2 Differentiation Revisited

Every graduate student knows what differentiation is, or does she? The essence of differentiation is the study of linear response. Many elementary properties of derivatives can be understood naturally from this point of view. Furthermore. this interpretation frees us from the elementary definition of derivatives, and paves the way to variational calculus and functional differentiation. However, these topics are postponed to **3**. In this section. we review differentiation, vector analysis. and curvilinear coordinates.

Key words: differentiation. chain rule, derivative as susceptibility. strong derivative, differentiation of complex function. partial differentiation. d'Alembert's solution for wave equation. moving coordinates. gradient, nabla. divergence. curl. Laplacian. Gauss-Stokes-Green's theorem, Poincaré's lemma. converse of Poincaré's lemma. Helmholtz-Hodge's theorem. Helmholtz-Stokes-Blumental's theorem. curvilinear coordinates. metric tensor.

Summary:
(1) To compute (strong) derivatives is to study linear responses (**2A.1**, **2A.4**. **2A.8**).
(2) The reader must be able to change freely the independent variables in PDE (**2B.3**. **2B.6**).
(3) 1D wave equation in free space can be solved via change of variables. The result is the famous d'Alembert's solution (**2B.4**).
(4) Geometrical meanings of *grad*. *div* and *curl* (**2C.1**, **2C.5**, **2C.8**) as well as their coordinate-free definitions must be understood clearly (**2A.5**. **2C.6**. **2C.9**).
(5) Gauss-Stokes-Green's theorem **2C.13**, Poincaré's lemma **2C.14**, and its converse (when the domain is singly connected) **2C.16** are crucial.
(6) If *curl* and *div* both vanish. the vector field is (essentially) constant. This can be shown by the Helmholtz-Hodge decomposition **2C.17**.
(7) The reader should be able to demonstrate various formulas of vector calculus (**2C.19**).
(8) Differential operators in orthogonal curvilinear coordinates **2D.3** must be understood without difficulty (**2D.7**, **2D.9**, **2D.10**).
(9) Do not use ∇ as a simple operator except in Cartesian coordinate system (**2C.12**).

Fermat's differentiation Stahl p15
$f(x+e) -f(x) =0 \implies \frac{f(x+e)-f(x)}{e}=0 \implies f[(x) =0$

Kepler also knew this sort of differentiation.

2.A Elementary Review

2A.1 What is differentiation? Let f be a function defined on an (open) interval I, and $a \in I$. If the following limit, denoted by $f'(a)$, exists, we say f is *differentiable* at a:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \tag{2.1}$$

$f'(a)$ is called the *differential coefficient* of f at a. If f is differentiable at $x = a$, then we have[64]

$$\delta f(a) \equiv f(a + \delta a) - f(a) \simeq f'(a)\delta a. \tag{2.2}$$

That is, we study the response of f against a small change of its variable. If a linear approximation of the response is reasonable for sufficiently small δa, we say f is differentiable. In other words, the essence of differentiation is the study of the linear response of f to a small perturbation of independent variables. This point of view will be exploited later (→**2A.4**, **3.7**), but we must note two immediate consequences of linearity, **2A.2** and **2A.3**.

Exercise. Perhaps, we should check our working knowledge of elementary calculus first.
(A) Discuss whether the following statements are true.[65] If correct, prove the statement.
(1) P_m is a polynomial in the following formula:

$$\frac{d^m}{dx^m}\left(\frac{1}{f}\right) = \frac{1}{f^{m+1}} P_m(f, f', \cdots, f^{(m)}), \tag{2.3}$$

where m is a positive integer.
(2) Let f be a differentiable function. If $f'(0) = 1$, then f is monotonic in a sufficiently small neighborhood of 0.
(3) Let f be a C^∞ function with $\lim_{x \to \infty} f(x) = 0$. Then, $\lim_{x \to \infty} f'(x) = 0$.
(B) Elementary differentiation questions:
(1) Let $x = e^t \cos t$ and $y = e^t \sin t$. Compute d^2y/dx^2 as a function of t.
(2) Compute the limits

$$\lim_{x \to 0} \left(\frac{1}{x} - \frac{x}{\sin^3 x}\right), \tag{2.4}$$

$$\lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x}. \tag{2.5}$$

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[64] $\simeq$ is used informally, but in these notes accurate meaning can always be attached. In the present case, $\simeq$ means equality ignoring $o[\delta a]$.

[65] B. R. Gelbaum and J. M. Olmsted, *Counterexamples in Analysis* (Holden-Day, 1964) is a useful book when you wish to think a delicate thing.

(3) Compute $d^m(e^{-1/x^2})/dx^m$ at $x = 0$ for all positive integers m.
(C) How many times are the following functions differentiable?
(1)

$$f(x) = x^{9/4} \text{ for } x \geq 0, \ 0 \text{ for } x < 0 \tag{2.6}$$

(2)

$$f(x) = x^3 \text{ for } x \geq 0, \ 0 \text{ for } x < 0 \tag{2.7}$$

(3)

$$f(x) = |x|^3. \tag{2.8}$$

(4)

$$y = e^{-1/x(1-x)} \text{ for } x \in [0,1], \ 0 \text{ otherwise} \tag{2.9}$$

(5)

$$f(x) = x^n \sin\frac{1}{x} \ \ (f(0) = 0), \tag{2.10}$$

where n is a positive integer.
(D) **Orthogonal polynomials**
They will be discussed in a unified way in **21**, but here, let us check some formulas related to them (generalized Rodrigues' formulas →**21A.6**)
(1) Demonstrate that $P_n^{(\alpha,\beta)}(x)$ defined as follows is an n-th order polynomial (called *Jacobi's polynomial*→**21A.6**) ($\alpha, \beta > -1$):

$$P_n^{(\alpha,\beta)}(x) = (1-x)^{-\alpha}(1+x)^{-\beta}\frac{(-1)^n}{2^n n!}\left(\frac{d}{dx}\right)^n \{(1-x)^{n+\alpha}(1+x)^{n+\beta}\}. \tag{2.11}$$

where $\alpha, \beta \in \boldsymbol{R}$.

In particular, $T_n(x) \equiv ((2n)!!/(2n-1)!!)P_n^{(-1/2,-1/2)}(x)$ are called the Chebychev polynomials (→**21B.9**), and $P_n^{(0,0)}(x) \equiv P_n(x)$ are called the Legendre polynomials (→**21B.2**).
(2)

$$H_n(x) = (-1)^n e^{x^2}\left(\frac{d}{dx}\right)^n e^{-x^2} \tag{2.12}$$

is an n-th order polynomial called the Hermite polynomial (→**21B.6**).
(3) $T_n(x) = \cos(n \arccos x)$ (Chebychev's polynomial) satisfies (→**21B.9**)

$$(1-x^2)\frac{d^2u}{dx^2} - x\frac{du}{dx} + n^2 u = 0. \tag{2.13}$$

(4) Laguerr's polynomial

$$L_n^{(\alpha)}(x) = \sum_{r=0}^{n}(-1)^r\binom{n+\alpha}{n-r}\frac{x^r}{r!} \tag{2.14}$$

satisfies

$$x\frac{d^2u}{dx^2} + (\alpha + 1 - x)\frac{du}{dx} + nu = 0. \tag{2.15}$$

(E) The following is a collection of standard special functions. They will not be stressed in the notes, but the reader should have enough analytical muscle to confirmed the relations in the following.

Γ is the Gamma function (→**9**), but we only need

$$\Gamma(x+1) = x\Gamma(x) \tag{2.16}$$

for positive real x (→**9.2**).

(1) Show that

$$J_m(x) = \left(\frac{x}{2}\right)^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(m+k+1)} \left(\frac{x}{2}\right)^{2k} \tag{2.17}$$

satisfies

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + \left(1 - \frac{m^2}{x^2}\right) u = 0. \tag{2.18}$$

J_m is called the Bessel function of order m (→**27A.1**).[66]

(2) Show that

$$I_\nu(x) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n!\Gamma(\nu+n+1)} \tag{2.19}$$

satisfies

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} - \left(1 + \frac{\nu^2}{x^2}\right) u = 0. \tag{2.20}$$

I_ν is called the *modified Bessel function* (of the first kind) (→**27A.23**).

(3) Demonstrate that

$$j_n(x) \equiv \sqrt{\frac{\pi}{2z}} J_{n+1/2}(x), \tag{2.21}$$

which is called the *spherical Bessel function* (→**27A.25**), satisfies

$$\frac{d^2u}{dx^2} + \frac{2}{x}\frac{du}{dx} + \left(1 - \frac{n(n+1)}{x^2}\right) u = 0. \tag{2.22}$$

(4) Show that Whitaker's function

$$M_{\kappa,\mu}(x) = x^{\mu+1/2} e^{-x/2} \sum_{n=0}^{\infty} \frac{\Gamma(2\mu+1)\Gamma(\mu-\kappa+n+1/2)x^n}{\Gamma(2\mu+n+1)\Gamma(\mu-\kappa+1/2)n!} \tag{2.23}$$

satisfies

$$\frac{d^2u}{dx^2} + \left(-\frac{1}{4} + \frac{\kappa}{x} - \frac{\mu^2 - (1/4)}{x^2}\right) u = 0. \tag{2.24}$$

(5) Show that Kummer's confluent hypergeometric function

$$F(\alpha,\gamma,x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n z^n}{(\gamma)_n n!} \tag{2.25}$$

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[66] Here, m can be any integer: if $m < 0$, terms with $m + k + 1 \leq 0$ are ignored. See **27A.2**.

satisfies

$$x\frac{d^2u}{dx^2} + (\gamma - x)\frac{du}{dx} - \alpha u = 0. \tag{2.26}$$

Here $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$, etc. with $(\alpha)_0 = 1$.
(6) Demonstrate that (cf. **27A.28**)

$$u(x) = J_\nu(e^x) \tag{2.27}$$

satisfies

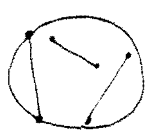
$$\frac{d^2u}{dx^2} + (e^{2x} - \nu^2)u = 0. \tag{2.28}$$

[Hint: see Exercise (1) above or **27A.1**.]
(7) The following Kelvin's function

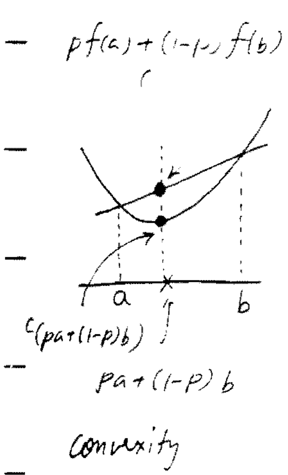
$$\mathrm{ber}\, x = \sum_{n=0}^{\infty} \frac{(-1)^n}{((2n)!)^2}\left(\frac{x}{2}\right)^{4n} \tag{2.29}$$

satisfies

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} - iu = 0. \tag{2.30}$$



Discussion.
(A) A **convex function** is a function such that the set $\{(x,y) : y \geq f(x)\}$ is a convex set.[67] A convex function must be a continuous function. The concept of convex function is very important in physics, esp., in statistical physics. In 1873 Gibbs characterized the family of equilibrium states of a system which is compatible with thermodynamics as follows (in modern words): The totality of equilibrium states of a simple fluid is a once differentiable manifold, which is the graph of a convex function U (internal energy) of S (entropy) and V (volume).[68]
(1) If f is convex between a and b, then



$$f(pa + (1-p)b) \leq pf(a) + (1-p)f(b) \tag{2.31}$$

for any $p \in [0,1]$. This property can be used to define the convexity. This is a simple case of *Jensen's inequality*: f is convex if and only if

$$f\left(\sum_i \lambda_i x_i\right) \leq \sum_i \lambda_i f(x_i), \tag{2.32}$$

where $\sum_i \lambda_i = 1$ and $\lambda_i \geq 0$. (2) Show that e^x, $-\log x$, x^q ($q \geq 1$) are convex.
(3) A periodic convex function is a constant.
(4) Let f and g be convex. Then, $g \circ f$ is convex.[69] [Note that f and g need not be

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[67] A set A is a convex set if for any $x, y \in A$ the segment connecting x and y is inside A.

[68] R. T. Rockafeller, *Convex Analysis* (Princeton, UP, 1970) is the standard reference of the topic. Its use in statistical physics is explained in the introduction by A. S. Wightman in R. B. Israel, *Convexity in the Theory of Lattice Gases* (Princeton UP, 1979).

[69] $(g \circ f)(x) \equiv g(f(x))$, i.e., the compositin is denoted by $\circ$.

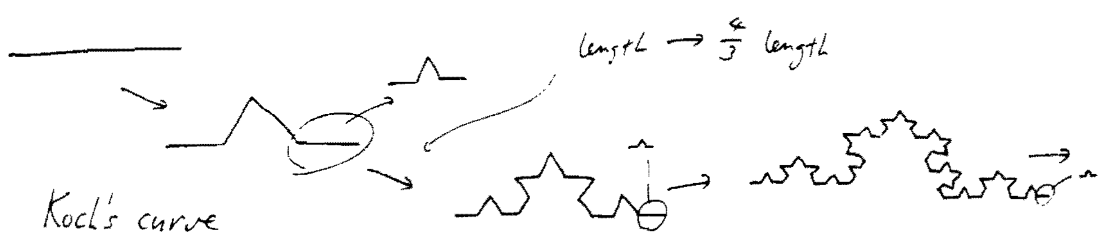
differentiable.]
(5) For $x > 0$ if $xf(x)$ is convex, so is $f(1/x)$. This is obvious, if $f''(x)$ exists. Is this true even if f is less smooth?
(B) **Pathological continuous functions.**
(1) **Weierstrass function.** The first example of nowhere differentiable continuous function was given by Weierstrass (→**13.3b**). An example is[70]

$$f(t) = \sum_{r=0}^{\infty} \frac{1}{r!} \sin((r!)^2 t). \tag{2.33}$$

The convergence is uniform, so that the limit must be a continuous function. To prove the nondifferentiability at any point, a detailed estimate is needed. See Körner Section 11. The lesson we should learn from such functions is that if we differentiate a function repeatedly many times, then we could encounter bizarre functions, because differentiation magnifies details (and generally reduces differentiability).
(2) **Takagi function.** Let $D(x)$ be the distance between x and the closest integer to it (That is, $D(x) = dist(x, \boldsymbol{Z})$).
(i) Illustrate the graph of $D(x)$.
(ii) Define

$$T(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} D(2^n x). \tag{2.34}$$

This is called the *Takagi* function, which is continuous, but nowhere differentiable. The function has self-similarity.[71]
Is any curve (except lines) which is self-similar nowhere differentiable?
(3) **Koch curves.** Many beautiful examples of bizarre curves can be found in B B Mandelbrot, *The Fractal Geometry of Nature* (Freeman, 1983). A simple example of nowhere differentiable (and consequently without length) curves is a Koch curve constructed by a self-similar substitution as illustrated below.



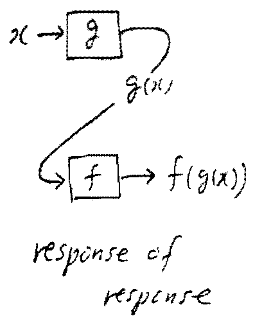
2A.2 Chain rule. Suppose the input to a system g (henceforth, a

[70] The original Weierstrass' functions are

$$f(t) = \sum_{r} \frac{1}{a^r} \sin b^r t$$

with b being an integer and b/a and a sufficiently large.
[71] (To the instructor) Its Hausdorff dimension is 1.

system and its response function are denoted by the same symbol) is x and we feed the output of g into another system f. Then the linear response of f to a small change in x must be the linear response of f to the 'linear response of g to the change of x.' This is the essence of the chain rule. Let $F = f \circ g$ (i.e., $F(x) = f(g(x))$). Then,



$$d(f \circ g)(x)/dx = f'(g(x))g'(x). \tag{2.35}$$

Exercise.
(1) Let F be a differentiable function, and define a sequence $\{x_n\}$ through $x_{n+1} = F(x_n)$. Compute dx_n/dx_1. In particular, if $F(x) = 2x$ for $x \in [0, 1/2]$ and $2(1-x)$ for $x \in (1/2, 1]$, then $|dx_n/dx_1| = 2^{n-1}$. This implies that x_n for large n (a long time asymptotic result) is extremely sensitive to a small change in the initial condition x_1. This is an important feature of deterministic chaos. Indeed, for this F, $\{x_n\}$ is a typical chaotic sequence.
(2) Demonstrate Leibniz' formula (→**A3.14**) with the aid of the binomial theorem.

2A.3 Linear responses can be superposed. If there are several parts to be changed by perturbation, then the overall perturbation effect is the superposition (→**1.4**) of all the responses of each part calculated as if other parts are intact. The simplest example is $d(fg)/dx = f'g + fg'$: the change of fg is the sum of the change of each part keeping the rest constant. Consider the following example:

$$\frac{d}{dt}\int_{f(t)}^{g(t)} h(x, t)dx, \tag{2.36}$$

where f, g and h are all well behaved. There are three places affected by the modification of the parameter t. Hence, the result should be the superposition of all three independent changes:

$$h(g(t), t)g'(t) - h(f(t), t)f'(t) + \int_{f(t)}^{g(t)} \frac{\partial h(x, t)}{\partial t} dx. \tag{2.37}$$

Exercise.
(A) Compute

$$\frac{d}{dt}\int_{\sin t}^{\log t} \cosh tx \, dx. \tag{2.38}$$

(B) Let f be a continuous function.
(1) Compute

$$\frac{d}{dx}\int_{-x^2/2}^{x^2/2} f(t)dt \tag{2.39}$$

(2) Find

$$\lim_{t \to 0} \frac{1}{2t}\int_{x-t}^{x+t} f(s)ds. \tag{2.40}$$

(C) Let $X(t)$ be a non-singular matrix whose elements are all differentiable. Demonstrate

$$\frac{d}{dt}det X(t) = det X(t) Tr[X(t)^{-1}X'(t)]. \tag{2.41}$$

[Hint: Use the following important relation: $det X(t) = \exp(Tr \ln X(t))$.]

2A.4 Generalization of differentiation, strong derivative. If we have a device to measure the size of the perturbation δa and the size of its effect δf (i.e., if we can reasonably say they are small (for example, if these quantities are vectors, we know how to evaluate their magnitudes[72]), then even if f is not an ordinary function and if a is not a number, we may be able to define the linear response. The relation between δf and δa should be linear. That is, if the response to $\delta_1 a$ of f is denoted by $\delta_1 f$ and that for $\delta_2 a$ $\delta_2 f$, then for any numbers α and β, the response of f to $\alpha\delta_1 a + \beta\delta_2 a$ is given by $\alpha\delta_1 f + \beta\delta_2 f$ (→**1.4**). If a relation between δa and δf satisfies this relation, we introduce a symbol Df and write

$$\delta f = Df[a]\delta a. \tag{2.42}$$

Here the dependence of Df on a is denoted by $[a]$. If such Df exists, we say f is *strongly differentiable*, and $Df[a]$ is called the *strong derivative* of f (at a). Notice that the linear operator (→**1.4**) $Df[a]$ is independent of the choice of the perturbation δa. This independence characterizes the strong differentiability.

We write $Df[a] = f'(a)$ when f is an ordinary real scalar function on a real number set $\boldsymbol{R}$ (→**2A.1**).

2A.5 Differentiation of function on space. Consider a smooth function $f : \boldsymbol{R}^3 \to \boldsymbol{R}$. Changing $\boldsymbol{r} \in \boldsymbol{R}^3$ slightly, we can study the linear response of $f(\boldsymbol{r})$, which is a scalar δf and must be a linear function of $\delta\boldsymbol{r}$. The derivative Df must be a vector such that (cf. **2A.6**)

$$\delta f \simeq Df \cdot \delta\boldsymbol{r}. \tag{2.43}$$

The vector Df is called the *gradient* of f at $\boldsymbol{r}$ (→**2C.2**).

If f is a function of x and y, we can write

$$\delta f = (f_x, f_y)(\delta x, \delta y)^T. \tag{2.44}$$

Here ^{T} denotes the transposition of the vector. Thus, we may write

$$Df = (f_x, f_y)^T. \tag{2.45}$$

2A.6 Warning. The existence of Df is much stronger than the condition for the existence of each f_x and f_y (→**2B.2**).

[72] We need a *norm* (→**3.3** footnote, **20.3**)

2A.7 Differentiation of vector valued function. Let $f = (f_1, f_2, f_3)^T$.[73] Then, $\boldsymbol{Df}$ must be a 3×3 matrix whose each row is $grad\, f_i$ ($i = x, y$, or z):

$$\boldsymbol{Df} = \frac{d\boldsymbol{f}}{d\boldsymbol{x}} = \begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} & \frac{\partial f_x}{\partial z} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} & \frac{\partial f_y}{\partial z} \\ \frac{\partial f_z}{\partial x} & \frac{\partial f_z}{\partial y} & \frac{\partial f_z}{\partial z} \end{pmatrix} \tag{2.46}$$

Componentwisely, we can write

$$(\boldsymbol{Df})_{ij} = \frac{\partial f_i}{\partial x_j} \equiv f_{i,j}. \tag{2.47}$$

This is, of course, consistent with the formal expression

$$d\boldsymbol{f} = \frac{d\boldsymbol{f}}{d\boldsymbol{x}} d\boldsymbol{x}. \tag{2.48}$$

As we will see, the trace of $\boldsymbol{Df}$ is called $div\, f$ (→**2C.6**).

Discussion: Hessian. Let $f(x_1, \cdots x_n)$ be a twice differentiable function. The matrix

$$Hess(f) = Matr.\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) \tag{2.49}$$

is called the *Hessian* of f at $(x_1, \cdots, x_n)$. If the point is an extremal point, then the Hessian determines its nature.

Exercise.
(A) Compute $\boldsymbol{Dv}$ for the following vector fields on $\boldsymbol{R}^3$:
(1) $\boldsymbol{v} = (e^y - x\cos(xz), 0, z\cos(xz))$.
(2) $\boldsymbol{v} = (y^2 \sin z, 2xy \sin z, xy^2 \cos z)$.
(B) If we superpose the two Coulomb electric fields due to point charges of $+q$ and $-q$ at the origin and at $(d, 0, 0)$, respectively, we can get the electric field created by an appropriate dipole moment. Find the matrix A such that the electric field due to the dipole moment $\boldsymbol{p}$ located at the origin is given by $A\boldsymbol{p}$.

2A.8 Differentiation of complex functions. A map from $\boldsymbol{C}$ to itself is usually called a *complex function*. If the following limit exists[74]

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}. \tag{2.50}$$

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[73] Whenever the components are written, we interpret the vectors to be column vectors.

[74] This means that the limit does not depend on how the origin is approached on the complex plane.

we say f is *differentiable* at z, or f is *holomorphic* at z. The limit is written as $f'(z)$ or df/dz and is called the *derivative* of f at z. Remember that this is a strong derivative (→**2A.4**). The condition that the limit does not depend on the direction along which the point $z+h$ reaches z is exactly the linearity requirement of the response. See **5.1-4**, **6.15**.

Exercise.
(1) Show that $f(z) = \bar{z}$ is not strongly differentiable. In complex analysis, we simply say f is not differentiable (→**5.1**).
(2) $z^n\bar{z}^m$ is strongly differentiable only when $m = 0$.

2.B Partial Differentiation Revisited

2B.1 Partial differentiation. We have already used $\partial/\partial t$, etc., in **1**. For simplicity, let $f(x, y)$ be a real-valued function defined in a region $D \subset \boldsymbol{R}^2$, and $(a, b) \in D$. If $f(x, b)$ is differentiable at a with respect to x, we say that $f(x, y)$ is *partially differentiable* with respect to x at (a, b), and the derivative is denoted by $f_x(a, b)$. More generally, if f is partial differentiable in D with respect to x, we may define $f_x(x, y)$:

$$f_x(x, y) = \lim_{h\to 0}\frac{f(x+h, y) - f(x, y)}{h} = \frac{\partial f}{\partial x} = \partial_x f.$$

If we write $z = f(x, y)$, $f_x(x, y)$ is written as $\partial z/\partial x$. $f_x(x, y)$ is called the *partial derivative* of f with respect to x. Usually, we do not explicitly write the variables kept constant (in this case y). We can analogously define $\partial f(x, y)/\partial y$.

Discussion [Hadamard's notation]. Hadamard intoduced a convenient set of notations to describe analysis of multivariable functions of $x = (x_1, \cdots, x_n)$. Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$. We write

$$|\alpha| \equiv \sum_i \alpha_i. \tag{2.51}$$

For $N = (N_1, \cdots, N_n)$,

$$x^N \equiv \prod_i x_i^{n_i}, \tag{2.52}$$

$$N! \equiv \prod_i N_i!. \tag{2.53}$$

Then, the partial differential operator is written as follows:

$$D^{\alpha} f(x) = \prod_i \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}} f(x) = \frac{\partial^{\alpha} f(x)}{\partial x^{\alpha}}. \tag{2.54}$$

(1) The multinomial theorem reads

$$|x|^n = \sum_N \frac{n!}{N!} x^N. \tag{2.55}$$

where the summation is over all N such that $|N| = n$.
(2) Taylor expansion reads

$$f(x+y) = \sum_N \frac{x^N}{N!} f^{(N)}(y). \tag{2.56}$$

Of course, $f^{(N)} \equiv D^N f$. For example,

$$e^{|x|} = \sum_N \frac{x^N}{N!}. \tag{2.57}$$

2B.2 Warning. Even if f_x and f_y exist at a point, f need not be continuous at the point.

Exercise.
(1) Make or sketch such an example.
(2) If f_{xy} is not continuous, then $f_{xy} = f_{yx}$ is not guaranteed. Compute f_{xy} and f_{yx} at the origin for

$$f(x,y) = \frac{xy(x^2 - y^2)}{x^2 + y^2} \text{ for } (x,y) \neq (0,0) \tag{2.58}$$

with $f(0,0) = 0$. If you wish, use Mathematica for this problem and report what you find.

2B.3 Change of variables. Suppose f is a well behaved function of x and t satisfying

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x^2} = 0, \tag{2.59}$$

where c is a positive constant. This is a 1D wave equation (→**1.2**, **a1D.11**). In this formula, $\partial/\partial x$ is the partial differentiation with t being kept constant, which is not explicitly written. We wish to change the variables from (x,t) to (X,Y) such that $X = x + ct$ and $Y = x - ct$. Now, Y is kept constant, when we write $\partial/\partial X$. With the aid of the chain rule

$$\frac{\partial}{\partial x} = \frac{\partial X}{\partial x}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial x}\frac{\partial}{\partial Y} = \frac{\partial}{\partial X} + \frac{\partial}{\partial Y}. \tag{2.60}$$

and

$$\frac{\partial}{\partial t} = \frac{\partial X}{\partial t}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial t}\frac{\partial}{\partial Y} = c\frac{\partial}{\partial X} - c\frac{\partial}{\partial Y}. \tag{2.61}$$

Or

$$\frac{\partial}{\partial X} = \frac{1}{2}\left\{\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right\},\quad \frac{\partial}{\partial Y} = -\frac{1}{2}\left\{\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right\}. \tag{2.62}$$

That is, we can rewrite the wave equation in the following form:

$$\frac{\partial^2 \psi}{\partial X \partial Y} = 0. \tag{2.63}$$

This implies that $\partial\psi/\partial Y$ is a function of Y alone:[75]

$$\frac{\partial \psi}{\partial Y} = \phi(Y). \tag{2.64}$$

Hence, ψ must be a sum of the function of Y only and X only. In other words, the most general solution of (2.59) is given by

$$\psi(x,t) = F(x+ct) + G(x-ct), \tag{2.65}$$

where F and G are differentiable functions. Notice that $F(x+ct)$ denotes a wave propagating in the $-x$-direction with speed c without changing its shape. We have found a general solution to the wave equation:

2B.4 D'Alembert's solution for 1-space wave equation. Consider (2.59) on the whole 1-space $\boldsymbol{R}$ and for all time $t \in (0,+\infty)$ with the initial condition $u(x,0) = f(x)$, and $\partial_t u(x,0) = g(x)$, where f is C^2 and g is C^1.[76] Then

$$u(t,x) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds. \tag{2.66}$$

This is called *d'Alembert's formula* and is a C^2-function. This is actually the unique solution of the problem.

[Demo] From (2.65) the functions F and G in the general solution are determined as follows:

$$F(x) + G(x) = f(x), \tag{2.67}$$
$$cF'(x) - cG'(x) = g(x). \tag{2.68}$$

[75] We assume well-behavedness of functions as much as we need to avoid technical complications.

[76] C^m denotes m-times continuously differentiable functions.

From (2.68) we get

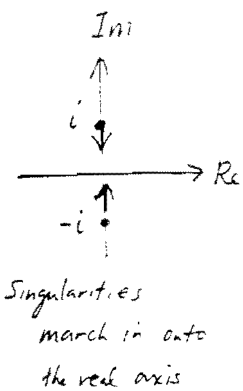
$$F(x) - G(x) = \frac{1}{c}\int_{x_0}^{x} g(\zeta)d\zeta + const, \tag{2.69}$$

where x_0 is an arbitrary base point. From (2.68) and (2.69), we can solve F and G as

$$F(x) = \frac{1}{2}\left[f(x) + \frac{1}{c}\int_{x_0}^{x} g(\zeta)d\zeta\right], \tag{2.70}$$

$$G(x) = \frac{1}{2}\left[f(x) - \frac{1}{c}\int_{x_0}^{x} g(\zeta)d\zeta\right]. \tag{2.71}$$

Here the integration constant in (2.69) is absorbed into the choice of x_0. This gives the desired formula.



Discussion.
(A) Formally apply the method to derive d'Alembert's formula to the initial value problem $u(x,0) = f(x)$ and $\partial_y u(x,0) = g(x)$ of the Laplace equation to derive

$$u(x,y) = \frac{1}{2}[f(x+iy) + f(x-iy)] - \frac{i}{2}\int_{x-iy}^{x+iy} g(s)ds \tag{2.72}$$

The formula tells us that the fate of the solution is determined by the behavior of the functions on the complex plane. For example, if $f(x) = 1/(1+x^2)$, then singularities appear in the solution which cannot be controlled by the initial condition (not well posed →**28.3**).

(B) Solve the forced 1D wave equation on $\boldsymbol{R}$

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u = Q(x,t) \tag{2.73}$$

with the initial condition $u = f(x)$ and $\partial_t u = 0$ (Use $x \pm t$) (→**30.7**).

Exercise.
(A) For a 1D wave equation, if the initial condition is nonzero only on a compact subset of $\boldsymbol{R}$, then so is the solution for any $t > 0$ (→**30.3**).
(B) All the spherically symmetric solutions to the 3-wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0 \tag{2.74}$$

in the whole space-time have the following form (see **2B.4** for Δ, esp., (2.158)):

$$u(\boldsymbol{x},t) = \frac{F(|\boldsymbol{x}| - ct) + F(|\boldsymbol{x}| + ct)}{|\boldsymbol{x}|}. \tag{2.75}$$

(C)
(1) Find the solution to the 1-space wave equation ($c = 1$) on $\boldsymbol{R}$ with the following initial data:

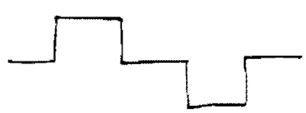
$$u_{t=0} = \cosh^{-2} x, \quad \partial_t u_{t=0} = \cosh^{-2} x \tanh x. \tag{2.76}$$

(2) Find the solution to the 1-space wave equation ($c = 1$) on $\boldsymbol{R}$ with the initial condition

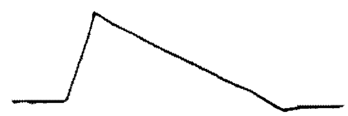
$$u_{t=0} = 0,\ \partial_t u|_{t=0} = A \operatorname{sech} x. \tag{2.77}$$

Write A in terms of the total energy (→**a1D.12** or **30.4**).
(3) Illustrate the solution of the wave equation for the following initial displacement with zero initial velocity.

(i)



(ii)



(D) Obtain the solution under the Cauchy condition given on the line $x = at$ as $u(x, x/a) = f(x)$ and $\partial_t u(x, x/a) = g(x)$. What happens if $a = \pm c$?
(E) Find the general solution to
(1)

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \sin x \cos t. \tag{2.78}$$

(2)

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \sin(x - t). \tag{2.79}$$

2B.5 Wave equation with boundary condition. Consider the *initial value problem* for 1-space wave equation on $[0, L]$

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right) u = 0. \quad x \in (0, L),\ t \in (0, \infty). \tag{2.80}$$

The initial condition is

$$u = f(x). \quad \frac{\partial u}{\partial t} = g(x). \quad \text{for } t = 0.\ x \in [0, L]. \tag{2.81}$$

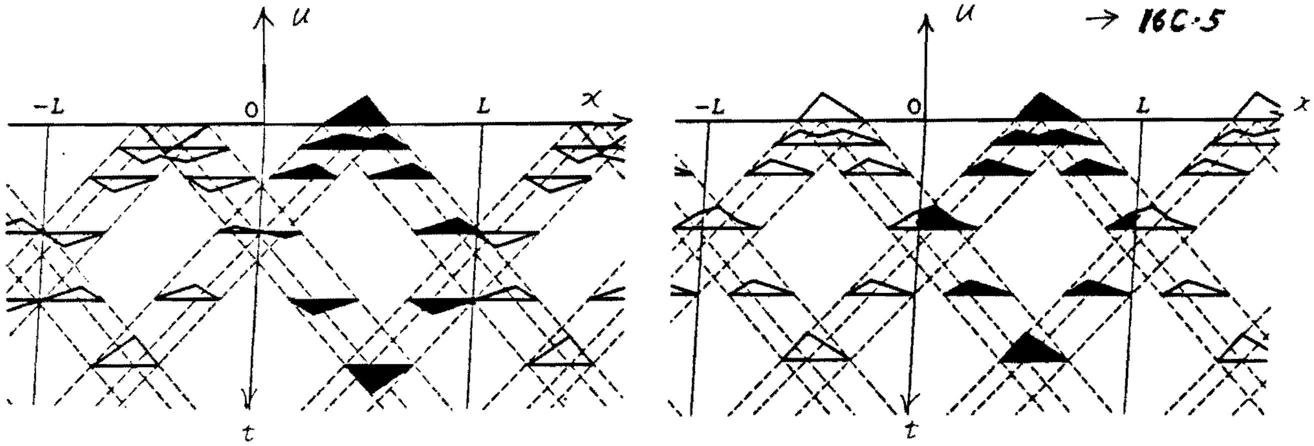
where f is a twice and g is a once differentiable function, and the boundary condition is $u = 0$ at $x = 0$ and $x = L$ for all $t > 0$. In this case the boundary condition implies from (2.65) $F(-ct) + G(ct) = 0$ and $F(L - ct) + G(L + ct) = 0$ for all $t > 0$. Thus $F(x) = -G(-x)$ and $F(x + L) = -G(-x + L)$.[77] Following **2B.4**, we get

$$u(t, x) = \frac{1}{2}\left[f(x + ct) - f(-x + ct) + \frac{1}{c}\int_{-x+ct}^{x+ct} g(\zeta)d\zeta\right]. \tag{2.82}$$

[77] $F(x) = -G(-x + 2L) = F(x + 2L)$.

We notice that $F(x) = F(x+2L)$. That is, F must be a periodic function of period $2L$. This is the source of Daniel Bernoulli's idea (→**1.6**).

See the following example. (This is an example of the method of images → 16C·5



homogeneous Dirichlet homogeneous Neumann

(from Ezawa)

2B.6 Moving coordinates. Consider the following equation

$$\frac{\partial \psi}{\partial t} + c\frac{\partial \psi}{\partial x} = D\frac{\partial^2 \psi}{\partial x^2}. \tag{2.83}$$

where c and D are positive constants. If $c = 0$, the equation is 2D diffusion equation, which should describe the relaxation of ψ back to 'equilibrium.' Let us rewrite this equation with the aid of the moving coordinate $X = x - ct$. To do so, the easiest way is to rewrite the equation in terms of t and X as the new independent variables. It is advantageous to introduce new time $T = t$ to minimize confusion. We get

$$\frac{\partial \psi}{\partial T} = D\frac{\partial^2 \psi}{\partial X^2}. \tag{2.84}$$

Thus, we understand the meaning of (2.83): it is a diffusion process advected by the flow of constant speed c to the positive x-direction.

Exercise.
(A) Rewrite the following equation with the aid of the moving coordinate $X = x - vt$, and find the general solution

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = F(x - vt), \tag{2.85}$$

where F is a well-behaved function.
(B) Consider the following (original) Fisher equation:

$$\frac{\partial v}{\partial t} = D\frac{\partial^2 v}{\partial x^2} + v(1 - v). \tag{2.86}$$

(1) Rewrite the equation as seen from the moving coordinate with velocity v.
(2) Find the equation for a steady moving front propagating with speed v.
(3) How can you show that there is such a wave front for sufficiently large v?.

2B.7 d'Alembert 1717-1783.[78] He was born as an illegitimate son of a salon hostess and a cavalry officer, abandoned on the steps of the Saint Jean-Le-Rond in Paris by his mother, but was quickly located by his father. who found him a home with a humble glazier, named Rousseau. His father. though never revealed his identity, provided an annual annuity of 1200 livres and also helped him to enroll a prestigious school. Collége de Quatre-Nations. where he developed an aversion for religions. He started his mathematical study in ca 1738. He learned mathematics largely by himself. later writing that mathematics was the only occupation really interested him.

In 1739. he started submitting papers to Paris Academy of Science. and was elected a member in 1741. In 1743. he published his most famous scientific work. *Traité de Dynamique*. in which he formulated his principle. From 1744 for three years he developed partial differential equations as a branch of calculus. inventing the wave equation (→**30**). His study of fluid dynamics is also a breakthrough (e.g.. d'Alembert's paradox). However. d'Alembert's quickly written papers were poorly understood. When Euler (→**4.4**) refined these ideas and wrote masterful expositions that did not give d'Alembert ample credit. he was furious.

After 1750. his interest turned increasingly beyond mathematics. and served as the science editor of *Encyclopédie* for seven years. but he resigned in 1758. due to his article on Genevan pastors who "no longer believe in the divinity of Jesus Christ. ⋯." He was accepted to the French Academy in 1754. He worked zealously to enhance its dignity and was made perpetual secretary in 1772. As his scientific and literary fame spread. Friedrich the Great wanted him to be the president of the Berlin Academy in 1764. d'Alembert recommended Euler for the position. This healed a rift that had developed for more than a decade. He subsequently declined the offer of Catherine the Great as well, refusing to leave the cultural capitol. Paris.

D'Alembert. though himself discouraged about the future of mathematics. helped encourage Lagrange (→**3.5**) and Laplace (→**33.3**) to launch their careers.

He stressed the importance of continuity. which led him to the considerations of limits. Almost alone in his time. he understood derivatives as ratios of limits of quotients of increments. He clearly recognized that all the complex numbers can be written as the sum of real and

[78] Mainly based on p479- of R. Calinger. *Classics of Mathematics* (Prentice-Hall. 1995). Read the original for his much more colorful private life. etc.

imaginary parts.

2.C Vector Analysis Revisited

2C.1 Gradient. Suppose we have a sufficiently smooth function $f : D \to \boldsymbol{R}$. where $D \subset \boldsymbol{R}^2$ is a region. We may imagine that $f(P)$ for $P \in D$ is the altitude of the point P on the island D. Since we assume the landscape to be sufficiently smooth, at each point on D there is a well defined direction $\boldsymbol{n}$ of the steepest ascent and the slope (magnitude) $s(\geq 0)$. That is. at each point on D, we may define the *gradient vector* $s\boldsymbol{n}$. which will be denoted by a vector $grad\, f$ (cf. —bf 2A.5).

Exercise.
(1) Compute $grad(r^{-2})$.
(2) Compute $grad\,(fg)$.

2C.2 Coordinate expression of $grad\, f$. Although $grad\, f$ is meaningful without any specific coordinate system (i.e.. the concept is coordinate-free). in actual calculations. introduction of a coordinate system is often useful. The 3-space version of gradient reads as follows. Choose a Cartesian coordinate system O-xyz.

$$grad\, f = \left(\frac{\partial f}{\partial x}. \frac{\partial f}{\partial y}. \frac{\partial f}{\partial z}\right). \tag{2.87}$$

or

$$grad\, f = \boldsymbol{i}\frac{\partial f}{\partial x} + \boldsymbol{j}\frac{\partial f}{\partial y} + \boldsymbol{k}\frac{\partial f}{\partial z}. \tag{2.88}$$

2C.3 Remark. Note that to represent $grad\, f$ (in 3-space) in terms of numbers, we need two devices: one is a coordinate system to specify the point in D with three numbers. which allow us to describe f as a function of three independent variables; the other device is the basis vectors spanning the three dimensional vector '$grad\, f$' at each point on D (i.e.. spanning the tangent space at each point of D). In principle. any choice is fine. but practically. it is wise to choose these base vectors to be parallel to the coordinate directions at each point. In the choice of **2C.2**, the coordinate system has globally the same coordinate directions at every point on D. and the basis vectors of the tangent space are chosen to be parallel to these directions, so again globally uniformly chosen. Nonuniformity of the choice of the basic vectors causes complications. We must be very careful (→**2C.7**, **2C.12** for a warning). especially when we formally use operators explained below.

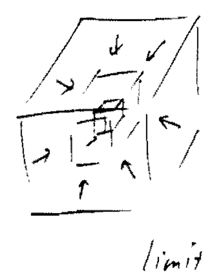
2C.4 Nabla or del. (2.88) suggests that *grad* is a map which maps f to the gradient vector at each point in its domain (if f is differentiable). We often write this linear operator (→**1.4**) as ∇, which is called *nabla*,[79] but is often read 'del' in the US. We write $grad\, f = \nabla f$. ∇ has the following expression if we use the Cartesian coordinates

$$\nabla \equiv \sum_{k=1}^{n} \boldsymbol{e}_k \frac{\partial}{\partial x_k}, \tag{2.89}$$

where x_k is the k-th coordinate and $\boldsymbol{e}_k$ is the unit directional vector in the k-th coordinate direction.

2C.5 Divergence. Suppose we have a flow field (velocity field) $\boldsymbol{u}$ on a region $D \in \boldsymbol{R}^3$. Let us consider a convex region (→**2A.1** Discussion (A)) $V \subset \boldsymbol{R}^3$ which may be imagined to be covered by *area elements* $d\boldsymbol{S}$ which can be identified with a vector whose magnitude $|d\boldsymbol{S}|$ is the area of the area element, and whose direction is parallel to the outward normal direction of the area element. Then $\boldsymbol{u} \cdot d\boldsymbol{S}$ is the rate of the volume of fluid going out through the area element in the unit time. The area integral

$$\int_{\partial V} d\boldsymbol{S} \cdot \boldsymbol{u}$$



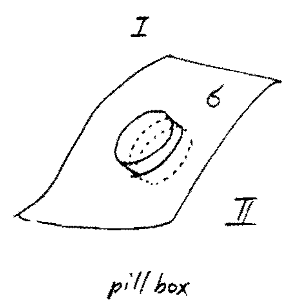
is the total amount of the volume of the fluid lost from the region V. The following limit, if exists, is called the *divergence* of the vector field $\boldsymbol{u}$ at point P and is written as $div\, \boldsymbol{u}$:

$$div\, \boldsymbol{u} \equiv \lim_{|V| \to 0} \frac{\int_{\partial V} \boldsymbol{u} \cdot d\boldsymbol{S}}{|V|}. \tag{2.90}$$

where the limit is taken over a nested sequence of convex volumes[80] converging to a unique point P. $div\, \boldsymbol{u}$ is the rate of loss of the quantity carried by the flow field $\boldsymbol{u}$ per unit volume (i.e., the loss rate density).

Discussion

The electric displacement satisfies $div\, \boldsymbol{D} = \rho$ (→**a1F.10**), where ρ is the charge density. At the boundary of two media I and II is a surface charge of density σ. Let $\boldsymbol{n}$ be the unit normal vector of the interface pointing from I to II. Show



$$(\boldsymbol{D}_I - \boldsymbol{D}_{II}) \cdot \boldsymbol{n} = \sigma. \tag{2.91}$$

See Section 1.5 of Jackson, *Classical Electrodynamics* (Wiley, 1975) for similar examples.

[79] 'Nabla' is a kind of harp (Assyrian harp).

[80] with piecewise smooth surfaces

2C.6 Cartesian expression of *div.* From (2.90) assuming the existence of the limit, we may easily derive the Cartesian expression for *div*. Choose as V a tiny cube whose surfaces are perpendicular to the Cartesian coordinates of O-xyz. We immediately get

$$div\,\boldsymbol{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}. \tag{2.92}$$

div has the following coordinate-free definition:

$$div\,\boldsymbol{u} = Tr\left(\frac{d\boldsymbol{u}}{d\boldsymbol{x}}\right). \tag{2.93}$$

Exercise. Find the vector potential (→**2C.16**) for $\boldsymbol{r}/r^3$, if any.

2C.7 Operator *div.* (2.92) again suggests that *div* is a linear operator (→**1.4**) which maps a vector field to a scalar field. Comparing (2.89) and (2.92) allows us to write

$$div\,\boldsymbol{u} = \nabla \cdot \boldsymbol{u}.$$

This 'abuse' of nabla is allowed only in the Cartesian coordinates (why? →**2C.3**). Generalization to n-space is straightforward.

Exercise. Compute $div(\boldsymbol{r}/r)$.

2C.8 Curl. Let $\boldsymbol{u}$ be a vector field as in **2C.5**. Take a singly connected[81] compact surface S in $\boldsymbol{R}^3$ whose boundary is smooth. The boundary closed curve with the orientation according to the right-hand rule is denoted by ∂S (see Fig.). Consider the following line integral along ∂S:

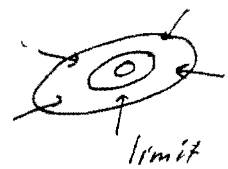
$$\int_{\partial S} \boldsymbol{u} \cdot d\boldsymbol{l}.$$

where $d\boldsymbol{l}$ is the line element along the boundary curve. Let us imagine a straight vortex line and take S to be a disc perpendicular to the line such that its center is on the line. Immediately we see that this integral is the strength of the vortex whose center (singular point) goes through S. Therefore, the following limit, if exists, describes the 'area' density of the $\boldsymbol{n}$-component of the vortex (as in the case of angular velocity,

[81] A region is *singly connected*, if, for any given pair of points in the region, any two curves connecting them are homotopic. That is, they can be smoothly deformed into each other without going out the region.

the direction of vortex is the direction of the axis of rotation with the right-hand rule):

$$\boldsymbol{n}\cdot curl\,\boldsymbol{u} = \lim_{|S|\to 0}\frac{\int_{\partial S}\boldsymbol{u}\cdot d\boldsymbol{l}}{|S|}, \tag{2.94}$$



where the limit is over the sequence of smooth surfaces which converges to point P with its orientation in the $\boldsymbol{n}$-direction. If the limit exists, then obviously there is a vector $curl\,\boldsymbol{u}$ called *curl* of the vector field $\boldsymbol{u}$.

2C.9 Cartesian expression of curl. If we assume the existence of the limit (2.94), we can easily derive the Cartesian expression for $curl\,\boldsymbol{u}$. We have

$$curl\,\boldsymbol{u} = \left(\frac{\partial u_z}{\partial y}-\frac{\partial u_y}{\partial z}, \frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}, \frac{\partial u_x}{\partial y}-\frac{\partial u_y}{\partial x}\right). \tag{2.95}$$

or

$$curl\,\boldsymbol{u} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ u_x & u_y & u_z \end{vmatrix} = \nabla\times\boldsymbol{u}. \tag{2.96}$$

This 'abuse' of the nabla symbol is admissible only with the Cartesian coordinates (→**2C.3**).

Componentwisely, we can write (with the summation convention)

$$(curl\,\boldsymbol{u})_i = \epsilon_{ijk}\partial_j u_k. \tag{2.97}$$

where ϵ_{ijk} is defined as $\epsilon_{123}=1$ and $\epsilon_{ijk}=sgn(ijk)$, where $sgn(ijk)$ is the sign of the permutation: if (ijk) is obtained from (123) with even number of exchanges of symbols, it is $+1$, and otherwise -1.[82] Notice that

$$(\boldsymbol{a}\times\boldsymbol{b})_i = \epsilon_{ijk}a_j b_k. \tag{2.98}$$

A useful formula is

$$\epsilon_{ijk}\epsilon_{abk} = \delta_{ia}\delta_{jb}-\delta_{ib}\delta_{ja}. \tag{2.99}$$

(The summation convention is implied.)

Exercise.
(1) Let $\boldsymbol{v} = (x^2z, -xy^3z^2, xy^2z)$. Compute $div\,\boldsymbol{v}$ and $curl\,\boldsymbol{v}$.
(2) Show

$$div(f\boldsymbol{v}) = grad\,f\cdot\boldsymbol{v} + f div\,\boldsymbol{v}. \tag{2.100}$$
$$curl(f\boldsymbol{v}) = grad f\times\boldsymbol{v} + f curl\boldsymbol{v}. \tag{2.101}$$

[82] e.g., $(213)=-1$, and $(312)=+1$.

(3) Compute

$$curl\,(\mu \times r/r^3)\,. \tag{2.102}$$

(4) Show

$$curl\,curl\,u = grad\,div u - \sum \frac{\partial^2}{\partial x_i^2} u.(??) \tag{2.103}$$

See **2C.12** about the Laplacian applied on a vector.

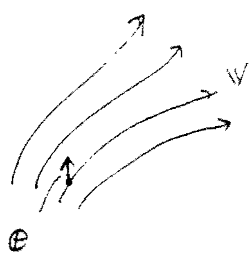
The coordinate free definition can be written as follows. Compute the strong derivative du/dx (→**2A.7**) . Denote its skew symmetric part[83] as $(du/dx)_-$.[84] Then

$$\left(\frac{du}{dx}\right)_- v = \frac{1}{2} curl\, u \times v, \tag{2.104}$$

where v is an arbitrary 3-vector. See **a1D.6**.

Discussion

Let us study the motion of a small vector e near the origin flowing with a flow field specified by v. We have



$$\frac{dr}{dt} = v. \tag{2.105}$$

If e is small, its deformation is governed by

$$\frac{de}{dt} = v(e) - v(0) = \left(\frac{dv}{dr}\right)_0 e. \tag{2.106}$$

where the (strong) derivative of the velocity field is evaluate at the origin. For a very small time δt, we can solve this equation as

$$e(\delta t) = \left(1 + \delta t \left(\frac{dv}{dr}\right)_0\right) e(0). \tag{2.107}$$

We can separate the velocity derivative into the symmetric (+) and anti (or skew) symmetric part (−) as

$$\left(\frac{dv}{dr}\right)_0 = \left(\frac{dv}{dr}\right)_+ + \left(\frac{dv}{dr}\right)_-, \tag{2.108}$$

where

$$\left(\frac{dv}{dr}\right)_\pm \equiv \frac{1}{2}\left[\left(\frac{dv}{dr}\right)_0 \pm \left(\frac{dv}{dr}\right)_0^T\right]. \tag{2.109}$$

---

[83] Let A be a square matrix. $2A_- \equiv A - A^T$ is called its skew symmetric part.

[84]

$$\left(\frac{du}{dx}\right)_- = \begin{pmatrix} 0 & \partial_y u_x - \partial_x u_y & \partial_z u_x - \partial_x u_z \\ \partial_x u_y - \partial_y u_x & 0 & \partial_z u_y - \partial_y u_z \\ \partial_x u_z - \partial_z u_x & \partial_y u_z - \partial_z u_y & 0 \end{pmatrix}.$$

Ignoring higher order terms, we can rewrite (2.107) as

$$e(\delta t) = \left(1 + \delta t \left(\frac{dv}{dr}\right)_+\right)\left(1 + \delta t \left(\frac{dv}{dr}\right)_-\right) e(0). \tag{2.110}$$

This tells us that we may separately study the effects of the symmetric and of the skew symmetric parts.

(1) Demonstrate that the symmetric part changes the volume of a (small) cube C spanned by e_x, e_y, and e_z. The changing rate of the volume is given by *div* v (→**2C.6**).

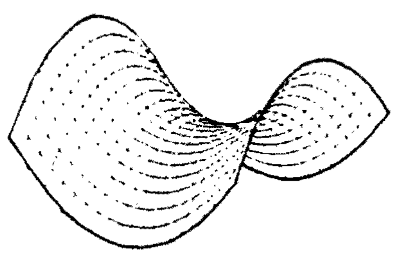
(2) Demonstrate that the skew part does not change the volume of the cube C. It rotates the cube with the angular velocity *curl* $v/2$. This is (2.104) above.

2C.10 Potential field, potential, solenoidal field, irrotational field. If a vector field u allows an expression $u = grad\,\phi$, then the field is called a *potential field* and ϕ is called its *potential.* A field without divergence, *div* $u = 0$, is called a divergenceless or *solenoidal field.* The field without curl, *curl* $u = 0$, is called an *irrotational field.*

2C.11 Laplacian, harmonic function. The operator Δ defined by

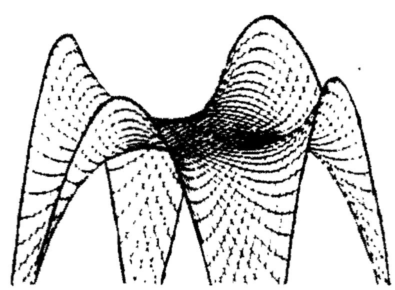
$$\Delta f \equiv div\, grad\, f \tag{2.111}$$

is called the *Laplacian*, and is often written as ∇^2. Δ is defined for a scalar function. A function f satisfying $\Delta f = 0$ in a region D is called a *harmonic function* in D (→**5.6**). According to our understanding of the Laplacian (→**1.13**) a harmonic function is a function which is invariant under the spatial moving average (→**29.4-5**). Hence, intuitively, no local extrema should exist. Graphs of harmonic functions.



(a)

$u = x^2 - y^2$



(b)

$u = x^4 - 6x^2y^2 + y^4$

2C.12 Laplacian for vector fields. If we formally calculate *curl curl* u

in the Cartesian coordinates. then we have (→(??)

$$curl\, curl\, \boldsymbol{u} = grad\, div\, \boldsymbol{u} - \nabla^2 \boldsymbol{u}.$$

Since the formal calculation treating ∇ as a vector is legitimate only in the Cartesian coordinate system (cf. **2C.3**), this calculation is meaningful only in the Cartesian system. In particular, $\nabla^2 \boldsymbol{u} = (\Delta u_x, \Delta u_y, \Delta u_z)$ is meaningful only in this coordinate system. However, the other two terms in the above equality are coordinate-free. Hence, we *define* $\Delta \boldsymbol{u}$ as

$$\Delta \boldsymbol{u} \equiv grad\, div\, \boldsymbol{u} - curl\, curl\, \boldsymbol{u}. \tag{2.112}$$

2C.13 Theorem [Gauss-Stokes-Green's theorem]. From our definitions of divergence and curl (→**2C.5**, **2C.8**), we have[85]

(1) Gauss' theorem.

$$\int_{\partial V} \boldsymbol{u} \cdot d\boldsymbol{S} = \int_V div\, \boldsymbol{u}\, d\tau, \tag{2.113}$$

where V is a region in the 3-space and $d\tau$ is the volume element.
(2) Stokes' theorem.[86]

$$\int_{\partial S} \boldsymbol{u} \cdot d\boldsymbol{l} = \int_S curl\, \boldsymbol{u} \cdot d\boldsymbol{S}. \tag{2.114}$$

where S is a compact surface in 3-space.
(3) In 2-space. Stokes' theorem reduces to *Green's theorem* (→**6.8**)

$$\int_{\partial D} (udx + vdy) = \int_D \left(-\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) dxdy. \tag{2.115}$$

where u and v are differentiable functions of x and y.

Exercise.[87]
(A)
(1) Let $S = \{(x, y, z) \,|\, 4x^2 + y^2 + z = 1, -3 \le z\}$ and $\boldsymbol{v}(x, y, z) = (3xy + 7y + x, y, z + 3)$. What is $\int_S \boldsymbol{v} \cdot d\boldsymbol{S}$?
(2) Let $S = \{(x, y, z) \,|\, x^2 + y^2 + 4z^6 = 4, 0 \le z\}$ and $\boldsymbol{v}(x, y, z) = (e^y, z, x^2)$. What is $\int_S \boldsymbol{v} \cdot d\boldsymbol{S}$?
(3) Compute

$$\int_S \boldsymbol{v} \cdot d\boldsymbol{S} \tag{2.116}$$

[85] Here the boundaries ∂V, ∂S and ∂D below must be sufficiently smooth, and the vector field must be (piecewise) C^1.

[86] George Gabriel Stokes, 1819-1903.

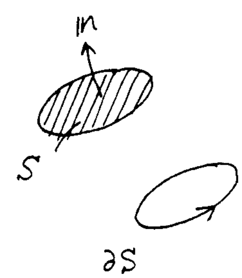
[87] From K Fukaya, *Electromagnetic Fields and Vector Analysis* (Iwanami, 1995), p98.

for $S = \{(x,y,z) \mid x^2 + y^2 + 4z^6 = 4, 0 \le z\}$ and $\boldsymbol{v}(x,y,z) = (e^y, z, x^2 + \cos y)$ (this is not a misprint).
(B) Prove Green's formula (??) (→**16A.19**).

2C.14 Poincaré's lemma.[88]

(1) $div\, curl \boldsymbol{A} = 0$,
(2) $curl\, grad\, \phi = 0$.
[Demo] Let V be a compact region of $\boldsymbol{R}^3$ whose boundary ∂V is sufficiently smooth. Notice that $\partial^2 V = \emptyset$. With the aid of the Gauss-Stokes-Green theorem (→**2C.13**), we have

$$\int_V d\tau\, div\, curl\, \boldsymbol{A} = \int_{\partial V} curl\, \boldsymbol{A} \cdot d\boldsymbol{S} = \int_{\partial^2 V} \boldsymbol{A} \cdot d\boldsymbol{\ell} = 0. \tag{2.117}$$



To demonstrate (2), take a surface S whose boundary ∂S is sufficiently smooth. Then, Stokes' theorem and the definition of *grad* tell us

$$\int_S curl\, grad\, \phi \cdot d\boldsymbol{S} = \int_{\partial S} grad\, \phi \cdot d\boldsymbol{\ell} = 0. \tag{2.118}$$

2C.15 Remark: differential forms. Notice that these relations are due to the topologically trivial fact that the boundary of a boundary is an empty set ($\partial^2 V = \emptyset$). These are examples of the general formula $d^2\omega = 0$, where ω is a *differential form*. I. M. Singer and J. A. Thorpe, *Lecture Notes on Elementary Topology and Geometry* (Scott, Foresman and Company, 1967) is strongly recommended. B. Schutz, *Geometrical Methods of Mathematical Physics* (Cambridge UP, 1980) is less modern, but may still be good for physicists who are not interested in elegance and depth of mathematical ideas. The Gauss-Stokes-Green theorem has the following unified expression

$$\int_M d\omega = \int_{\partial M} \omega. \tag{2.119}$$

where M is a n-manifold (which must be sufficiently smooth), and ω is a differential form. Notice that this is a natural extension of the fundamental theorem of calculus:

$$\int_{[a,b]} df = f(b) - f(a) (= \int_{[a],[b]} f). \tag{2.120}$$

Poincaré's lemma $d^2\omega = 0$ follows from $\partial^2 M = \emptyset$. d and ∂ are, in a certain sense, dual (Good symbols reveal deep relations. This duality is the duality between cohomology and homology. The references cited above will tell the reader about this a bit.).

2C.16 Converse of Poincaré's lemma holds.

[88]Henri Poincaré, 1854-1912.

(1) If a vector field $\boldsymbol{F}$ is *irrotational* (i.e., $curl\,\boldsymbol{F} = 0$) in a singly connected region, there is a potential function ϕ such that $\boldsymbol{F} = grad\,\phi$.
(2) If a vector field $\boldsymbol{F}$ is *solenoidal* (i.e., $div\,\boldsymbol{F} = 0$) in a singly connected region, then there is a vector field (called a *vector potential*) $\boldsymbol{A}$ such that $\boldsymbol{F} = curl\,\boldsymbol{A}$. □
(1) can be demonstrated easily by calculation. Do not overlook the importance of the shape of the region. In the language of differential forms, the converse of Poincaré's lemma can be written as: if $d\omega = 0 \Rightarrow$ there is a differential form ϕ such that $\omega = d\phi$
[Demo of (1)] Define

$$\phi(x) \equiv \int_0^1 dt\boldsymbol{F}(t\boldsymbol{x} + (1-t)\boldsymbol{x}_0)\cdot(\boldsymbol{x} - \boldsymbol{x}_0). \tag{2.121}$$

The assumption of (1) implies that for any closed curve C in the region $\int_C \boldsymbol{F}\cdot d\boldsymbol{\ell} = 0$. That is, the line integral of $\boldsymbol{F}$ along a smooth curve in the region D connecting two points $\boldsymbol{x}_0 \in D$ and $\boldsymbol{x} \in D$ does not depend on smooth paths connecting these two points. Hence ϕ is a well-defined function of $\boldsymbol{x}$. Check that actually $grad\,\phi = \boldsymbol{F}$.
Perhaps the clearest way to demonstrate (2) is to use the Helmholtz-Hodge theorem **2C.17** below. The condition of (2) with the aid of **2C.14** implies that $\boldsymbol{F}$ can be written as (2.124) with $\Delta\phi = 0$ such that $\phi \to 0$ at infinity. We will see later that only $\phi \equiv 0$ satisfies this condition (→Liouville's theorem **29.13**).

Exercise
(A)
(1) Show that the following 3-vector field has a vector potential and construct it.

$$\boldsymbol{v} = (e^y - x\cos(xz), 0, z\cos(xz)). \tag{2.122}$$

(2) Show that the following 3-vector field has a scalar potential and find it

$$\boldsymbol{v} = (y^2 \sin z, 2xy\sin z, xy^2\cos z). \tag{2.123}$$

(3) Find the vector potential of $\boldsymbol{v} = (-y/(x^2+y^2), x/(x^2+y^2), 0)$.
(4) Find a potential for $\boldsymbol{v} = f(r)\boldsymbol{r}$.
(B) Construct an example of an irrotational vector field on an appropriate domain which does not have any scalar potential.

2C.17 Theorem [Helmholtz-Hodge]. Let $\boldsymbol{F}$ be a vector field which is once differentiable, and its first order derivatives vanish at infinity. Then, there is a scalar field ϕ and a solenoidal (i.e., $div\,\boldsymbol{A} = 0$) vector field $\boldsymbol{A}$ such that[89]

$$\boldsymbol{F} = grad\,\phi + curl\,\boldsymbol{A}. \tag{2.124}$$

□
This can be rewritten with the aid of **2C.14** as

[89] We need a condition to control the 'size' of $\boldsymbol{F}$ near infinity: For example, $|\boldsymbol{F}| \sim 1/r^2$ is a good condition. Such a condition is needed because we must solve the Poisson equation to find ϕ and $\boldsymbol{A}$ (cf. **26B.4**).

2C.18 Theorem [Helmholtz-Stokes-Blumental]. Let $\boldsymbol{F}$ be a vector field which is once differentiable, and its first order derivatives vanish at infinity. Then, there is the following decomposition of $\boldsymbol{F}$:

$$\boldsymbol{F} = \boldsymbol{U} + \boldsymbol{V}, \quad curl\, \boldsymbol{U} = 0, \; div\, \boldsymbol{V} = 0. \tag{2.125}$$

□

Discussion.
Check the following formal result: Let

$$\phi(\boldsymbol{r}) = -\frac{1}{4\pi}\int_V \frac{div\boldsymbol{F}}{r} d\tau. \tag{2.126}$$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{4\pi}\int_V \frac{curl\boldsymbol{F}}{r} d\tau. \tag{2.127}$$

where r is the distance between the volume element $d\tau$ and $\boldsymbol{r}$. Furthermore, V is the finite domain containing the supports of $curl\boldsymbol{F}$ and $div\boldsymbol{F}$. Then

$$\boldsymbol{F} = grad\, \phi + curl\boldsymbol{A}. \tag{2.128}$$

2C.19 Formulas of vector calculus.
(1) $grad\, \boldsymbol{A}\cdot\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} + (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + \boldsymbol{B}\times curl\,\boldsymbol{A} + \boldsymbol{A}\times curl\,\boldsymbol{B}$.
(2) $div\,(\boldsymbol{A}\times\boldsymbol{B}) = curl\,\boldsymbol{A}\cdot\boldsymbol{B} - curl\,\boldsymbol{B}\cdot\boldsymbol{A}$.
(3) $curl(\boldsymbol{A}\times\boldsymbol{B}) = (div\,\boldsymbol{B})\boldsymbol{A} - (div\,\boldsymbol{A})\boldsymbol{B} + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{B}$.
In particular. $curl(\boldsymbol{A}\times\boldsymbol{r}/2) = \boldsymbol{A}$. if $\boldsymbol{A}$ is constant.
(4) $(\boldsymbol{C}\cdot\nabla)(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{A}\times(\boldsymbol{C}\cdot\nabla)\boldsymbol{B} - \boldsymbol{B}\times(\boldsymbol{C}\cdot\nabla)\boldsymbol{A}$.
(5) $\boldsymbol{C}\cdot grad(\boldsymbol{A}\cdot\boldsymbol{B}) = \boldsymbol{A}\cdot(\boldsymbol{C}\cdot\nabla)\boldsymbol{B} + \boldsymbol{B}\cdot(\boldsymbol{C}\cdot\nabla)\boldsymbol{A}$.
(6) $div(grad\, f \times grad\, g) = 0$.

Exercise.
Demonstrate all the formulas. In $curl(\boldsymbol{A}\times\boldsymbol{r}/2) = \boldsymbol{A}$. $\boldsymbol{A}$ must be constant. If not, what is the result? [Perhaps. the componentwise demonstration like (2.104) is the easiest.]

2.D Curvilinear Coordinates

2D.1 Curvilinear coordinates, metric tensor. The role of a coordinate system in 3-space is to assign uniquely a numerical vector $(q^1.q^2.q^3)$ to each point in $\boldsymbol{R}^3$. Thus the Cartesian coordinates of the point x^1, x^2, x^3 are unique functions of (q^1, q^2, q^3). Let $(q^1 + dq^1, q^2 + dq^2, q^3 + dq^3)$ be a point an infinitesimal distance away from (q^1, q^2, q^3). The distance between these two points ds can be written as the following quadratic form:

$$ds^2 = \sum_{i,j} g_{ij} dq^i dq^j, \tag{2.129}$$

where

$$g_{ij} \equiv \sum_k \frac{\partial x^k}{\partial q^i} \frac{\partial x^k}{\partial q^j}, \tag{2.130}$$

which is called the *metric tensor.*

2D.2 Riemann geometry. The Riemann geometry (→**7.15**) is the geometry determined by the metric tensor. M. Spivac, *Comprehensive Introduction to Differential Geometry* (Publish or Perish, Inc., Berkeley. 1979). vol. II. Chapter 4 contains Riemann's epoch-making inaugural lecture (English translation) with a detailed mathematical paraphrase of the lecture. "What did Riemann say?". According to Dedekind.[90] Gauss (→**6.17**) sat at the lecture which surpassed all his expectations. in the greatest astonishment. and on the way back from the faculty meeting he spoke to Wilhelm Weber (Riemann's lifelong patron). with the greatest appreciation. and with an excitement rare for him, about the depth of the idea presented by Riemann.

Read for a nice introduction to Riemann geometry an overview by Kazdan in Bull. Amer. Math. Soc. **33**, 339 (1996).

2D.3 Orthogonal curvilinear coordinate system. At each point (q^1, q^2, q^3). call the direction of the tangent to the i-th coordinate the i-th coordinate direction at (q^1, q^2, q^3) [e.g.. the direction of the tangent to the second coordinate is the direction parallel to $(q^1, q^2 + dq^2, q^3) - (q^1, q^2, q^3)$]. If at every point all the coordinate directions are orthogonal to each other. we call the coordinate system an *orthogonal curvilinear coordinate system.* In this case. the metric tensor is always diagonal at every point:

$$g_{ij} = \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix}. \tag{2.131}$$

where

$$h_i = \sqrt{\sum_k \left(\frac{\partial x_k}{\partial q^i}\right)^2}. \tag{2.132}$$

[90] Julius Wilhelm Richard Dedekind. 1831-1916.

2D.4 Cylindrical coordinates. $(q^1, q^2, q^3) = (r, \varphi, z)$, and

$$\begin{aligned} x &= r\cos\varphi, \\ y &= r\sin\varphi, \\ z &= z. \end{aligned} \tag{2.133}$$

From (2.132) we have $h_1 = 1$, $h_2 = r$, and $h_3 = 1$.

2D.5 Spherical coordinates. $(q^1, q^2, q^3) = (r, \theta, \varphi)$, and

$$\begin{aligned} x &= r\sin\theta\cos\varphi, \\ y &= r\sin\theta\sin\varphi, \\ z &= r\cos\theta. \end{aligned} \tag{2.134}$$

From (2.132) we have $h_1 = 1$, $h_2 = r$, and $h_3 = r\sin\theta$.

2D.6 Elliptic cylindrical coordinates.[91] $(q^1, q^2, q^3) = (\xi, \eta, \varphi)$, and for some positive real c

F

$$\begin{aligned} x &= c\sqrt{(\xi^2 - 1)(1 - \eta^2)}\cos\varphi, \\ y &= c\sqrt{(\xi^2 - 1)(1 - \eta^2)}\sin\varphi, \\ z &= c\xi\eta. \end{aligned} \tag{2.135}$$

From (2.132) we have

$$h_1 = c\sqrt{\frac{\xi^2 - \eta^2}{\xi^2 - 1}}, \quad h_2 = c\sqrt{\frac{\xi^2 - \eta^2}{1 - \eta^2}}, \quad h_3 = c\sqrt{(\xi^2 - 1)(1 - \eta^2)}, \tag{2.136}$$

where ξ and η can also be defined as

$$\xi = \frac{r_1 + r_2}{2c}, \quad \eta = \frac{r_1 - r_2}{2c}. \tag{2.137}$$

Discussion.
(A) Compute h_i (→**2D.3**) for the toroidal coordinates (α, β, φ) where

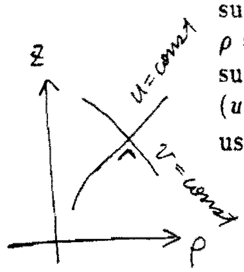
F

$$x = \frac{c\sinh\alpha\cos\varphi}{\cosh\alpha - \cosh\beta}, \quad y = \frac{c\sinh\alpha\sin\varphi}{\cosh\alpha - \cosh\beta}, \quad z = \frac{c\sin\beta}{\cosh\alpha - \cosh\beta}. \tag{2.138}$$

Here $\alpha \in [0, \infty)$, $\beta, \varphi \in (-\pi, \pi]$. What is the general shape of $\beta =$ constant surface?
(B) Introduce u and v variables that are related to ρ and z as

$$\rho = F_1(u, v), \quad z = F_2(u, v) \tag{2.139}$$

[91] This is a natural coordinate system for the Schrödinger equation for H_2^+ molecular ion.



such that u =const. and v =const. curves are orthogonal on the (ρ, z)-plane, and $\rho = 0$ is among such curves. Rotating the plane around the z-axis, we can make surfaces orthogonal to each other. Therefore, if we introduce the rotation angle φ, (u, v, φ) is a orthogonal curvilinear coordinate system in 3-space. Its relation to the usual Cartesian system is given by $\rho = \sqrt{x^2 + y^2}$ and $\varphi = \tan^{-1}(x/y)$.

$$x = F_1(u, v)\cos\varphi, \tag{2.140}$$
$$y = F_1(u, v)\sin\varphi, \tag{2.141}$$
$$z = F_2(u, v). \tag{2.142}$$

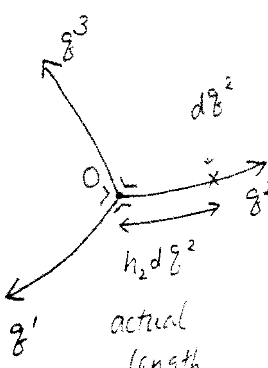
(1) For this system show that

$$h_1 = \sqrt{\left(\frac{\partial F_1}{\partial u}\right)^2 + \left(\frac{\partial F_2}{\partial u}\right)^2},\ h_2 = \sqrt{\left(\frac{\partial F_1}{\partial v}\right)^2 + \left(\frac{\partial F_2}{\partial v}\right)^2},\ h_3 = \rho. \tag{2.143}$$

(2) For elliptic cylindrical coordinates, the choice is

$$\rho = a\sqrt{(u^2 - 1)(1 - v^2)},\ \ z = auv \tag{2.144}$$

2D.7 Gradient in orthogonal curvilinear coordinates. Consider an infinitesimal cube whose apices are at $(q^1 + \theta_1 dq^1, q^2 + \theta_2 dq^2, q^3 + \theta_3 dq^3)$, where $\theta_i = 0$ or 1. The lengths of the edges of the cube are $|h_1 dq^1|$, $|h_2 dq^2|$, and $|h_3 dq^3|$. From the geometrical definition of *grad* ($\to$**2C.1**), we have



$$(grad\,\phi)_1 = \frac{1}{h_1}\frac{\partial \phi}{\partial q^1},\ (grad\,\phi)_2 = \frac{1}{h_2}\frac{\partial \phi}{\partial q^2},\ (grad\,\phi)_3 = \frac{1}{h_3}\frac{\partial \phi}{\partial q^3}. \tag{2.145}$$

Here, 1, 2 and 3 denote the components of the vector in the 1, 2 and 3 coordinate directions, respectively.

Exercise.
(A) Find the velocity and acceleration components along the coordinate directions of a particle in
(1) spherical coordinates,
(2) elliptical cylindrical coordinates.
[Hint. Find the relation between the unit vectors of the curvilinear and Cartesian coordinates.]
(B) Demonstrate

$$\frac{\partial}{\partial x} = \sin\theta\cos\varphi\frac{\partial}{\partial r} + \cos\theta\cos\varphi\frac{1}{r}\frac{\partial}{\partial \theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}, \tag{2.146}$$
$$\frac{\partial}{\partial y} = \sin\theta\sin\varphi\frac{\partial}{\partial r} + \cos\theta\sin\varphi\frac{1}{r}\frac{\partial}{\partial \theta} - \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}, \tag{2.147}$$
$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{1}{r}\frac{\partial}{\partial\theta}. \tag{2.148}$$

2D.8 Volume element in orthogonal curvilinear coordinates. From the consideration above obviously the volume element $d\tau$ is given by

$$d\tau = h_1 h_2 h_3 dq^1 dq^2 dq^3. \tag{2.149}$$

Exercise. Compute the volume element for the elliptic cylindrical coordinates.

2D.9 Divergence and curl in orthogonal curvilinear coordinates. From the geometrical definitions of these quantities (→**2C.5**, **2C.8**). we get

$$div\, \boldsymbol{A} = \frac{1}{h_1 h_2 h_2}\left[\frac{\partial}{\partial q^1}(h_2 h_3 A_1) + \frac{\partial}{\partial q^2}(h_3 h_1 A_2) + \frac{\partial}{\partial q^3}(h_1 h_2 A_3)\right], \tag{2.150}$$

$$(curl\, \boldsymbol{A})_1 = \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial q_2}(h_3 A_3) - \frac{\partial}{\partial q_3}(h_2 A_2)\right]. \tag{2.151}$$

$(curl\, \boldsymbol{A})_2$ and $(curl\, \boldsymbol{A})_3$ are obtained from (2.151) by cyclic permutations of the indices. Notice that in these formulas A_i are the actual projection of the vector $\boldsymbol{A}$ on the i-th coordinate direction.

Exercise.
(1) Compute $curl$ and div of $\boldsymbol{A} = r^2\boldsymbol{e}_r$, where $\boldsymbol{e}_r$ is the unit coordinate vector parallel to the radius in the spherical coordinates. How about if $\boldsymbol{e}_r$ is the unit vector parallel to the radius in the cylindrical coordinates?
(2) Show in the spherical coordinates that

$$curl\left(\frac{\cot\theta \boldsymbol{e}_\varphi}{r}\right) = -\frac{\boldsymbol{e}_r}{r^2}. \tag{2.152}$$

2D.10 Laplacian in orthogonal curvilinear coordinates. Combining (2.145) and (2.150). we get for the Laplacian ($\Delta \equiv div\, grad$)

$$\Delta = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\frac{h_2 h_3}{h_1}\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}\frac{h_3 h_1}{h_2}\frac{\partial}{\partial q_2} + \frac{\partial}{\partial q_3}\frac{h_1 h_2}{h_3}\frac{\partial}{\partial q_3}\right]. \tag{2.153}$$

For the cylindrical coordinates. we have

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}. \tag{2.154}$$

Notice that

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}. \tag{2.155}$$

For the spherical coordinates, we have

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}L^2 \tag{2.156}$$

with

$$L^2 \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \tag{2.157}$$

Notice that

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} = \frac{1}{r}\frac{\partial^2}{\partial r^2}r. \tag{2.158}$$

Exercise. Derive the formula for the Laplacian in the elliptic cylindrical coordinates.