# 18 Separation of Variables –Rectangular Domain –

As the first section devoted to solve second order PDE explicitly, boundary value problems on rectangular regions are considered. The essence of separation of variables is the expansion of the solution into Fourier series.

**Key words**: separation of variables. eigenvalue problem, Poisson's formula.

**Summary**:
(1) How to construct appropriate eigenvalue problems is the key to separation of variable (**18.1-2**).

**18.1 Separation of variables: general strategy.**[270] Suppose we wish to solve a PDE of the form

$$(L_1(x) + L_2(y))u(x, y) = 0. \tag{18.1}$$

where $L_1$ and $L_2$ are linear differential operators (→**1.4**) such that $L_1(x)f(y) = L_2(y)g(x) = 0$ for any function $f$ and $g$. If we assume

$$u(x, y) = X(x)Y(y). \tag{18.2}$$

then

$$YL_1X + XL_2Y = 0. \tag{18.3}$$

or we conclude that

$$(L_1X)/X = -(L_2Y)/Y. \tag{18.4}$$

(1) **[Separating step]**. The LHS of (18.4) depends only on $x$ and the RHS only on $y$. so this equality implies that both sides must be constant:

$$(L_1X)/X = -(L_2Y)/Y = \lambda \tag{18.5}$$

[270]This method was first employed by Daniel Bernoulli around 1755 to solve the wave equation. A more abstract setting and a general theory will be given later (→**23**).

where $\lambda$ is a constant (Sometimes called a *separation constant.*
(2) [**Eigenvalue problem**]. We must split the auxiliary conditions accordingly to obtain two problems which depend only on one of the variables.
If the boundary condition is homogeneous for, say, $x$-direction, then $L_1X = \lambda X$ becomes an *eigenvalue problem*, because the nonzero solution usually exists only for very special values (eigenvalues) of $\lambda$. For each eigenvalue, we have a nontrivial solution denoted by $X_\lambda(x)$.
(3) [**Inhomogeneous boundary problem**]. For such $\lambda$, we must solve the second problem $L_2Y = -\lambda Y$ under appropriate auxiliary conditions, which are usually not homogeneous. Let us denote its solution by $Y_\lambda(y)$.
(4) [**Superposition**]. Since our problem is linear, the superposition principle ($\rightarrow$**1.45**) tells us that $\sum_\lambda X_\lambda(x)Y_\lambda(y)$ is also a solution.
If any smooth function can be expanded as a linear combination of $X_\lambda$ (i.e., if the set $\{X_\lambda\}$ is complete), then we will be able to solve the problem generally.[271] If $\{X_\lambda\}$ is the set of trigonometric functions, the theory of Fourier series ($\rightarrow$**17**) can be fully exploited as Fourier expected ($\rightarrow$**1.6**). In summary, the essence of the separation of variables is to use a problem-adapted Fourier expansion.

**18.2 Illustration: 2D Laplace, Dirichlet**. Solve the following two-dimensional Laplace equation on $[0,1] \times [0,1]$:

$$\partial_x^2\psi + \partial_y^2\psi = 0 \text{ on } [0,1] \times [0,1] \tag{18.6}$$

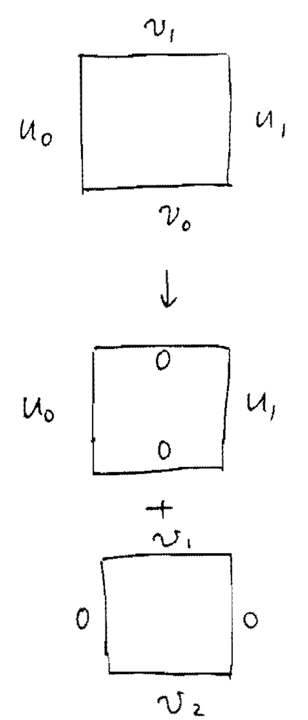
with the inhomogeneous Dirichlet condition

$$\psi(0,y) = u_0(y), \quad \psi(1,y) = u_1(y), \quad \psi(x,0) = v_0(x), \quad \psi(x,1) = v_1(x). \tag{18.7}$$

(1) [**Separating step**] We use the superposition principle ($\rightarrow$**1.4**) to split the problem as

$$\partial_x^2\psi + \partial_y^2\psi = 0 \text{ on } [0,1] \times [0,1], \tag{18.8}$$

$$\psi(0,y) = u_0(y), \quad \psi(1,y) = u_1(y), \quad \psi(x,0) = 0, \quad \psi(x,1) = 0. \tag{18.9}$$



separating step

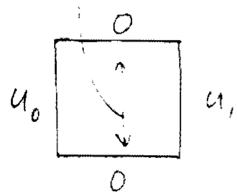
[271] We must be able to show that the series converges uniformly, and we can freely exchange the order of the infinite summation and differentiation, etc. A condition for $\sum_{n=1}^{\infty} u_n(x)$ to be termwisely differentiable is:
(i) $u_n(x)$ is $C^1$.
(ii) the series is pointwise convergent.
(iii) $\sum_{n=1}^{\infty} u_n'(x)$ is uniformly convergent.
Physicists usually do not care about these things, believing that their solutions are always well-behaved (e.g., sufficiently smooth). Indeed, often they are right, and that is why physicists do not pay much attention to mathematicians' careful statements.

and

$$\partial_x^2 \psi + \partial_y^2 \psi = 0 \text{ on } [0,1] \times [0,1], \tag{18.10}$$

$$\psi(0.y) = 0, \quad \psi(1,y) = 0, \quad \psi(x,0) = v_0(x), \quad \psi(x,1) = v_1(x). \tag{18.11}$$

eigenvalue problem direction




The separation of boundary conditions expects eigenvalue problems in all but one coordinate directions. (Further decomposition is possible, but usually there is no need or no merit.) Here we only solve the first set, since the second set is analogous. The solution to the original equation is the sum of the solutions to these split problems.

(2) **[Eigenvalue problem]** (18.8)+(18.9) has a homogeneous boundary condition perpendicular to the $y$ direction (i.e., $\psi = 0$ at $y = 0$ and $y = 1$). Therefore. we should study the eigenvalue problem of $\partial_y^2$ under the homogeneous boundary condition. Solving the eigenvalue problem

$$\frac{d^2u}{dy^2} = -\mu u, \quad u(0) = u(1) = 0, \tag{18.12}$$

we get $\mu = \pi^2 n^2$ for $n = 1, 2, \cdots$ with the corresponding eigenfunction $\sin n\pi y$. We know the totality of such functions is a complete set according to **17.16(1)**. Notice that the sign of the separation constant $\mu$ is dictated by the requirement that (18.12) becomes an eigenvalue problem (the solutions must be oscillatory).

(3) **[Inhomogeneous boundary problem]** Therefore. superposition principle tells us that the solution must have the following form:

$$\psi = \sum_{n=1}^{\infty} Q_n(x) \sin n\pi y. \tag{18.13}$$

where $A_n(x)$ satisfies

$$\frac{d^2 Q_n(x)}{dx^2} = n^2 \pi^2 Q_n(x). \tag{18.14}$$

(4) **[Superposition]** The general solution to (18.14) is $A_n \sinh n\pi x + B_n \cosh n\pi x$. so that the general form of the solution to our problem reads

$$\psi = \sum_{n=1}^{\infty} (A_n \sinh n\pi x + B_n \cosh n\pi x) \sin n\pi y. \tag{18.15}$$

The inhomogeneous boundary conditions at $x = 0$ and $x = 1$ requires

$$\sum_{n=1}^{\infty} B_n \sin n\pi y = u_0(y), \tag{18.16}$$

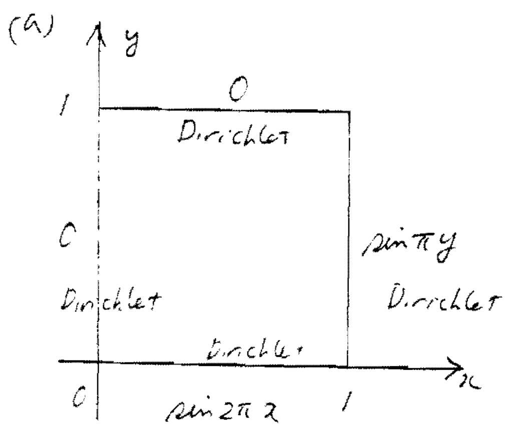
$$\sum_{n=1}^{\infty} (A_n \sinh n\pi + B_n \cosh n\pi) \sin n\pi y = u_1(y). \tag{18.17}$$

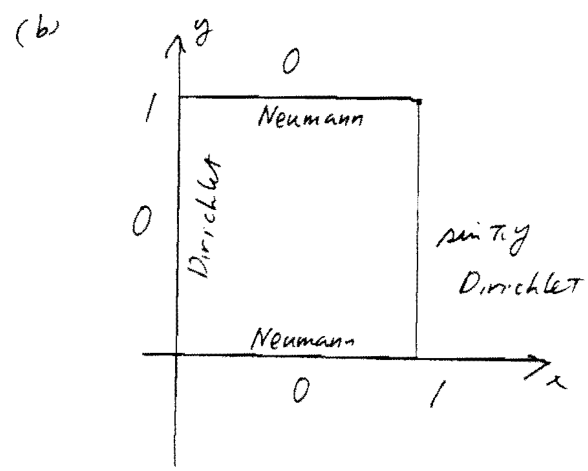
We can determine $B_n$ and $A_n$ from these equations. following **17.13**(1).[272]

**Exercise.**
(1) Consider

$$\frac{\partial^2 u}{\partial t^2} + a^2 \frac{\partial^4 u}{\partial x^4} = 0 \tag{18.18}$$

for $x \in [0. L]$ and $t \geq 0$.
(i) Discuss possible boundary conditions to single out the solution.
(ii) Assume that on the boundary $u$ and $\partial_x^2 u$ vanish and the initial condition is $\partial_t u(x.0) = 0$ and $u(x.0) = f(x)$.
(2) Solve the Laplace equation for the following boundary conditions. Before solving these problems. you must be able to guess the approximate shapes of the solutions.




## 18.3 Laplace equation: Dirichlet condition.

$$\Delta\psi = 0 \text{ on } [0. a_x] \times [0. a_y] \times [0. a_z] \tag{18.19}$$

with the Dirichlet boundary condition

$$\begin{aligned} \psi(0. y. z) &= f_x(y. z). \quad \psi(a_x. y. z) = g_x(y. z). \\ \psi(x. 0. z) &= f_y(x. z). \quad \psi(x. a_y. z) = g_y(x. z). \\ \psi(x. y. 0) &= f_z(x. y). \quad \psi(x. y. a_z) = g_z(x. y). \end{aligned} \tag{18.20}$$

Procedure **18.2(1)** gives. for example, the problem

$$\Delta\psi = 0. \text{ on } [0. a_x] \times [0. a_y] \times [0, a_z], \tag{18.21}$$

$$\psi(0. y. z) = \psi(a_x. y. z) = \psi(x. 0. z) = \psi(x. a_y. z) = 0, \tag{18.22}$$

$$\psi(x. y. 0) = f_z(x. y). \ \psi(x. y. a_z) = g_z(x, y). \tag{18.23}$$

[272] Of course. $u_0$ and $u_1$ must be Fourier-expandable (→**17.5**).

($x, y, z$ in the boundary conditions must be in the domain of the problem, of course.) Thus the relevant eigenvalue problem analogous to that appearing in **18.2(2)** is

$$(\partial_x^2 + \partial_y^2)u = -\mu^2 u \tag{18.24}$$

with the homogeneous Dirichlet boundary condition $u(0,y) = u(a_x, y) = u(x,0) = u(x,a_y) = 0$. This can be separated further, and the superposition principle asserts

$$\psi = \sum_{m,n}(A_{m,n}\sinh\mu_{m,n}z + B_{m,n}\cosh\mu_{m,n}z)\sin\frac{m\pi x}{a_x}\sin\frac{n\pi y}{a_y}. \tag{18.25}$$

where $\mu_{m,n}^2 = (m\pi)^2/a_x^2 + (n\pi)^2/a_y^2$. The unknown constants $A_{m,n}$ and $B_{m,n}$ are fixed with the aid of **17.16(1)**.

If, for example, $a_x$ is not finite, the summation over $m$ in (18.25) becomes an integral (*Fourier sine transform*) (→**32A.8**).

The full solution to our problem is obtained by summing all three solutions to inhomogeneous problems in the $x$, $y$ and $z$ directions resulted from the splitting.

**Exercise.**
Consider the Laplace equation on a square $[0,L]\times[0,L]$ with the boundary conditions[273]

$$u(0,y) = 0,\ \ u(L,y) = A\sin(2\pi x/L),\ \ u(x,0) = 0,\ \ u(x,L) = B\sin(2\pi x/L). \tag{18.26}$$

### 18.4 Laplace equation: Neumann condition.

$$\Delta\psi = 0 \text{ on } [0,a_x]\times[0,a_y]\times[0,a_z] \tag{18.27}$$

with the Neumann boundary condition

$$\begin{aligned}\partial_x\psi(0,y,z) &= f_x(y,z),\ \ \partial_x\psi(a_x,y,z) = g_x(y,z),\\ \partial_y\psi(x,0,z) &= f_y(x,z),\ \ \partial_y\psi(x,a_y,z) = g_y(x,z),\\ \partial_z\psi(x,y,0) &= f_z(x,y),\ \ \partial_z\psi(x,y,a_z) = g_z(x,y).\end{aligned} \tag{18.28}$$

**18.2**(1) gives, for example, the problem

$$\Delta\psi = 0 \text{ on } [0,a_x]\times[0,a_y]\times[0,a_z], \tag{18.29}$$

$$\partial_x\psi(0,y,z) = \partial_x\psi(a_x,y,z) = \partial_y\psi(x,0,z) = \partial_y\psi(x,a_y,z) = 0, \tag{18.30}$$

$$\partial_z\psi(x,y,0) = f_z(x,y),\ \ \partial_z\psi(x,y,a_z) = g_z(x,y). \tag{18.31}$$

[273] L91.

Thus the relevant eigenvalue problem is

$$(\partial_x^2 + \partial_y^2)u = -\mu^2 u \tag{18.32}$$

with the homogeneous Neumann boundary condition $\partial_x u(0, y) = \partial_x u(a_x, y) = \partial_y u(x, 0) = \partial_y u(x, a_y) = 0$. This can be separated further, and eventually we get

$$\psi = \sum_{m,n} (A_{m,n} \sinh \mu_{m,n} z + B_{m,n} \cosh \mu_{m,n} z) \cos \frac{m\pi x}{a_x} \cos \frac{n\pi y}{a_y}. \tag{18.33}$$

where $\mu_{m,n}^2 = (m\pi)^2/a_x^2 + (n\pi)^2/a_y^2$. The unknown constants $A_{m,n}$ and $B_{m,n}$ are fixed with the aid of **17.16(2)**.

If the region is not bounded. then the summation over $m$ and/or $n$ becomes integration (*Fourier cosine transform*→**32A.8**).

**18.5 Diffusion equation.** Consider

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \tag{18.34}$$

for $x \in (0, l)$ and for $t > 0$ with the initial condition $u(x, 0) = A$ for $x \in (0, l)$ and the boundary condition $u(0, t) = B$ and $u(l, t) = C$ for $t > 0$. where $A, B, C$ are constants.[274]
A clever (and standard) trick is to convert the problem to a homogeneous boundary value problem by introducing

$$v = u - \left(\frac{C - B}{l}x + B\right). \tag{18.35}$$

We have

$$\frac{\partial v}{\partial t} = D\frac{\partial^2 v}{\partial x^2} \tag{18.36}$$

for $x \in (0, l)$ and for $t > 0$ with the initial condition $v(x, 0) = (B - C)x/l + A - B$ for $x \in (0, l)$ and the boundary condition $v(0, t) = 0$ and $v(l, t) = 0$ for $t > 0$. Thus we may assume the following solution

$$v(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi}{l} x. \tag{18.37}$$

Notice that the above method works even when $A, B, C$ are time-dependent.

[274]With this delicate choice of the space-time positions to impose the auxiliary conditions. we need not worry about the compatibility among $A, B, C$.

**Exercise.**
(1) Find the solution of the 1d diffusion equation for $x \in [0, \pi]$ and $t \geq 0$ with a homogeneous Neumann condition and the initial condition $u(x, 0) = \sin^2 x$.
(2) Find the solution of the 1d diffusion equation for $x \in [0, \pi]$ and $t \geq 0$ with the initial condition $u(x, 0) = x$, and a homogeneous Dirichlet boundary condition.
(3) Solve the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{18.38}$$

on $[0, 1]$ with the initial condition $u(x, 0) = \sin(\pi x/2)$ and the boundary conditions $u(0, t) = 0$ and

$$\left.\frac{\partial u}{\partial x}\right|_{x=1} = -\frac{1}{\nu} u(1, t), \tag{18.39}$$

where $\nu$ is a constant (i.e., a homogeneous Robin condition).
[Hint: Let $\mu_n$ be the $n$-th zero of $\tan x + \nu x = 0$ arranged in the increasing order. Then,

$$\int_0^1 \sin(\mu_n x) \sin(\mu_m x) dx = \delta_{m,n} \frac{1 + \nu \cos^2 \mu_n}{2}. \tag{18.40}$$

]
(6) There is a thermally isolated ring of radius $\ell$ whose thermal diffusivity is $D$. The initial temperature distribution is given by

$$T(0, x) = T_0 \cos \frac{2x}{\ell}, \tag{18.41}$$

where $x$ is the coordinate along the ring. Find $T(t, x)$.
(7) Considering the diffusion equation on an appropriate interval with a homogeneous Dirichlet condition, show

$$\sum_{n=-\infty}^{\infty} \cos(2\pi n x) e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{-(x-n)^2/t}. \tag{18.42}$$

(cf. the Poisson sum formula **32C.2**)
(8) There is a thin rod of length $\ell$ occupying between $x = 0$ and $x = \ell$ whose thermal diffusivity is $D$. The temperature at one end, say, at $x = 0$ is given as a function of time as $T(x = 0, t) = T_0 e^{-\alpha t}$ ($\alpha > 0$, constant), and the other end is maintained at $T_0$ for all $t > 0$. Initially the temperature is given by $T(x, 0) = T_0 \sin(3\pi x/\ell)$. Find the temperature field for $t > 0$.

**18.6 Obtaining Poisson's formula.** Consider the Laplace equation on a disk of radius $a$ centered at the origin (cf. **2D.10**):

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{18.43}$$

The boundary condition is a Dirichlet condition: $u(a, \theta) = f(\theta)$ which is a smooth periodic function with period $2\pi$.

We can of course follow the honest separation strategy, but we may assume that $u$ can be Fourier expanded (as the reader can guess (→**1.13**. **13C.6**(2). **5.6**, **16D.10**), harmonic functions are very smooth (→**29.10**). so we can do this with confidence) as

$$u(r,\theta) = \frac{A_0(r)}{2} + \sum_{n=1}^{\infty}[A_n(r)\cos n\theta + B_n(r)\sin n\theta]. \tag{18.44}$$

Putting this into the equation, we obtain the following ODE for the coefficients:

$$\begin{aligned} \frac{d^2A_n}{dr^2} + \frac{1}{r}\frac{dA_n(r)}{dr} - \frac{n^2}{r^2}A_n(r) &= 0 \ (n = 0, 1, 2, \cdots), \quad (18.45) \\ \frac{d^2B_n}{dr^2} + \frac{1}{r}\frac{dB_n(r)}{dr} - \frac{n^2}{r^2}B_n(r) &= 0 \ (n = 1, 2, \cdots). \quad (18.46) \end{aligned}$$

From this we get the following solutions that are finite at the origin (→**11B.14**):

$$A_n(r) = A_n r^n, \ B_n(r) = B_n r^n. \tag{18.47}$$

where $A_n$ and $B_n$ are constants. With the aid of the boundary condition at $r = a$. these coefficients are uniquely fixed as (→**17.1**)

$$A_n = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\cos n\phi\, d\phi. \ B_n = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\sin n\phi\, d\phi. \tag{18.48}$$

Hnece. our solution (18.44) reads:

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\left(1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\phi - \theta)\right) d\phi \tag{18.49}$$

or summing the series. we finally obtain *Poisson's formula* (for $r < a$) (→**16D.8**):

$$u(r,\theta) = \frac{1}{2}\int_0^{2\pi} f(\phi)\frac{a^2 - r^2}{a^2 - 2ar\cos(\phi - \theta) + r^2}d\phi. \tag{18.50}$$

**18.7 1-space wave equation.** Let us consider, as an example, the following 1-wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x} \tag{18.51}$$

for $x \in [0, a]$ and $t \geq 0$, where $c$ is a positive constant. The auxiliary conditions are:
the initial condition: $u(x, 0) = f(x)$. $\partial_t u(x, 0) = 0$ for $x \in [0, a]$,

the boundary conditions: $u(0,t) = u(a,t) = 0$ for $t \geq 0$.
We know that the solution, if exists, is unique for smooth initial conditions (→**1.19**).
Again, we can immediately proceed as in **18.6** to assume the solution in the following form (cf. **17.16(1)**, **1.6**):

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{a}\right). \tag{18.52}$$

Now the boundary conditions have been taken into account. The initial condition requires that

$$a_n(0) = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx, \tag{18.53}$$

and $a_n'(0) = 0$. The wave equation is translated into a set of infinite ODEs:

$$\frac{d^2 a_n(t)}{dt^2} = -c^2 \frac{n^2\pi^2}{a^2} a_n(t). \tag{18.54}$$

Thus, we get

$$a_n(t) = a_n(0) \cos\left(\frac{cn\pi t}{a}\right). \tag{18.55}$$

Recover d'Alembert's formula in **2B.3** from this.

**Exercise.**
(1) Find the solution for

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} + u \tag{18.56}$$

for $x \in [0,\pi]$ with a homogeneous Dirichlet boundary condition and the initial condition $u(x,0) = \sin x$ and $\partial_t u(x,0) = 0$.
(2) Solve 1-d wave equation with the wave speed $c$ under the initial condition

$$u(0,x) = \sin\frac{3\pi}{2\ell}x, \ \partial_t u(0,x) = 0 \tag{18.57}$$

with the boundary condition $u(t,0) = 0$ and $\partial_x u(t,0) = 0$ for $t > 0$ (i.e., $x = 0$ is fixed and $x = \ell$ is open).
(3) There is a string of length $l$ whose both ends are fixed. A concentrated force $A\sin\omega t$ is applied at $x = c$ on the string. Let the density of the string be $\rho$ and its tension $T$. Then, the speed is given by $c^2 = T/\rho$ (→**a1D.11**).
(4) A uniform flexible chain is hanging along the z-axis. Let $u$ be the displacement of the chain in the $xz$-plane, hanging from the origin. Then

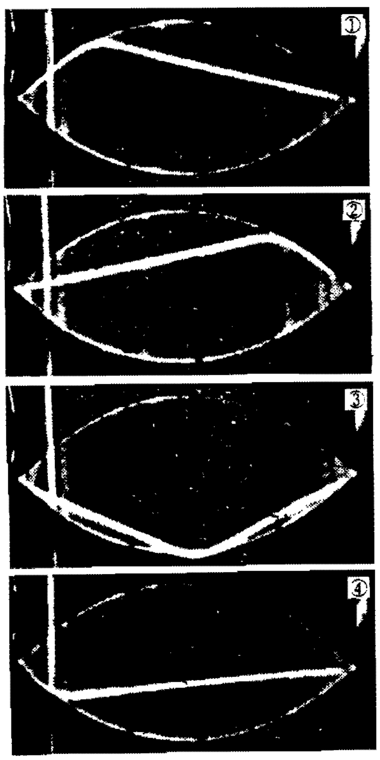
$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right). \tag{18.58}$$

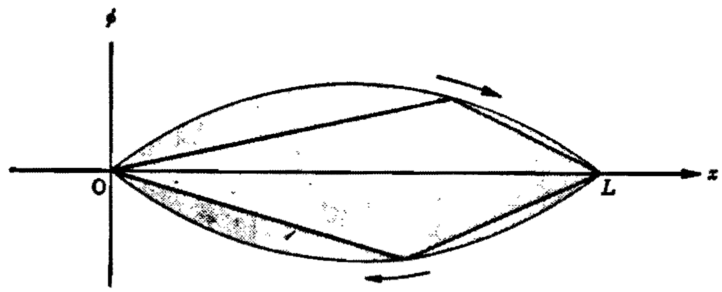
Solve this with the aid of the separation of variables. The equation for the spatial function becomes Bessel's equation (→**27A.1**) in this case.

**Dsicussion.**
The shape of the string of a violin at time $t$ takes the form in the figure; it looks like a ˆ . The breaking pont moves with a constant velocity and its trajectory is on a parabola (see the photos). The formula for the shape is

$$\phi(x,t) = C\sum_{n=1}^{\infty}\frac{1}{n^2}\sin k_n w \sin\omega_n t, \tag{18.59}$$

where $k_n = \pi n/L$ and $\omega_n = ck_n$ with the wave speed $c$. $C$ is a constant dependent on the loudness of the sound. Demonstrate the statement about the shape (esp. the motion of the breaking point) from this formula




vibration of violin string