

Quantum mechanics should be added.

Quote Zwillinger

APPENDIX a1 CONVENTIONAL PHYSICS EXAMPLES

Notice: This is not a course of physics, so the contents of this Appendix will never be stressed in the course. This Appendix should not be regarded as a substitute of a course on physics of continuum bodies and classical fields; it is a mere quick reference to refresh the reader's knowledge.

These Appendices give actual (physical) situations where representative PDE arise. Derivation of many PDE is considered from the point of view of the local description of conservation laws. After all, the equations of motion are expressions of the conservation of momentum, the diffusion equation is an expression of mass conservation, etc. Uniqueness of the solutions is discussed from the physics point of view. Vector analysis is freely used, so those who have some difficulties should consult **2C**. Most topics in these appendices are basic and rather dull (= conventional physics). Since the lecturer has no interest in most of these problems, they will not be discussed in this course.

a1A Balance Equation

Key words: continuum, volume element, conservation law, flux density.

a1A.1 Continuum description. Macroscopic objects are often treated as continuum bodies. This description is valid when the motion or the deformation of the body is extremely gentle at the microscopic level. Therefore, the motion must be sufficiently slow, and the deformation must be sufficiently small (i.e., the relative displacement of adjacent molecules or atoms must be much smaller than their sizes).^{33,34}

a1A.2 Volume element. The basic element of continuum description is the *volume element*, whose size is much smaller than the macroscopic representative scale

³³Consequently, in many cases in which continuum description is admissible, we may safely assume local thermodynamic equilibrium.

³⁴It should not be forgotten, however, that continuum mechanics treats macroscopic bodies as a collection of geometrical points, so the microscopic rotational degree of freedom is completely ignored. To describe microscopic degrees of freedom, the general strategy is to introduce more fields such as the field of an order parameter in statistical physics.

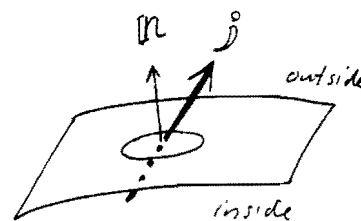
(say, the smallest scale of the macroscopic observation), but much larger than the atomic scale. Thus the volume element contains macroscopic number of atoms or molecules. The existence of a length scale l much larger than the microscopic length scale (e.g., the atom size) a and much smaller than the system size (or representative macroscopic scale) L is the prerequisite for the concept of volume element to be meaningful. This is also the condition that makes continuum descriptions meaningful.

a1A.3 Local conservation law. All the basic equations of conservation laws have the following form:

$$\text{increase in a volume element} = \text{net influx} + \text{production rate.} \quad (1.61)$$

which is generally called the *balance equation* (\rightarrow a1A.7). If a quantity under consideration is a conserved quantity, then there is no production or source term (the second term in RHS). To make the first term in RHS explicit, we need the concept of:

a1A.4 Flux density. A *flux density* of a quantity is a vector whose direction is the direction of the flow of the quantity, and whose magnitude is the rate of the quantity passing through the unit surface whose normal coincides with the flow direction. The flux density often consists of two parts, the *advective* and *conductive* parts. The advective part is the reversible (macroscopic and/or mechanical) part: something is carried by a flow, passively riding on it. Thus advection flux density = density \times flow velocity (\rightarrow a1A.6). The conductive part is due to microscopic or irreversible effects (\rightarrow a1B.1).



a1A.5 Sign convention for flux, “going out is positive.” Take a small area element³⁵ on the surface of a volume. The sign of the flux through this area element is said to be positive, if the flux density vector j points outward through the area. That is, if $j \cdot n > 0$, where n is the outward normal unit vector of the area element, then we say the flux is positive. Under this convention the net influx to a volume element can be written as $-\text{div } j$ (\rightarrow 2C.5).

a1A.6 Examples of flux density. If there is a flow whose velocity is v , which carries a quantity whose density is ρ , then its flux density carried by the flow is ρv . This is the general form of the advective flux density. Suppose ρ is the mass density. Then, ρv is the mass flux density, if v is the velocity field of a fluid (this velocity field should be the average velocity (=total momentum/total mass) of the particles in the volume element). Therefore, ρv is also the momentum density. There must be a corresponding flux density: $\rho v_x v$ is the (advective) flux density of the x -component of the momentum (\rightarrow a1E.1).

a1A.7 General form of local conservation law. For any flux density j , $\text{div } j$ is the rate of decrease of the quantity carried by the flux per unit time per unit volume (\rightarrow 2C.5). That is, this is the rate of loss of the density ρ due to the flux at the point the divergence is calculated. If the quantity cannot be produced or

³⁵An area element is specified by a vector parallel to the outward normal vector of the area with the length equal to the area.

annihilated (i.e., conserved as mass in the non-relativistic situation), we must have

$$\frac{\partial \rho}{\partial t} = -\text{div } j. \quad (1.62)$$

This is called the *local conservation equation*. If the quantity could be produced (or dissipated), then we add the *source term* σ which is the rate of production $\sigma > 0$ implies production, and $\sigma < 0$ loss) of the quantity per time per volume:

$$\frac{\partial \rho}{\partial t} = -\text{div } j + \sigma. \quad (1.63)$$

This is the general form of the *balance equation*.

a1B Diffusion Equation and Laplace Equation

Key words: diffusion equation, advection, conduction, Laplace equation, reaction-diffusion equation, conservation of probability

a1B.1 Linear phenomenological law. For spatially gently varying situations the conduction flux density (\rightarrow a1A.4) is often assumed to be proportional to the gradient of the density of the quantity to be transported. This is the *linear phenomenological law* of transport.

(i) If there is a spatially gently inhomogeneous temperature field T , there must be a heat flux density j proportional to $\text{grad } T$ (\rightarrow 2C.1). Thus

$$j = -\kappa \text{ grad } T. \quad (1.64)$$

where κ is called the *heat conductivity*. This is called *Fourier's law*.

(ii) If there are spatially gently inhomogeneous distributions of chemicals c_i in a medium, there must be diffusion currents j_i of these substances:

$$j_i = -\sum_j D_{ij} \text{ grad } c_j. \quad (1.65)$$

where D_{ij} are called *diffusion constants*. This is *Fick's law*. The off-diagonal coefficients are especially called *cross-diffusion constants*. The matrix $\text{Matr.}(D_{ij})$ must be symmetric according to Onsager's reciprocity principle,³⁶ and must be positive definite due to the second law of thermodynamics. In particular, all the diagonal elements must be positive. However, some off-diagonal elements could be negative.³⁷

³⁶L. Onsager, Phys. Rev. **37**, 405 (1931); **38**, 2265 (1931); S. R. de Groot, and P. Mazur, *Non-Equilibrium Thermodynamics* (Dover, 1984).

³⁷This negativity could cause interesting pattern formations in reaction-diffusion systems (\rightarrow a1B.4). See, for example, Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer 1984).

(iii) Other examples of conduction include the Newtonian viscosity (\rightarrow a1E.3), and Ohm's law (\rightarrow a1F.16).

a1B.2 Heat conduction, diffusion equation. If there is a change δT of temperature T , then there is an (internal) energy density change of $\rho C \delta T$, where ρ is the mass density, and C is the specific heat. Hence, the energy conservation law (\rightarrow a1A.7) has the following expression:

$$\frac{\partial \rho C T}{\partial t} = -\text{div}(-\kappa \text{grad} T). \quad (1.66)$$

If all the materials constants ρ , C and κ can be regarded constant, we reach

$$\frac{\partial T}{\partial t} = D_T \Delta T, \quad (1.67)$$

where $D_T \equiv \kappa/\rho C$ is called the *thermal diffusion constant*. An equation of this type is called the *diffusion equation* and was derived for the first time by Fourier (\rightarrow 1.7). Where there is strong dissipation, we often encounter this or related equations (e.g., Stokes' equation for viscous fluid \rightarrow a1E.10).

a1B.3 Laplace equation. We expect that the system reaches a steady state if the boundary condition is fixed. The steady temperature distribution of a uniform body must obey the Laplace equation according to (1.67):

$$\Delta T = 0. \quad (1.68)$$

Any function satisfying this equality is called a *harmonic function* (\rightarrow 2C.11).

a1B.4 Reaction-diffusion equations, Fisher's equation. If there are chemical reactions, the source term must describe their effects. The source terms are often modeled with naive mass-action laws. For example, if there is a ternary reaction $2A + B \rightarrow C$, then $\sigma_A = -2kc_A^2 c_B$ and $\sigma_B = -kc_A^2 c_B = -\sigma_C$ with k being some positive constant (*rate constant*), where c_A is the concentration of A, etc. In this way many nonlinear source terms can be modeled or interpreted as chemical reactions.³⁸ We have, generally,

$$\frac{\partial c}{\partial t} = D \Delta c + f(c). \quad (1.69)$$

This type of equation is called Fisher's equation after R. A. Fisher³⁹ who used this equation to study the propagation of an allele type in a population in population genetics, or the Fisher-KPP equation after Kolmogorov,⁴⁰ Petrovskii⁴¹ and Piskunov who studied in depth the propagation waves in this equation. There are many interesting equations which exhibits various spatio-temporal phenomena.⁴²

³⁸In lower spatial dimensions, there is a complication due to concentration fluctuation, and such simple laws of mass action do not hold.

³⁹Ronald Aylmer Fisher, 1890-1962.

⁴⁰Andrei Nikolaevich Kolmogorov, April 3, 1903 (Tambov)-October 20, 1989 (Moscow). See V. M. Tikhomirov, *Russ Math Surveys* **43**(6) 1-39 (1988).

⁴¹Ivan Georgievich Petrovskii, 1901-1973.

⁴²For beautiful examples, see H Meinhardt, *The Algorithmic Beauty of the Sea Shell* (Springer, 1995), which contains a diskette for PC.

a1B.5 Conservation of probability. Let $P(x)$ be the probability density of an event x (that is, the probability of an event specified by the parameter between x and $x+dx$ is given by $P(x)dx$). Suppose the probability is time dependent. Since the total sum (integral) of the probability density must always be unity ($\int P(x)dx = 1$), if x changes continuously, then P must obey a conservation law (\rightarrow **a1A.7**):

$$\frac{\partial P}{\partial t} = -\text{div} j_P. \quad (1.70)$$

where j_P is the probability flux density. Many statistical mechanically interesting examples have this form. The Smoluchowski equation and the Fokker-Planck equation are discussed in Appendix **a1C**.

a1C Diffusion and Brownian Motion

Key words: Brownian motion. Smoluchowski equation. Schrödinger equation. Fokker-Planck equation.

a1C.1 Diffusion equation and Brownian particles. Suppose there is an ensemble of numerous non-interacting particles undergoing random motions (Brownian motion) which we wish to describe collectively. The most natural way may be to use the probability density distribution $P(x, t)$ which tells us that the fraction of particles in the volume element $d^d x$ (in d -space) at time t is given by $P(x, t)d^d x$. P obeys (1.70).

We may imagine that the ensemble of particles is an ideal gas, because the particles do not interact. Thermodynamically, the induced flow velocity due to the uneven pressure is written with the aid of the chemical potential μ of the particles as $-D \text{grad } \mu$, where D is a phenomenological constant called diffusion constant. μ must be the chemical potential of an ideal gas, that is, $\mu = +k_B T \ln(\text{pressure})$, where k_B is the Boltzmann constant, and T the absolute temperature. For convenience, henceforth, we will use the energy unit for which $k_B T$ is unity. The pressure at position x at time t is proportional to the number density of the particles (recall $p = nRT/V$), which in turn is proportional to $P(x, t)$. Therefore, ignoring an additive constant, we get

$$\mu = \ln P(x, t). \quad (1.71)$$

Since the probability is advected by the probability flow, we conclude

$$j_P = P(x, t) \times (-D \text{grad } \ln P(x, t)) = -D \nabla P. \quad (1.72)$$

Putting this into (1.70), we obtain a diffusion equation

$$\frac{\partial P}{\partial t} = D \Delta P. \quad (1.73)$$

a1C.2 Smoluchowski equation. If Brownian particles are in a sufficiently viscous fluid and are under an external force field \mathbf{F} , then its equation of motion, which is overdamped, reads

$$\frac{dx}{dt} = \gamma \mathbf{F} + (\text{noise}). \quad (1.74)$$

a1C.1 may be considered as the case without the external force. Therefore, the probability flux density consists of two parts: $\mathbf{j}_P = \gamma \mathbf{F} P - D \nabla P$, the systematic part (the advection part \rightarrow a1A.4) due to the external force $P \times \gamma \mathbf{F}$ and the noise contribution (1.72) (the conductive part \rightarrow a1A.4). (1.70) becomes

$$\frac{\partial P}{\partial t} = D \Delta P - \gamma \nabla \cdot (\mathbf{F} P). \quad (1.75)$$

This is called the *Smoluchowski equation*.⁴³

a1C.3 Smoluchowski equation vs. Schrödinger equation. A Smoluchowski equation with a conservative external force $\gamma \mathbf{F} = -\text{grad } U$:

$$\frac{\partial P}{\partial t} = \text{div} (P \text{ grad } U) + D \Delta P. \quad (1.76)$$

can be rewritten as

$$\frac{\partial \psi}{\partial t} = \left[\frac{1}{2} \Delta U - \frac{1}{4D} (\nabla U)^2 \right] \psi + D \Delta \psi. \quad (1.77)$$

where

$$\psi = P e^{U/2D}. \quad (1.78)$$

(1.77) is an Euclidized Schrödinger equation with the potential $V \equiv \Delta U/2 - (\nabla U)^2/4D$.

a1C.4 Fokker-Planck equation. A more general theory for Brownian particles does not assume the overdamped situation like the Smoluchowski equation (\rightarrow a1C.2). The equation of motion for a particle is

$$\begin{pmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{v}} \end{pmatrix} = \begin{pmatrix} \mathbf{v} \\ -\zeta \mathbf{v} + \mathbf{F}/m + \text{noise} \end{pmatrix}. \quad (1.79)$$

where \mathbf{r} is the position, \mathbf{p} the momentum, \mathbf{F} the external force, m the mass of the particle, and ζ the friction coefficient. **a1C.2** and **a1C.3** tell us that the probability flux density (now it is a 6 dimensional vector) should have the following form:

$$\mathbf{j}_P = P \begin{pmatrix} \mathbf{v} \\ -\zeta \mathbf{v} + \mathbf{F}/m \end{pmatrix} + P \begin{pmatrix} 0 \\ -D \nabla_{\mathbf{v}} \ln P \end{pmatrix}. \quad (1.80)$$

⁴³If the system is in equilibrium and \mathbf{F} has a potential U (such that $\mathbf{F} = -\text{grad } U$), then (1.75) must have the unique equilibrium distribution $\propto \exp(-U)$ as its steady state (we choose the energy unit such that $k_B T = 1$). This imposes a relation between D and γ called the *fluctuation-dissipation theorem of the second kind*: $D = \gamma$. Check this.

where $\nabla_{\mathbf{v}}$ is the gradient operator for velocity \mathbf{v} . The nabla operator (\rightarrow 2C.4) in our context is

$$\begin{pmatrix} \nabla_{\mathbf{r}} \\ \nabla_{\mathbf{v}} \end{pmatrix}, \quad (1.81)$$

so that (1.70) reads

$$\frac{\partial P}{\partial t} = -\mathbf{v} \cdot \nabla_{\mathbf{r}} P - \nabla_{\mathbf{v}} \cdot \left(\frac{\mathbf{F}}{m} P \right) + \zeta \nabla_{\mathbf{v}} \cdot (\mathbf{v} P) + D \Delta_{\mathbf{v}} P \quad (1.82)$$

This is called the *Fokker-Planck equation*. Notice that (1.75) and (1.82) consist of two parts, the second-order derivative term (diffusion term) describing the microscopic noise effect, and the first-order derivative terms just as the advection term of the Navier-Stokes equation (\rightarrow a1E.6) describing the convection or advection due to systematic flows.

a1C.5 Reduction of Fokker-Planck equation to Smoluchowski equation.

If the viscosity of the fluid is large, then the velocity part of the Fokker-Planck equation should settle down quickly to its equilibrium form (the Maxwell distribution $\propto \exp(-m\mathbf{v}^2/2)$). Hence, the slow motion of the particle position described by the Fokker-Planck equation must be given by the Smoluchowski equation (\rightarrow a1C.2). A systematic expansion of the solution in terms of $1/\zeta$ can be used. However, a 'quick and dirty' way is as follows. Introduce the marginal distribution $p(\mathbf{r}, t) = \int d\mathbf{v} P(\mathbf{r}, \mathbf{v}, t)$. Integrating the Fokker-Planck equation with respect to \mathbf{v} , we get

$$\frac{\partial p}{\partial t} = -\text{div } \mathbf{p}, \quad (1.83)$$

where \mathbf{p} is the momentum density (\rightarrow a1A.6, a1E.1)

$$\mathbf{p}(\mathbf{r}, t) = \int d\mathbf{v} \mathbf{v} P(\mathbf{r}, \mathbf{v}, t). \quad (1.84)$$

We have assumed that P decays sufficiently quickly in the $|\mathbf{v}| \rightarrow \infty$ limit. Multiply \mathbf{v} to the Fokker-Planck equation and integrate over \mathbf{v} . Since we choose $k_B T$ to be unity $\mathbf{v}\mathbf{v}$ averaged over P must be I/m , where I is the unit matrix. Thus we get

$$-\frac{1}{m} \nabla_{\mathbf{r}} \mathbf{p} + \frac{\mathbf{F}}{m} \mathbf{p} = -\zeta \mathbf{p}. \quad (1.85)$$

Putting this \mathbf{p} into (1.83), we get a Smoluchowski equation as expected. (Notice that $\gamma = D = 1/m\zeta$)

Here, QM.CM, and Liouville eq., relativity

a1D Equation of Motion of Continuum Body

Key words: momentum balance, stress tensor, equation of motion of continuum body, strain tensor, constitutive equation, Hooke's law, wave equation, energy integral.

a1D.1 Momentum balance, Force as negative momentum flux. The essence of the classical equation of motion is the momentum balance. Take an area element and imagine that a force \mathbf{F} (per area) is acting outwardly on the element. Then, Newton's equation of motion tells us that the volume element gains momentum with the rate equal to the force. That is, through the area element, momentum is flowing into the volume element at the rate of \mathbf{F} . In other words, we may interpret force (per area) as the negative momentum flux density through the surface.

a1D.2 Stress tensor. Let σ_{ij} be the i -th component of the force per unit area acting on the j -th plane (i.e., the plane perpendicular to the j -th coordinate direction: its normal vector points the positive direction of the j -th axis) (for example, $\sigma_{xx} > 0$ implies that there is an outward normal force on the plane perpendicular to the x -axis). The second rank tensor $\sigma = \text{Matr.}(\sigma_{ij})$ is called the *stress tensor*.⁴⁴ For example, $\sum_j \sigma_{ij} dS_j$ is the i -th component of the force acting on the area element dS ($\rightarrow 2C.5$). This force is due to the environment surrounding the volume element.

Discussion [Stress tensor]:

(1) Stress tensors must be symmetric. The moment of the forces acting on the volume V can be written as (summation convention is used)

$$\int_V \left(\frac{\partial \sigma_{ij}}{\partial x_j} x_k - \frac{\partial \sigma_{kj}}{\partial x_j} x_i \right) d\tau = \int_V \frac{\partial (\sigma_{ij} x_k - \sigma_{kj} x_i)}{\partial x_j} d\tau - \int_V (\sigma_{ik} - \sigma_{ki}) d\tau. \quad (1.86)$$

where τ is the volume element, and we have used $\partial_j x_k = \delta_{jk}$. The first term can be converted into the surface integral with the aid of Gauss' theorem ($\rightarrow 2C.13$). Since we assume the forces on the volume V are acting on its surface, there must not be any volume integral term left. Hence $\sigma_{ij} = \sigma_{ji}$. That is, the stress tensor must be a symmetric second rank tensor.

(2) Stress tensor is meaningful only when the forces are 'short-ranged.' As we have seen, the concept of stress tensor should be useful only when the force on a volume element can be considered as surface forces. Consider a horizontal plane cutting through a body, and calculate the force between the upper half and lower half of the body through the plane. If we use an atomistic model of matter, then the force must be written as

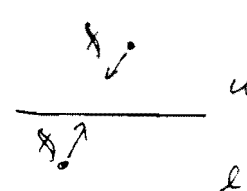
$$\mathbf{F} = \sum_u \sum_l \mathbf{f}_{ul}. \quad (1.87)$$

where u runs over all the atoms on the upper side and l on the lower side of the plane and \mathbf{f} is the force between atoms. Only when this sum is proportional to the area of the plane, can we successfully use the concept of stress tensor.

Exercise.

Show that this requires the forces \mathbf{f} to decay faster than r^{-4} asymptotically. Thus gravity, and the unscreened Coulomb force must be excluded. They become

⁴⁴A second rank tensor $\{t_{ij}\}$ is a quantity which have the same symmetry as the product of two vectors $\{a_i b_j\}$. That the stress tensor is a second rank tensor can be seen from the fact that $-p v_i v_j$ can be a part of σ_{ij} as discussed in **a1A.6**. An easy introduction to tensor can be found in *The Feynman Lectures of Physics*, vol. II, Section 31.



the so-called *bodily forces*.

a1D.3 Force on a volume. The i -component of the total force acting on the volume V can be written as

$$\int_{\partial V} \sigma_{ij} dS_j. \quad (1.88)$$

where ∂V is the boundary of V . From this, we conclude that the i -component of the force per volume due to this stress tensor is (with the aid of Gauss' theorem \rightarrow 2C.13)

$$\sum_j \frac{\partial}{\partial x_j} \sigma_{ij}. \quad (1.89)$$

This must be the increase (\rightarrow a1D.1) of the i -th component of the momentum density in the volume element surrounding the point \mathbf{x} .

a1D.4 Basic equation of continuum mechanics. A deformation of a body can be described by the displacement of a point in space. If \mathbf{x} is displaced to \mathbf{x}' , $\mathbf{x}' - \mathbf{x} = \mathbf{u}$ is called the *displacement vector*. Let ρ be the mass density, and the displacement of the volume element at \mathbf{x} be $\mathbf{u}(\mathbf{x})$. Then, Newton's law = the momentum balance condition tells us

$$\rho \ddot{u}_i = \frac{\partial \sigma_{ij}}{\partial x_j} + F_i. \quad (1.90)$$

where $\mathbf{F} = (F_x, F_y, F_z)$ is the bodily external force (like gravity) per unit mass. To close this equation we need the relation between the displacement field \mathbf{u} (and its derivatives) and the stress tensor (field) σ . The relation depends on system specificities. Such relations sensitive to materialistic details are called *constitutive relations*, and they are described by *constitutive equations*. Often we must study the detailed structure of matter to obtain reasonable constitutive equations.

a1D.5 Deformation of a body. If the displacement vector \mathbf{u} is given all over the body, the deformation is completely specified. However, uniform translation and rotation should be removed if we wish to describe the true deformation of a continuum body. First of all, the absolute value of \mathbf{u} is not important. Thus its derivative $U \equiv \{\partial_i u_j\}$ (a second rank tensor) is more relevant. In this way we can remove translation. However, rigid rotation must be removed as well.

a1D.6 Strain tensor. U in a1D.5 can be uniquely split into the symmetric and skew-symmetric parts, U_+ and U_- : $U_+ \equiv (U + U^T)/2$ and $U_- \equiv (U - U^T)/2$.⁴⁵ Notice that (see 2C.8 for curl)

$$2U_- \mathbf{a} = \text{curl } \mathbf{u} \times \mathbf{a} \quad (1.91)$$

for any 3-vector \mathbf{a} .⁴⁶ It is clear that the skew part describes rotation (see Discussion

⁴⁵ T implies transposition.

⁴⁶Here the following useful relation is used.

$$\begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (1.92)$$

in 2C.9). Therefore, a small deformation of a body is described by the symmetric part of the spatial derivative of the displacement vector. The *strain tensor* is defined as

$$u_{ij} \equiv \frac{1}{2} \left(\frac{\partial u_j}{\partial x^i} + \frac{\partial u_i}{\partial x^j} \right). \quad (1.93)$$

Exercise. Demonstrate that the trace of the strain tensor describes the volume change. (Remember that $Tr(\partial \mathbf{u} / \partial \mathbf{r})$ may be considered to be a coordinate free definition of $div \mathbf{u}$.) (→2C.6)

a1D.7 Hooke's law. A linear relation between the stress (→a1D.2) and strain (→a1D.6) is called *Hooke's law*. Generally, we need a 6×6 tensor to relate all σ_{ij} to all u_{ij} (both have only 6 independent components due to symmetry). For a uniform isotropic material, there are only two independent parameters K (*compressibility*) and μ (*shear modulus*).⁴⁷

$$\sigma_{ij} = K \operatorname{div} \mathbf{u} \delta_{ij} + 2\mu \left(u_{ij} - \frac{1}{3} \delta_{ij} \operatorname{div} \mathbf{u} \right). \quad (1.94)$$

In terms of Young's modulus E and the Poisson ratio σ .⁴⁸

$$\mu = \frac{E}{2(1+\sigma)}, \quad K = \frac{E}{3(1-2\sigma)}. \quad (1.95)$$

Hooke's law (1.94) for an isotropic body is rewritten as

$$\sigma_{ij} = \frac{E}{1+\sigma} \left(u_{ij} + \frac{\sigma}{1-2\sigma} \operatorname{div} \mathbf{u} \delta_{ij} \right). \quad (1.96)$$

a1D.9 Equation of motion of isotropic elastic body: wave equation. With the aid of the definition of the strain tensor (→a1D.6 (1.93)) and (1.96), the equation of motion (1.90) (→a1D.4) is explicitly written as

$$\rho \ddot{\mathbf{u}} = \frac{E}{2(1+\sigma)} \Delta \mathbf{u} + \frac{E}{2(1+\sigma)(1-2\sigma)} \operatorname{grad} \operatorname{div} \mathbf{u} + \mathbf{F}. \quad (1.97)$$

where \mathbf{F} is a bodily external force. See 2C.12 for the Laplacian acting on vectors. Define

$$c_l = \sqrt{\frac{E(1-\sigma)}{\rho(1+\sigma)(1-2\sigma)}}, \quad c_t = \sqrt{\frac{E}{2\rho(1+\sigma)}}. \quad (1.98)$$

The equation of motion (1.97) without \mathbf{F} can be rewritten as

$$\ddot{\mathbf{u}} = c_l^2 \Delta \mathbf{u} + (c_l^2 - c_t^2) \operatorname{grad} \operatorname{div} \mathbf{u}. \quad (1.99)$$

where \times is the vector product. (1.91) can be considered as a coordinate-free definition of *curl*.

⁴⁷See Chapter 1 of Landau-Lifshitz. *Theory of Elasticity* (Pergamon) for much more details.

⁴⁸When $\sigma_{zz} = p$ and other components of the stress tensor are zero, u_{zz}/p is called *Young's modulus*. The ratio of longitudinal stretching vs transversal shrinking is called *Poisson's ratio*, and defined by $\sigma \equiv -u_{zz}/u_{zz}$.

Now, let us apply the Helmholtz-Stokes-Blumental (HSB) theorem (→2C.18), and split \mathbf{u} as $\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t$ with $\text{div } \mathbf{u}_l = 0$ and $\text{curl } \mathbf{u}_t = 0$. Putting this splitting into (1.99), we get

$$\ddot{\mathbf{u}}_l - c_l^2 \Delta \mathbf{u}_l = -(\ddot{\mathbf{u}}_t - c_t^2 \Delta \mathbf{u}_t). \quad (1.100)$$

This implies that both the divergence and the curl of $\ddot{\mathbf{u}}_l - c_l^2 \Delta \mathbf{u}_l$ and $\ddot{\mathbf{u}}_t - c_t^2 \Delta \mathbf{u}_t$ vanish. The Helmholtz-Hodge theorem (→2C.17) tells us that a vector whose divergence and curl both vanish must be zero.⁴⁹ Thus we get two separate *wave equations*:

$$\ddot{\mathbf{u}}_l = c_l^2 \Delta \mathbf{u}_l, \quad (1.101)$$

$$\ddot{\mathbf{u}}_t = c_t^2 \Delta \mathbf{u}_t. \quad (1.102)$$

\mathbf{u}_l describes the longitudinal wave, which allows the change of volume (→a1D.6 Exercise), and \mathbf{u}_t the transversal wave, which does not accompany any volume change.

a1D.10 Drumhead. Consider a membrane for which the deformation vectors are always perpendicular to the xy -plane. This condition is satisfied by a small transversal oscillation of a drumhead. Only u_z is non-zero, so that only σ_{zx} and σ_{zy} are non-zero. Hence, Hooke's law (1.94) reads

$$\sigma_{zx} = \mu \frac{\partial u_z}{\partial x}, \quad \sigma_{zy} = \mu \frac{\partial u_z}{\partial y}. \quad (1.103)$$

Putting this into (1.90), we get a wave equation in 2-space:

$$\rho \ddot{u}_z = \mu \left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} \right). \quad (1.104)$$

The equilibrium shape of the membrane is determined by the Laplace equation.

a1D.11 String under tension. Consider a string which is under tension T and which is confined in the xz -plane. We consider its displacement u in the z -direction perpendicular to the string. If the line mass density is ρ , then the acceleration in the z -direction of a segment $[x, x + dx]$ is given by $\partial^2 u / \partial t^2$. The force at x in the z -direction is given by $-T \partial u / \partial x|_x$, and that at $x + dx$ is given by $T \partial u / \partial x|_{x+dx}$. Hence, the total force acting on the segment in the z -direction is the sum of these two, i.e., $T(\partial^2 u / \partial x^2) dx$. Consequently, Newton's equation of motion reads

$$\rho \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2}. \quad (1.105)$$

⁴⁹To conclude this we must require that the waves we study are spatially localized. According to the Helmholtz-Hodge theorem any once differentiable vector field \mathbf{f} whose derivatives vanish at infinity can be decomposed as $\mathbf{f} = \text{grad } \phi + \text{curl } \mathbf{A}$ and $\text{div } \mathbf{A} = 0$, $\text{div } \mathbf{f} = \Delta \phi$, and $\text{curl } \mathbf{f} = -\Delta \mathbf{A}$, so that vanishing of the divergence and the curl of \mathbf{f} implies that ϕ and \mathbf{A} are harmonic. Since a harmonic function (→2C.11) which is finite throughout the space is constant (Liouville's theorem 29.13), $\mathbf{f} = 0$ is concluded. See that the explanation in Landau-Lifshitz is incomplete.

a1D.12 Energy integral. One of the most important properties of the wave equation (1.2) is that the following *energy integral*

$$E(t) \equiv \int_D \frac{1}{2} \left((\partial_t u)^2 + c^2 \sum_i (\partial_x u)^2 \right) dx \quad (1.106)$$

is conserved if the boundary condition is homogeneous (→20.5, 30.4).

beam equation: $u_t + u_{xxxx} = 0$

a1E Fluid Dynamics: Navier-Stokes Equation

Key words: Newtonian viscosity, long-time tail, continuity equation, Navier-Stokes equation, perfect fluid, Euler's equation, substantial derivative, potential flow, velocity potential, Stokes' approximation.

a1E.1 Momentum flux density. When we treat fluid, it is often convenient to introduce the momentum flux density Π . Let ρ be the mass density, and \mathbf{v} be the velocity field of the fluid. Then, $\rho\mathbf{v}$ is the momentum density. If this macroscopic momentum is never dissipated, the momentum is carried (=advected) only by the flow itself, so that $\Pi = \rho\mathbf{v}\mathbf{v}$ is the momentum flux density.⁵⁰ Usually, there is a flux that describes the dissipation Π' (or the conductive part of the momentum flux density) as well. Thus, generally we should write

$$\Pi = \rho\mathbf{v}\mathbf{v} + \Pi'. \quad (1.107)$$

In general, we need a microscopic consideration to set up the constitutive equation (→a1D.4) for the dissipative part Π' of the momentum flux density tensor in terms of the velocity field (→a1E.3).

a1E.2 Pressure; equation of motion. For isotropic fluids, the stress tensor must be isotropic, that is, it must be a constant matrix. We usually write it as

$$\sigma_{ij} = -p\delta_{ij}, \quad (1.108)$$

where p is called pressure. Pressure is defined to be positive when it pushes into the volume on whose surfaces the pressure is acting. Thus the sign convention is opposite to the stress tensor (→a1D.2, a1D.3), so that we need $-$ in this formula. Notice that $\sum_j \partial_j \sigma_{ij} = -\partial_i p = -(\text{grad } p)_i$. Thus the momentum balance (→a1D.4) becomes

$$\frac{\partial \rho v_i}{\partial t} = -\frac{\partial(\rho v_i v_j + \Pi'_{ij})}{\partial x^j} - \frac{\partial p}{\partial x^i} + F_i. \quad (1.109)$$

⁵⁰This is a collective expression of the flux density of the i -th component of the momentum $\rho v_i \mathbf{v}$ (→a1A.6).

where \mathbf{F} is the external bodily force per volume (if it is gravity, then $\mathbf{F} = -\rho g \mathbf{z}$, where \mathbf{z} is the unit vector pointing upward).

a1E.3 Newtonian viscosity. To close (1.109) we must express the conduction part of the momentum flux tensor Π' in terms of the velocity field. This is a problem of constitutive equations (\rightarrow a1D.4), but for small velocity fields, we can fix the functional form. There is no dissipation if the velocity field is uniform. Hence, it must be a function(al) of $\partial_i v_j$. We assume the linear relation between Π'_{ij} and these derivatives (called *Newtonian viscosity*). This should be admissible when the velocity varies spatially gently. Uniform rotation of fluid does not cause any dissipation, so that only the symmetric part of $\partial_i v_j$ should contribute to dissipation (see an analogous argument in a1D.6). Isotropy of fluid then dictates the general form of dissipation as a sum of the term accompanying the volume change and that without volume change (\rightarrow a1D.7 for an analogous argument):

$$-\Pi'_{ij} = \eta \left(\frac{\partial v_j}{\partial x^i} + \frac{\partial v_i}{\partial x^j} - \frac{2}{3} \delta_{ij} \sum_k \text{div } v \right) + \zeta \delta_{ij} \text{div } v. \quad (1.110)$$

Here η is called the *shear viscosity* and ζ the *volume viscosity*.⁵¹ Both are positive to be consistent with the second law of thermodynamics. Remember that this is a model supposedly appropriate for gently spatially dependent velocity fields. Hence, it is a legitimate question to ask whether this may be used in the study of turbulence.

a1E.4 Viscosity and long-time tail. Viscosity (\rightarrow a1E.3) is due to the diffusion of momentum imparted to the fluid surrounding the body: if there is no such diffusion, then the conservation of momentum implies that the body together with its immediate surrounding fluid would keep the initial total momentum, so that there would be no appreciable deceleration of the body. Generally speaking, diffusion spreads a quantity to the range of radius \sqrt{t} as a function of time.⁵² Hence, the motion of the body drags the surrounding mass of fluid of radius $\sim \sqrt{t}$ up to time t . That is, the fluid mass of order $t^{3/2}$ moves with the body. This implies that the speed of the body becomes of order $t^{-3/2}$, because the body moves with the surrounding fluid. That is, the velocity of a body in a viscous fluid does not decay exponentially, but much more slowly. This is called the *long-time tail*, and has actually been observed.⁵³

a1E.5 Mass conservation, continuity equation. The mass flux density is

⁵¹ $\zeta = 0$ for monatomic gases. For diatomic gases ζ and η are of the same order. Almost nothing is known about the volume viscosity of liquids. However, thanks to the very small compressibility of liquids, we can often ignore ζ .

⁵²For physicists, the dimensional analytic argument is the best. The only dimensionless quantity we can construct from t , x and D is tD/x^2 . Hence, the representative length scale at time t must be \sqrt{tD} or proportional to \sqrt{t} .

⁵³The algebraic decay was first discovered in a computer experiment: B. J. Alder and T. E. Wainright, Phys. Rev. A1, 18 (1970). For a clear real experimental demonstration, see K. Oohayashi, T. Kohno and H. Utiyama, Phys. Rev. A27, 2632 (1983).

obviously $\rho \mathbf{v}$ (if there is only one component, there is no diffusion). Hence the conservation of mass is (\rightarrow a1A.7)

$$\frac{\partial \rho}{\partial t} = -\operatorname{div} \rho \mathbf{v}. \quad (1.111)$$

This is also called the *continuity equation*.

a1E.6 Navier-Stokes equation. The momentum balance equation

$$\frac{\partial \rho \mathbf{v}}{\partial t} = - \sum_j \frac{\partial}{\partial x_j} \Pi_{ij} \quad (1.112)$$

is the equation of motion (\rightarrow a1D.4). The viscosity is modeled according to Newton (\rightarrow a1E.3). The resultant equation of motion of a fluid is called the *Navier-Stokes equation*:⁵⁴

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \frac{\eta}{\rho} \Delta \mathbf{v} + \frac{1}{\rho} \left(\zeta + \frac{\eta}{3} \right) \operatorname{grad} \operatorname{div} \mathbf{v} - \frac{1}{\rho} \operatorname{grad} p + \frac{1}{\rho} \mathbf{F}. \quad (1.113)$$

where \mathbf{F} is the bodily external force (per unit volume). We need the mass balance equation (1.111) and a constitutive equation (i.e., an equation of state) to close this equation for p .

Under the incompressibility condition ($\operatorname{div} \mathbf{v} = 0$), the term containing the volume viscosity disappears and we get

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \nu \Delta \mathbf{v} - \frac{1}{\rho} \operatorname{grad} p + \mathbf{F}. \quad (1.114)$$

where $\nu \equiv \eta/\rho$ is called the *kinematic viscosity*. In this case the pressure p is a mere auxiliary variable to enforce incompressibility.

a1E.7 Perfect fluid, Euler's equation.⁵⁵ If there is no dissipation effect in the fluid, the fluid is called an *ideal fluid* or *perfect fluid*, for which $\Pi' = 0$ in (1.107). Combining (1.109) and (1.111), we get

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \mathbf{F}. \quad (1.115)$$

This is called *Euler's equation*. (1.115) is not a closed equation because of p . We need an equation of state for the fluid. If the fluid is isothermal, we may regard p as a function of ρ only.⁵⁶ Hence, (1.111), (1.115) and $p = p(\rho)$ make a closed system of equations.

a1E.8 Substantial derivative. The changing rate of an observable f observed from an observer flowing with the fluid is given by

$$\frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla) f. \quad (1.116)$$

⁵⁴Louis Marie Henri Navier. 1785-1836.

⁵⁵Leonhard Euler. 1707-1783. See 4.4.

⁵⁶The fluid need not be in an isothermal condition. Often $pV^\alpha = \text{const.}$ is used.

Hence, it is convenient to introduce a special symbol for this derivative:

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla, \quad (1.117)$$

which is called the *substantial derivative*. Thus, the continuity equation reads

$$\frac{D\rho}{Dt} = -\rho \operatorname{div} \mathbf{v}, \quad (1.118)$$

and Euler's equation reads

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \mathbf{F}. \quad (1.119)$$

a1E.9 Incompressibility, potential flow. If the flow speed is much less than the sound velocity, we may treat the fluid as incompressible. That is, ρ is constant (if the fluid is isothermal). (1.111) implies

$$\operatorname{div} \mathbf{v} = 0. \quad (1.120)$$

In this case we do not need any equation of state: p is determined by the incompressibility condition.

If the vorticity (i.e., $\operatorname{curl} \mathbf{v}$) of the flow is 0 everywhere, the flow is called an *irrotational flow* or *potential flow*, since velocity has a potential (\rightarrow 2C.16): $\mathbf{v} = \operatorname{grad} \Phi$. Φ is called the *velocity potential*. A velocity potential for an incompressible fluid must obey:

$$\Delta \Phi = 0. \quad (1.121)$$

that is, Φ is harmonic (\rightarrow 2C.11).

a1E.10 Stokes' approximation. For very slow flows,⁵⁷ we ignore the advection term in (1.114). This is called the *Stokes approximation*, and the result

$$\frac{\partial \mathbf{v}}{\partial t} = \nu \Delta \mathbf{v} - \frac{1}{\rho} \operatorname{grad} p + \frac{1}{\rho} \mathbf{F}. \quad (1.122)$$

is called the *Stokes equation*. Notice that this is under the incompressibility condition.

Taking the divergence of the equation, we notice that p obeys the following Poisson equation (\rightarrow 1.2):

$$\Delta p = \operatorname{div} \mathbf{F}. \quad (1.123)$$

⁵⁷The extent of the importance of inertia, or the advection term is measured by the dimensionless quantity called the *Reynolds number* $Re \equiv VL/\nu$, where V is the representative velocity and L the representative length scale. For example, if a ball of radius a is moving at speed v in a stationary fluid with kinetic viscosity ν , the Reynolds number of the system is av/ν . The Stokes approximation is reliable for $Re \ll 0.1$.

a1F Electrodynamics

Key words: Coulomb's law, conservation of charge, Biot-Savart's law, Ampère's law, Faraday's law, displacement current, Maxwell's equation, vector potential, scalar potential, Poisson's equation, gauge symmetry, Coulomb gauge, Lorentz gauge, d'Alembertian, polarization, electrical displacement, dielectric constant, magnetization, magnetic permeability, supplementary equations, Ohm's law, telegrapher's equation, skin depth.

a1F.1 Empirical facts.

(1) **Priestley(1767)-Cavendish(1773)-Coulomb(1785)'s law.** The electric field \mathbf{E} due to a point charge Q at \mathbf{r} is given by

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{Q\mathbf{r}}{r^3}. \quad (1.124)$$

where ϵ_0 is a constant (called the *dielectric constant of the vacuum*).

(2) **Conservation of charge.**

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{i} = 0. \quad (1.125)$$

where ρ is the charge density, and \mathbf{i} is the current density (i.e., the charge flux density).

(3) **Non-existence of magnetic monopoles.**

(4) **Biot-Savart's law.** The magnetic flux \mathbf{B} created by a current density \mathbf{i} at \mathbf{r} is given by

$$\mathbf{B} = \frac{\mathbf{i} \times \mathbf{r}}{4\pi\epsilon_0 c^2 r^3}. \quad (1.126)$$

where c is a constant (called the speed of light in the vacuum).⁵⁸ This is equivalent to *Ampère's law*:

$$\int_{\partial S} \mathbf{B} \cdot d\mathbf{l} = \frac{1}{\epsilon_0 c^2} \int_S \mathbf{i} \cdot d\mathbf{S}. \quad (1.127)$$

where S is a smooth surface whose boundary curve is ∂S (pay attention to the orientation 6.4).

(5) **Faraday's law.**

$$\int_{\partial S} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S}. \quad (1.128)$$

Here the minus sign implies Lenz's law.

(6) **Superposition principle.** All the fields are linear (\rightarrow 1.4).

⁵⁸ $\mu_0 = 1/c^2\epsilon_0$ is called the *magnetic permeability*.

a1F.2 Differential forms of empirical facts. (1) and (6) imply

$$\operatorname{div} \mathbf{E} = \frac{\rho}{\epsilon_0}. \quad (1.129)$$

From (3) and (6) we obtain

$$\operatorname{div} \mathbf{B} = 0. \quad (1.130)$$

(4) implies

$$\operatorname{curl} \mathbf{B} = \frac{\mathbf{i}}{\epsilon_0 c^2} = \mu_0 \mathbf{i}. \quad (1.131)$$

where μ_0 is called the *magnetic permeability* of the vacuum. (5) implies

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \quad (1.132)$$

a1F.3 Displacement current. Taking the divergence of (1.131) with the aid of (1.130), we get

$$0 = \operatorname{div} \operatorname{curl} \mathbf{B} = \frac{1}{\epsilon_0 c^2} \operatorname{div} \mathbf{i}, \quad (1.133)$$

which contradicts charge conservation (1.125). To remove this inconsistency, Maxwell introduced the *displacement current* $\epsilon_0 \partial \mathbf{E} / \partial t$ into (1.131) as

$$\operatorname{curl} \mathbf{B} = \frac{1}{\epsilon_0 c^2} \left(\mathbf{i} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right). \quad (1.134)$$

This is indeed consistent with charge conservation thanks to (1.129).

a1F.4 Maxwell's equation in vacuum. Thus we get the following set of equations:

$$\operatorname{div} \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad (1.135)$$

$$\operatorname{div} \mathbf{B} = 0, \quad (1.136)$$

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (1.137)$$

$$c^2 \operatorname{curl} \mathbf{B} = \frac{\mathbf{i}}{\epsilon_0} + \frac{\partial \mathbf{E}}{\partial t}. \quad (1.138)$$

These are called *Maxwell's equations* (in the vacuum).⁵⁹

⁵⁹The fourth equation also reads

$$\frac{1}{\mu_0} \operatorname{curl} \mathbf{B} = \mathbf{i} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

a1F.5 Vector potential, scalar potential. Since $\text{div } \mathbf{B} = 0$, there is a vector \mathbf{A} called the *vector potential* such that $\mathbf{B} = \text{curl } \mathbf{A}$ ($\rightarrow 2\text{C.16}$).⁶⁰ Introducing this into (1.137), we get

$$\text{curl} \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0. \quad (1.139)$$

Hence ($\rightarrow 2\text{C.16}$), there is a function ϕ such that

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\text{grad } \phi. \quad (1.140)$$

ϕ is called the *scalar potential*.

a1F.6 Poisson's equation. For a static electric field (1.140) and (1.135) imply

$$\Delta \phi = -\frac{\rho}{\epsilon_0}. \quad (1.141)$$

This is called *Poisson's equation*. If there is no charge, then the scalar potential obeys Laplace's equation ($\rightarrow 1.2$), so it must be a harmonic function.

a1F.7 Gauge symmetry. In terms of the vector potential \mathbf{A} and the scalar potential ϕ , we obtain

$$\mathbf{B} = \text{curl } \mathbf{A}, \quad \mathbf{E} = -\text{grad } \phi - \frac{\partial \mathbf{A}}{\partial t}. \quad (1.142)$$

Consider the equations

$$\text{curl } \mathbf{a} = 0, \quad \text{grad } \varphi + \frac{\partial \mathbf{a}}{\partial t} = 0. \quad (1.143)$$

\mathbf{a} must have a potential such that $\mathbf{a} = -\text{grad } \Lambda$. Then, $\varphi = \Lambda$ satisfies the above set of equations. Obviously $\mathbf{A} + \mathbf{a}$ and $\phi + \varphi$ give the same \mathbf{E} and \mathbf{B} . Thus for any sufficiently smooth Λ the following transformation, called the *gauge transformation*,

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} - \text{grad } \Lambda, \quad (1.144)$$

$$\phi \rightarrow \phi' = \phi + \frac{\partial \Lambda}{\partial t} \quad (1.145)$$

does not alter physics. The independence of physics from the choice of Λ (gauge fixing) is called the *gauge symmetry* (or *gauge invariance*). Choosing Λ appropriately, we may be able to simplify problems. Typical gauges are

- (1) *Coulomb gauge*: $\text{div } \mathbf{A} = 0$.
- (2) *Lorentz gauge*:⁶¹

$$\text{div } \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0. \quad (1.146)$$

⁶⁰Since \mathbf{B} is an *axial vector*, \mathbf{A} must be a *polar vector* as \mathbf{r} or \mathbf{i} . Note that axial vectors are invariant under mirror symmetry, but polar vectors change their signs. As seen in the footnote of **a1D.6** axial vectors should be regarded as skew-symmetric tensors in 3-space.

⁶¹This choice is, in contrast to (1), Lorentz covariant.

a1F.8 Maxwell's equation in terms of potentials, electromagnetic wave.
From (1.140) and (1.135), we obtain

$$-\Delta\phi = \frac{\rho}{\epsilon_0} + \frac{\partial}{\partial t} \operatorname{div} \mathbf{A}. \quad (1.147)$$

From (1.138) and (1.140) we also obtain

$$c^2 \operatorname{curl}(\operatorname{curl} \mathbf{A}) = \frac{i}{\epsilon_0} + \frac{\partial}{\partial t} \left(-\frac{\partial \mathbf{A}}{\partial t} - \operatorname{grad} \phi \right). \quad (1.148)$$

or, using a standard formula of vector analysis ($\rightarrow 2C.12$)

$$c^2(\operatorname{grad} \operatorname{div} \mathbf{A} - \Delta \mathbf{A}) = \frac{i}{\epsilon_0} - \frac{\partial^2 \mathbf{A}}{\partial t^2} - \frac{\partial \operatorname{grad} \phi}{\partial t}. \quad (1.149)$$

Use the Lorentz gauge (1.146). Then (1.147) and (1.149) simplify as

$$\square \phi = \frac{\rho}{\epsilon_0}, \quad (1.150)$$

$$\square \mathbf{A} = \frac{i}{c^2 \epsilon_0}. \quad (1.151)$$

where \square is the *d'Alembertian*:

$$\square \equiv \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta. \quad (1.152)$$

If the field are not time dependent, notice that these potentials obey Poisson's equation ($\rightarrow a1F.6$). In the vacuum (without ρ and i) potentials obey wave equations ($\rightarrow 1.2$).

a1F.9 Polarization. When an electric field is applied to a material, generally, charges in it are displaced. To describe this we introduce the *polarization* vector \mathbf{P} as the density of dipole moments. A dipole moment can be interpreted as a flux (charge \times displacement), so \mathbf{P} is a kind of flux density ($\rightarrow a1A.4$). Thus we have

$$-\operatorname{div} \mathbf{P} = \rho_P. \quad (1.153)$$

where ρ_P is the charge density due to polarization. The minus sign is due to the definition of divergence (the gain of the volume element must be positive. Here the volume element is losing positive charges).

a1F.10 Electrical displacement.

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P} \quad (1.154)$$

is called the *electrical displacement*. With the aid of (1.135) and (1.153) we get

$$\operatorname{div} \mathbf{D} = \rho - \rho_P = \rho_T. \quad (1.155)$$

Here ρ is the charge density (total charge one can detect microscopically), so that $\rho_T = \rho - \rho_P$ is the charge density eliminating the polarization effect. ρ_T is sometimes called the *true charge density*.⁶² Also this charge can increase or decrease due to electrical current, so that the continuity equation for the electrical current reads

$$\frac{\partial \rho_T}{\partial t} = -\text{div } \mathbf{i}. \quad (1.156)$$

a1F.11 Dielectric constant. The relation between \mathbf{D} and \mathbf{E} is generally complicated (may not even be linear). The relation is materials-dependent. For isotropic materials under small electric fields we often assume the following linear *constitutive equation*

$$\mathbf{D} = \epsilon \mathbf{E}. \quad (1.157)$$

where ϵ is called the *dielectric constant*.

a1F.12 Magnetization. The density of magnetic moment \mathbf{M} is called *magnetization*. This corresponds to polarization **a1F.9**. Thus

$$\text{div } \mathbf{M} = -\rho_M \quad (1.158)$$

is understood as the 'excess' magnetic 'charge' (positive monopole density). However, since monopoles do not exist,

$$\text{curl } \mathbf{M} = \mu_0 \mathbf{i}_M \quad (1.159)$$

is physically better, where \mathbf{i}_M is the density of ring currents.

a1F.13 Magnetic field, magnetic permeability. In analogy to \mathbf{E} , the field \mathbf{H} called the *magnetic field* is introduced as

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}. \quad (1.160)$$

where $\mu_0 = 1/\epsilon_0 c^2$ is called the *magnetic permeability of the vacuum*. Often the following constitutive equation is used

$$\mathbf{B} = \mu \mathbf{H}. \quad (1.161)$$

where μ is called the *magnetic permeability*.

a1F.14 Maxwell's equation in material. Replacing (1.125) with (1.156) and repeating the argument to introduce the displacement current (\rightarrow **a1F.3**), we find that the displacement current is $\partial \mathbf{D} / \partial t$. Therefore, Maxwell's equations now read (cf. **a1F.4**)

$$\text{div } \mathbf{D} = \rho_T. \quad (1.162)$$

$$\text{div } \mathbf{B} = 0, \quad (1.163)$$

$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \quad (1.164)$$

$$\text{curl } \mathbf{H} = \mathbf{i} + \frac{\partial \mathbf{D}}{\partial t}. \quad (1.165)$$

⁶²Macroscopically, if the local average of charge distribution is made with respect to the volume element whose boundary does not cut any molecules, then we observe $\rho - \rho_P$.

a1F.15 Supplementary equations. Notice that (1.162) and (1.163) do not contain time. Take the divergence of (1.165) to get

$$\operatorname{div} \mathbf{i} + \frac{\partial \operatorname{div} \mathbf{D}}{\partial t} = 0. \quad (1.166)$$

Using the charge conservation (1.156), this can be rewritten as

$$\frac{\partial}{\partial t}(-\rho_T + \operatorname{div} \mathbf{D}) = 0. \quad (1.167)$$

Analogously, we can get

$$\frac{\partial}{\partial t} \operatorname{div} \mathbf{B} = 0. \quad (1.168)$$

These equations imply that if (1.162) and (1.163) hold once in the past, they are always satisfied. Thus they are called *supplementary equations*.

a1F.16 Ohm's law. We need the constitutive equations (1.157) and (1.161) to close the equations in **a1F.14**. We need one more constitutive equation for \mathbf{i} . The simplest linear law is *Ohm's law*

$$\mathbf{i} = \sigma \mathbf{E}. \quad (1.169)$$

where σ is called the *conductivity*. Combining (1.156) and (1.169) we get the equation that governs the evolution of the charge density in a conductor:

$$\frac{\partial \rho}{\partial t} = -\frac{\sigma}{\epsilon} \rho. \quad (1.170)$$

Notice that this is NOT a diffusion equation (\rightarrow 1.2).⁶³

a1F.17 Electromagnetic wave in matter, telegrapher's equation. Differentiating (1.165) with respect to t and using all three linear constitutive equations (1.157), (1.161) and (1.169), we get

$$\frac{\partial^2 \mathbf{D}}{\partial t^2} + \frac{\sigma}{\epsilon} \frac{\partial \mathbf{D}}{\partial t} = -\frac{1}{\epsilon \mu} \operatorname{curl} \operatorname{curl} \mathbf{D}. \quad (1.171)$$

That is,

$$\frac{\partial^2 \mathbf{D}}{\partial t^2} + \frac{\sigma}{\epsilon} \frac{\partial \mathbf{D}}{\partial t} = \frac{1}{\epsilon \mu} \Delta \mathbf{D} - \frac{1}{\epsilon \mu} \operatorname{grad} \operatorname{div} \mathbf{D}. \quad (1.172)$$

Analogously, we get

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} = \frac{1}{\epsilon \mu} \Delta \mathbf{B} + \frac{1}{\epsilon} \operatorname{curl} \mathbf{i}. \quad (1.173)$$

If we use (1.164) and Ohm's law (1.169), this finally becomes

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} + \frac{\sigma}{\epsilon} \frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\epsilon \mu} \Delta \mathbf{B}. \quad (1.174)$$

⁶³The relaxation time ϵ/σ is of order 10^{-18} sec for usual conductors, but for good insulators this could be 10^8 sec.

For a 1-space problem this equation is called the *telegrapher's equation*. If $\sigma = 0$ (as an insulator), then these equation becomes the wave equation. For a conductor, for which σ is large, these equations become the diffusion equation:

$$\frac{\partial \psi}{\partial t} = \frac{1}{\sigma \mu} \Delta \psi. \quad (1.175)$$

This means that the electromagnetic wave cannot invade deeply into the matter, if σ is large. The characteristic length scale $1/\sqrt{\sigma \mu \omega}$ is called the *skin depth* for the wave of angular frequency ω .