulli's home (cf. **1.6**).

He became the first professor of mathematics, succeeding D. Bernoulli in 1733, who returned to Switzerland. From 1733 to 1741, Euler immersed himself in research with enthusiasm despite hostility from the Russian nobility and from the Orthodox Church which opposed Copernican astronomy. He precisely computed $\zeta(2) = \sum(1/n^2) = \pi^2/6$ (see Discussion below). He gained an European-wide reputation with this and with his first book *Mechanica* (1736). During this period he found $e^{ix} = \cos x + i \sin x$ and $e^{i\pi} + 1 = 0$ (→**4.3**). He also introduced beta and gamma functions (→**9.1**).

In 1741 he accepted the invitation of Friedrich the Great to join the Brandenburg Society (Berlin Academy of Sciences after 1744) (→**2B.7**). He was the director of its mathematical section from 1744 to 1765. He was at the peak of his career during this period. In the mid-1750s Euler tutored Lagrange (→**3.5**) by correspondence and selflessly withheld from publication the part of his work on the calculus of variations (→**3.2-4**)so that Lagrange might receive due credit for his contribution to the subject.

After disagreeing with the king over academic freedom. Euler returned to Russia in 1766. where Catherine the Great made him a generous offer. A cataract and its maltreatment made him totally blind by 1771 (he had lost his right eye sight in 1735). but his productivity at least in number of pages increased: he dictated books to a small group of collaborators. doing calculations in his head involving as many as 50 decimal places. He died of a brain hemorrhage in 1784.

Euler was chiefly responsible for differential equations. and calculus of variation with Lagrange. He pioneered differential geometry and topology (Euler's polyhedral formula: $v - e + f = 2$). His colleagues dubbed him "analysis incarnate." His disciplinary intuition never failed when he used infinite series. even though its general theory was to be created by Cauchy (→**6.11. 17.18(2)**). Euler found the prime number theorem in 1752. although he could not prove it, which was to be rediscovered and proved by Gauss (→**6.17**). Most of his number theoretic results appeared in his correspondence with his best friend in St. Petersburg. Christian Goldbach (famous for his conjecture: every even number is a sum of two prime numbers. This is mentioned in Hilbert's 8th problem (→**20.4**). and is still open).

Discussion.

(1) Euler often used 'algebraic formalism' (the belief that algebraic expressions are always correct whatever numbers replace the symbols in them) to obtain nontrivial results. The following illustrates his approach to compute $\zeta(2)$ (→**7.15**), etc. [An example of the 'modern version' of 'algebraic formalism' is illustrated in **17.3a** Discussion.]

Euler tried to extend the factorization of polynomials to more general functions. Sine has zeros at $n\pi$ for all $n \in \boldsymbol{Z}$. Therefore, he guessed

$$\sin z \propto z\left(1-\frac{z^2}{\pi^2}\right)\left(1-\frac{z^2}{4\pi^2}\right)\cdots. \tag{4.11}$$

We know $\sin z \simeq z$ for small z, so that the proportionality constant should be 1:

$$\sin z = z\left(1-\frac{z^2}{\pi^2}\right)\left(1-\frac{z^2}{4\pi^2}\right)\cdots. \tag{4.12}$$

Admitting this relation (which is actually correct) and expanding the both sides in the Taylor series, we obtain

$$z-\frac{z^3}{3!}+\cdots = z\left\{1-\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\cdots\right)z^2+\cdots\right\}. \tag{4.13}$$

Comparing the coefficients on both sides, Euler obtained

$$\zeta(2) \equiv 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{6}. \tag{4.14}$$

In this way the value of the zeta function for even positive integers can be obtained. Obtain $\zeta(4) = \pi^4/90$.
(2) Show

$$\int_0^\infty \frac{\log(1-x)}{x}dx = -\frac{\pi^2}{6}. \tag{4.15}$$

4.5 Complex trigonometric functions: For real x we know (→**4.3**)

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}, \quad \sin x = \frac{e^{ix}-e^{-ix}}{2i}. \tag{4.16}$$

Therefore, the following definitions for $\forall z \in \boldsymbol{C}$ are suggested (thisis an analytic continuation →**7.10**):

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}, \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}. \tag{4.17}$$

These functions are entire functions (→**4.2**). We can easily demonstrate:
(i) $\overline{\cos z} = \cos\bar{z}$, $\overline{\sin z} = \sin\bar{z}$.
(ii) $\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$, $\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$. Other complex trigonometric functions such as tan, sec, etc., can be defined analogously.
(iii) $\sin(z+2n\pi) = \sin z$, $\cos(z+2n\pi) = \cos z$, where $n \in \boldsymbol{Z}$. 2π is the primitive period of these functions.
(iv) $d\sin z/dz = \cos z$, $d\cos z/dz = -\sin z$.

(v) $\cos^2 z + \sin^2 z = 1$.
(vi) $\sin 2z = 2 \sin z \cos z$.
In short. all the real formulas can be straightforwardly extended to complex cases (this is due to the principle of invariance of functional relations (→**7.6**)).

Discussion.
If f has a Taylor expansion around a real point whose coefficients are all real, then $\overline{f(z)} = f(\overline{z})$.[113]

Exercise.
(1) Is $|\cos z| \le 1$ correct?
(2) Is $Im(e^{iz}) = \sin z$ correct?
(3) Demonstrate

$$\sin n\theta = \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n}{2k+1} \sin^{2k+1}\theta \cos^{n-2k-1}\theta. \tag{4.18}$$

and

$$\cos n\theta = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \sin^{2k}\theta \cos^{n-2k}\theta. \tag{4.19}$$

Here $[X]$ implies the largest integer not exceeding X (the Gauss symbol).
(4) Demonstrate

$$\tan(x+iy) = \frac{\sin 2x}{\cos 2x + \cosh 2y} + i\frac{\sinh 2y}{\cos 2x + \cosh 2y}. \tag{4.20}$$

4.6 Complex hyperbolic functions. Complex hyperbolic functions are defined by

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}. \tag{4.21}$$

Other complex hyperbolic functions as $\tanh z$ etc., can be defined analogously.
(i) $\overline{\cosh z} = \cosh \overline{z}$, $\overline{\sinh z} = \sinh \overline{z}$.
(ii) $d\cosh z/dz = \sinh z$. $d\sinh z/dz = \cosh z$.
(iii) $\cosh^2 z - \sinh^2 z = 1$.
(iv) $\sin iz = i \sinh z$. $\cos iz = \cosh z$.
Again. in short. all the real formulas can be straightforwardly extended to complex cases (this is due to the principle of invariance of functional relations (→**7.6**)).

[113] To complete the demonstration. we need **7.6**.

Exercise.
Show

$$\cosh iz = \cos z, \quad \sinh iz = i \sin z \tag{4.22}$$

$$\tan iz = i \tanh z, \quad \tanh iz = i \tan z \tag{4.23}$$

4.7 Logarithm: $e^z = w$ can be solved for any $w \in \boldsymbol{C} \setminus \{0\}$. Let the polar form (→**a4.6**) of w be $|z|(\cos \arg z + i \sin \arg z) = |z|e^{i \arg z}$. *Logarithm* is defined as the inverse function of e^z, so we define

$$\log z = \log|z| + i \arg(z). \tag{4.24}$$

Here $\log|z|$ is computed as the usual real logarithm. $\arg z$ is multivalued. and consequently $\log z$ is multivalued. When we choose the principal value of $\arg z$ (i.e.. Arg z. →**a4.7**). the value of $\log z$ is written (sometimes) as Log z and is called the *principal value* of $\log z$:

$$\mathrm{Log} z = \log|z| + i\mathrm{Arg}\ z. \tag{4.25}$$

log is an example of many-valued functions. In a nbh (not including 0) of $z(\neq 0)$ (4.24) defines infinitely many maps. They are called *branches* (→**8A.3-4**). Thus log is an *infinitely many-valued function* (→**8A.4**).

Discussion.
Let

$$f(z) \equiv \mathrm{Log} z - \mathrm{Log}(z-1). \tag{4.26}$$

where Log is the principal value of log. Describe the function $\phi(x)$ defined by

$$\phi(x) \equiv \lim_{\epsilon \to +0} \frac{1}{2\pi i}[f(x+i\epsilon) - f(x-i\epsilon)]. \tag{4.27}$$

where ϵ is a positive number tending to 0. and x is real (**8B.16** may be a hint. See also **6.1** Exercise (1).).

4.8 Warning: $\log(\alpha\beta)$ is not simply equal to $\log\alpha + \log\beta$. In particular. $\log z^2$ is not equal to $2\log z$: even the sets of possible values are different. However. $\log z^2$ and $\log z + \log z$ have the same sets of possible values. but the equality between these quantities is generally false. Even $\log 1 + \log 1 = \log 1$ is false.

Exercise.
(1) $\log 4$ [$= 2\mathrm{Log}2 + 2n\pi i$].
(2) $\log[(1+i)/\sqrt{2}]$ [$= \pi/4 + 2n\pi i$].

4.9 Power. α^β is defined by

$$\alpha^\beta \equiv \exp(\beta \log\alpha). \tag{4.28}$$

Due to the many-valuedness (→**4.7**) of $\log\alpha$, α^z describes many single-valued functions. On the other hand, z^β generally describes a single many-valued function. For z^b ($b \in \boldsymbol{R}$) we have three different cases:
(i) If $b \in \boldsymbol{Z}$, then the multiplicative uncertainty of z^b, i.e., $e^{b2n\pi i}$ ($n \in \boldsymbol{Z}$) is always 1, so z^b is single-valued.
(ii) If $b \in \boldsymbol{Q} \setminus \boldsymbol{Z}$, then there exists the irreducible fraction representation $b = p/q$ ($p, q \in \boldsymbol{Z}$, $q \neq 1$ and $q > 0$). Hence, the multiplicative uncertainty factor $e^{b2n\pi i}$ ($n \in \boldsymbol{Z}$) takes different values only for $n = 0, 1, \cdots, q-1$. Consequently, z^b is a q-valued function. For example, when we write $\sqrt{z}$, this means $z^{1/2}$, so it is a double-valued function. For example, $\sqrt{1} = \pm 1$, not 1.
(iii) If $b \in \boldsymbol{R} \setminus \boldsymbol{Q}$, then $e^{b2n\pi i}$ is distinct for all the values of $n \in \boldsymbol{Z}$. That is, in this case z^b is an infinitely many-valued function.

Exercise.
(A) Find all the values of the following ($n \in \boldsymbol{Z}$):
(1) $(-2)^{\sqrt{2}}$ [$= 2^{\sqrt{2}}\{\cos((2n+1)\sqrt{2}\pi) + i\sin((2n+1)\sqrt{2}\pi)\}$]
(2) 1^{-i} [$= e^{2n\pi}$].
(B) Compute i^i and find the range of $|i^i|$.
(C) Is $|a^b| \le |a|^{|b|}$ correct?

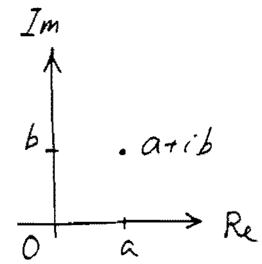
4.10 Examples and warnings. If a complex (non-integer) power function is involved, we must compute them <u>before</u> applying any operation to it. For example, when we wish to compute $\log\alpha^\beta$ we must compute $\alpha^\beta = \exp(\beta\log\alpha)$ first.
(i) $i\log i$ and $\log i^i$ are different.
(ii) $\sqrt{1} + \sqrt{1} = 2\sqrt{1}$ fails to be true.
(iii) $(z^b)^c = z^{bc}$ is not generally true, even when $b, c \in \boldsymbol{R}$.
(iv) $(z_1 z_2)^\alpha = z_1^\alpha z_2^\alpha$ is not generally true, where $z_1, z_2, \alpha \in \boldsymbol{C}$.
(v) 2^i has infinitely many different values which can be indefinitely large or indefinitely close to 0.
(vi) $d\alpha^z/dz = \alpha^z\log\alpha$ holds only when the same branch (→**8A.4**) of $\log\alpha$ is used on both sides.

4.11 Inverse trigonometric functions: Solving (4.17), we can easily demonstrate

$$\arcsin z = i\log(-iz + \sqrt{1-z^2}), \tag{4.29}$$
$$\arccos z = i\log(z + \sqrt{z^2-1}). \tag{4.30}$$

Here $\sqrt{\ }$ and log must be understood as many-valued functions (→**4.7**, **4.9**).

APPENDIX a4 Complex Numbers



a4.1 Complex number. A *complex number* is an expression of the form $a + ib$, where $a, b \in \boldsymbol{R}$ and i is called the *imaginary unit*.[114] The set of all the complex numbers is denoted by $\boldsymbol{C}$. The property of 'i' is fixed by the following arithmetic rules, which force $i^2 = -1$.

Discussion.
Discuss the relation (due to Hamilton) between $a + ib$ and the matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \tag{4.31}$$

Consider, for example, multiplication. This 'justification' of complex numbers is due to Cayley.

a4.2 Arithmetic rules of complex numbers. Let $\alpha = a + ib$ and $\beta = c + id$, where $a, b, c, d \in \boldsymbol{R}$.
(i) $\alpha = \beta \iff a = c$ and $b = d$.
(ii) $\alpha \pm \beta = (a + c) \pm i(b + d)$
(iii) $\alpha\beta = (ac - bd) + i(ad + bc)$
(iv) $\alpha/\beta = (ac + bd)/(c^2 + d^2) + i(ad - bc)/(c^2 + d^2)$.
Thus $a + ib$ is no more a symbol but actually the sum of a and $i \times b$.[115]
Notice that $\alpha\beta = 0 \iff \alpha = 0$ or $\beta = 0$.

Exercise. Find the real and imaginary parts (→**4.5-6**).
(1) $\sin 2i$ $[= i \sinh 2]$
(2) $\cos(1 + i)$ $[= \cos 1 \cosh 1 - i \sin 1 \sinh 1]$
(3)

$$\tan(2 - i) \quad \left[= \frac{\sin 4 - i \sinh 2}{2(\cos^2 2 + \sinh^2 1)}\right].$$

a4.3 Real part, imaginary part. Let $\alpha = a + ib$ with $a, b \in \boldsymbol{R}$. a is called the *real part* of α and is denoted by $\Re\alpha$. ib is called the *imaginary part* of α and is denoted by $\Im\alpha$. If $\Re\alpha = 0$, then α is called a *purely imaginary number*.

a4.4 Conjugate complex number. Let $\alpha = a + ib \in \boldsymbol{C}$. $a - ib$ is called the *conjugate complex number* of α (or *complex conjugate* of α) and is denoted by $\overline{\alpha}$.

---

[114] This form was given as a general expression of complex numbers by d'Alembert (→**2B.7**) in a prize paper, but the popularity of complex numbers was never large until Gauss's proof of the fundamental theorem of algebra (→**a4.5**, **6.17**). d'Alembert himself remained silent about this in the *Encyclopèdie*.

[115] These rules imply that with + and ×, $\boldsymbol{C}$ becomes a commutative field. Its zero element is $0 + i0$ and its unit element is $1 + i0$.

(i) $\overline{\overline{\alpha}} = \alpha$.
(ii) $\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$.
(iii) $\overline{\alpha\beta} = \overline{\alpha}\overline{\beta}$
(iv) $\Re\alpha = (\alpha + \overline{\alpha})/2$, $\Im\alpha = (\alpha - \overline{\alpha})/2i$.

a4.5 Complex plane. If on a plane with rectangular coordinate axes a complex number $\alpha = a + ib$ is represented by a point (a, b), then the plane is called the *complex plane*. The abscissa is called the *real axis*, and the ordinate is called the *imaginary axis*.

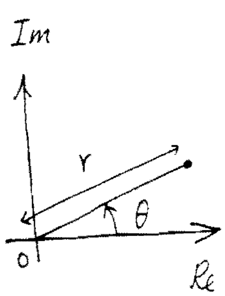
Discussion.
Let α, β, and γ be distinct complex numbers. A necessary and sufficient condition for these three points on the complex plane to make a regular triangle is

$$\alpha^2 + \beta^2 + \gamma^2 = \alpha\beta + \beta\gamma + \gamma\alpha. \qquad (4.32)$$

Exercise.
(1) Draw the images of lines parallel to the real (imaginary) axis due to $1/z$ (→**10.3**).
(2) Demonstrate that

$$\left|\frac{z - a}{\overline{a}z - 1}\right| < 1. \qquad (4.33)$$

if $|z| < 1$ (→**10.12**).



a4.6 Polar form of complex number, modulus and argument. A point on a complex plane may be represented by the polar coordinates (r, θ) with the origin and the real axis as the pole and the generating line. For $\alpha = a + ib$ $r = \sqrt{a^2 + b^2} = \sqrt{\alpha\overline{\alpha}}$ is called the *absolute value* or the *modulus* of α and is denoted by $|\alpha|$. $\theta \equiv \arctan(b/a)$ is called the *argument* of α, and is denoted by $\arg\alpha$ (this is defined for $\alpha \neq 0$). $\alpha = r(\cos\theta + i\sin\theta)$ is called the *polar form* of α.

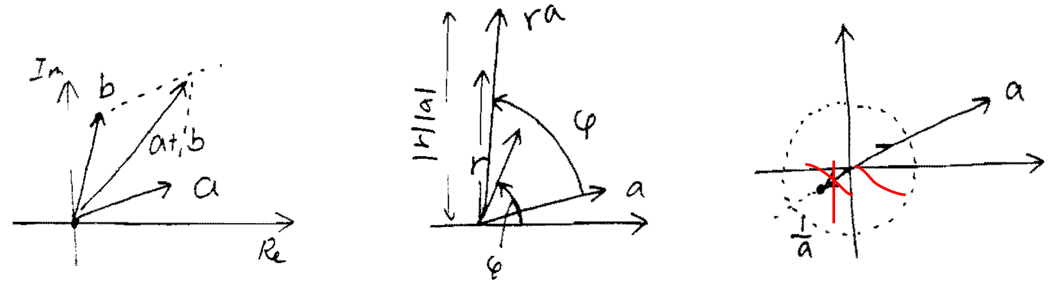
The set of all the complex numbers satisfying $|z| = 1$ is called the *unit circle*.

Exercise.
(1) The absolute value should be computed as $|a| = \sqrt{a^* a}$. For example, try to compute the absolute value of $(1 + 2i)/(3 + 5i)$.
(2) Find the polar form of $(1 + i)^n + (1 - i)^n$.
(3) Let $x \in \boldsymbol{R}$ and a be a complex constant. Compute the derivative of $arg(x + a)$ with respect to x.

a4.7 Principal value of argument. The argument of a complex number z ($\neq 0$) cannot be chosen uniquely, because arctan is only locally one-to-one. When the value is chosen in $[0, 2\pi)$ or $(-\pi, \pi]$, this is called the principal value of the argument, and is denoted by Arg z.

a4.8 Elementary properties of absolute values
(i) $|\alpha| = 0 \iff \alpha = 0$,
(ii) $\alpha\overline{\alpha} = |\alpha|^2$,
(iii) $|\alpha\beta| = |\alpha||\beta|$.
(iv) $|\alpha| \sim |\beta| \leq |\alpha + \beta| \leq |\alpha| + |\beta|$,[116]
where the second inequality is called the *triangle inequality.*
(v) $|\Re\alpha| \leq |\alpha|$. $|\Im\alpha| \leq |\alpha|$.

a4.9 Graphic representation of arithmetic operations: See the figures.



a4.10 Limit: We define

$$\lim_{n\to\infty} \alpha_n = \alpha \iff \lim_{n\to\infty} |\alpha_n - \alpha)| = 0. \tag{4.34}$$

Or, more precisely, we say $\{\alpha_n\}$ *converges* to α, if for any positive ϵ we can find a positive integer N such that

$$n > N \iff |\alpha_n - \alpha| < \epsilon. \tag{4.35}$$

a4.11 Convergence of series: Consider an infinite series $\sum_{i=0}^{\infty} \alpha_i$. Define its partial sum as $\beta_n \equiv \sum_{i=0}^{n} \alpha_i$. We say that the series is *convergent* if the sequence $\{\beta_n\}$ converges.

[116] $a \sim b$ means the absolute value of the difference of a and b.