16 Green's Function for PDE – Elementary Approach

Green's idea is illustrated with simple (but representative) examples. We first construct free space Green's functions. With the aid of the image source method, we construct Green's functions for simple domains as well. Dirichlet boundary value problems for the Laplacian are discussed toward the end.

Key words: free space Green's function. method of descent. method of images. reflection principle. image source, conformal map. Green's theorem

Summary:
(1) Free space Green's functions can be constructed as spherically symmetric solutions (**16A.2-4**. **16B.4**. **16C.1**).
(2) Inhomogeneous equations can be solved with the aid of Green's functions (**16A.21**. **16D.9**).
(3) Green's functions for simple domains may be constructed with the aid of the method of images (**16A.7-17**. **16B.9**. **16C.5**).
(4) In 2-space. conformal maps can be fully exploited to construct Green's function for the Laplace equation. Neumann problems can be reduced to Dirichlet problems (→**16D.2**). and the latter can be mapped on a Dirichlet problem on the unit disk (**16D.3**. **16D.11**. **16D.12**). Look up 'stylebooks' of conformal maps.

16.A Green's Function for Laplace Equation

16A.1 Free space Green's function for Laplace equation. We have already seen (→**14.20**) that in 3-space

$$-\Delta G(\boldsymbol{x}|\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y}) \tag{16.1}$$

where

$$G(\boldsymbol{x}|\boldsymbol{y}) = \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}. \tag{16.2}$$

This is a fundamental solution (→**14.2**) of the Laplace equation. and is the Coulomb potential created by a 'unit' charge located at $\boldsymbol{y}$. We can say the Coulomb potential is the Green's function for the Laplace equation with the condition that the solution vanishes at infinity ($\to$**14.23**(4)).

16A.2 Spherical symmetric solution of Laplace equation. Suppose $f(r)$ is spherically symmetric, where $r \equiv |\boldsymbol{x}|$. Then, in d-space (see Remark below)

$$\Delta f(r) = \frac{1}{r^{d-1}}\frac{d}{dr}r^{d-1}\frac{d}{dr}f(r). \tag{16.3}$$

If we assume that the solution vanishes at infinity, for $d > 2$ a spherical symmetric solution to $\Delta f = 0$ is given by

$$f \propto \frac{1}{r^{d-2}}. \tag{16.4}$$

The solution is singular at the origin, and we can show $\Delta f(r) \propto \delta(\boldsymbol{x})$ ($\to$**16A.3**, cf. **14.20**).

Remark. There are two ways to obtain (16.3). One is to use the d-space version of **2D.10**. Let $q_1 = r$ and the remaining angular coordinates to be $q_2, \cdots, q_d$. Then ($\to$**2D.3**) $h_1 = 1$ and the other h_j's are all proportional to r. It is easy to generalize **2D.7** and **2D.9** to d-space, so the Laplacian can be written as

$$\Delta = \frac{1}{h_1\cdots h_d}\sum_j \frac{\partial}{\partial q_j}\frac{h_1\cdots h_d}{h_j^2}\frac{\partial}{\partial q_j}. \tag{16.5}$$

If this is applied to a function of r only, then we have only to pay attention to the $j = 1$ term. Since we need not worry about the angular coordinates, what matters is the fact that $h_1 \cdots h_d \propto r^{d-1}$. We obtain (16.3).

A cleverer method, is to apply

$$\Delta = \sum \frac{\partial^2}{\partial x_i^2} \tag{16.6}$$

to $f(r)$. The chain rule gives us

$$\frac{\partial}{\partial x_i}f(r) = f'(r)\frac{\partial r}{\partial x_i} = f'(r)\frac{x_i}{r}. \tag{16.7}$$

Here, $\partial r/\partial x_i = x_i/r$ is used, which can be best obtained from $rdr = \boldsymbol{x}\cdot d\boldsymbol{x}$. Differentiate this again, and we get the desired formula.

Exercise.
Let D be the unit ball centered at the origin in $\boldsymbol{R}^3$. Suppose

$$\Delta u = 1 \tag{16.8}$$

inside the ball, and $u = 1$ on ∂D.
(1) Show that if there is a solution, it is unique.

(2) Find a spherically symmetric solution.

16A.3 Demonstration of $\Delta(1/r^{d-2}) \propto \delta(\boldsymbol{x})$. To demonstrate this, we go to the basic: let φ be a test function (→**14.8**)

$$\begin{aligned}\langle \Delta f.\varphi\rangle &= \int \left(\frac{1}{r^{d-1}}\frac{\partial}{\partial r}r^{d-1}\frac{\partial}{\partial r}r^{2-d}\right)\varphi(\boldsymbol{x})r^{d-1}drd\omega, &(16.9)\\ &= S_{d-1}\int\left(\frac{d}{dr}r^{d-1}\frac{d}{dr}r^{2-d}\right)\overline{\varphi}(r)dr. &(16.10)\end{aligned}$$

Here $d\omega$ is the solid angle element in d-space (i.e., the area element of the unit $(d-1)$-sphere). S_{d-1} is the area (volume) of $(d-1)$-unit sphere. and the overline denotes the angular average just as in **14.20**. Continuing the calculation. we get

$$\langle \Delta f.\varphi\rangle = -S_{d-1}\int r^{d-1}\left(\frac{d}{dr}r^{2-d}\right)\frac{d\overline{f}}{dr}dr = (2-d)S_{d-1}\overline{f}(0). \quad (16.11)$$

Since $\overline{f}(0) = f(\mathbf{o})$. we have demonstrated

$$-\Delta r^{2-d} = (d-2)S_{d-1}\delta(\boldsymbol{x}). \quad (16.12)$$

16A.4 Coulomb potential in d-space. This tells us the general form of the free space Green's function of the d-Laplace equation:

$$G(\boldsymbol{x}|\boldsymbol{y}) = \frac{1}{(d-2)S_{d-1}|\boldsymbol{x}-\boldsymbol{y}|^{d-2}}. \quad (16.13)$$

This is the d-Coulomb potential. Here the area of $(d-1)$-sphere (the 'skin' of d-ball) is given by

$$S_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \quad (16.14)$$

which is obtained from the volume $V(r)$ of the d-ball of radius r (→**9.10** Exercise (B)) as

$$S_{d-1} = \left.\frac{dV(r)}{dr}\right|_{r=1}. \quad (16.15)$$

(16.13) for $d=3$ is. of course. in agreement with the 3-space result.

For $d=2$. if we ignore 'infinite' constants. taking the $d\to 2$ limit in the formula. we arrive at the correct formula for 2-space:

$$G(\boldsymbol{x}|\boldsymbol{y}) = -\frac{1}{2\pi}\ln|\boldsymbol{x}-\boldsymbol{y}|. \quad (16.16)$$

Obtain this formula directly.

Exercise.
Express the 2-space Green's function (16.16) in the polar coordinates. Then (→**16D.10**), show

$$G(\boldsymbol{x}|\boldsymbol{y}) = -\frac{1}{2\pi}\log r_> + \sum_{m=1}^{\infty} \frac{\cos m(\phi_x - \phi_y)}{m}\left(\frac{r_<}{r_>}\right)^m, \tag{16.17}$$

where ϕ_x (resp., ϕ_y) is the polar coordinate of $\boldsymbol{x}$ (resp., $\boldsymbol{y}$), and $r_> \equiv \max\{|\boldsymbol{x}|, |\boldsymbol{y}|\}$, and $r_< \equiv \min\{|\boldsymbol{x}|, |\boldsymbol{y}|\}$. An ananolgous formula for 3-space can be obtained in terms of spherical harmonics as seen in **26A.14**.

16A.5 Method of descent. In d-space, if we assume that the system is translationally symmetric along one coordinate direction, then the cross-section of this solution perpendicular to this direction should be indistinguishable from the $(d-1)$-space result. That is, averaging over one direction of d-space results gives $(d-1)$-space results. This method to obtain lower dimensional results is called the *method of descent.*

If we integrate the d-Coulomb potential (16.13) over x_d, then we should get the $(d-1)$-Coulomb potential, because $\int dx_d \delta_d(\boldsymbol{x}-\boldsymbol{y}) = \delta_{d-1}(\boldsymbol{x}'-\boldsymbol{y}')$, where $\boldsymbol{x}'$ is $\boldsymbol{x}$ with its x_d-component suppressed. The best way to demonstrate this is to use the exponentiation trick explained in **9.10** and to integrate over x_d:

$$\int_{-\infty}^{+\infty} dx_d \frac{1}{(a^2+x_d^2)^{(d-2)/2}} = \int_0^{+\infty} dt \int_{-\infty}^{+\infty} dx_d \frac{1}{\Gamma((d-2)/2)} t^{(d-2)/2-1} e^{-(a^2+x_d^2)t}, \tag{16.18}$$

$$= \frac{\sqrt{\pi}}{\Gamma((d-2)/2)} \int_0^{+\infty} dt\, t^{(d-3)/2-1} e^{-a^2 t} \tag{16.19}$$

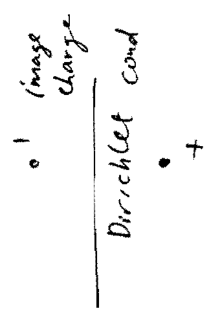
$$= \frac{\sqrt{\pi}}{\Gamma((d-2)/2)} \Gamma\left(\frac{d-3}{2}\right) \frac{1}{a^{d-3}}, \tag{16.20}$$

where $a^2 = x_1^2 + \cdots + x_{d-1}^2$. With the aid of this and the fundamental functional relation **9.2** of the Gamma function, we eventually obtain the $(d-1)$-Coulomb potential. This is a good exercise.

16A.6 Green's function in (semi)bounded space. If the domain of the equation is (semi)bounded, then to satisfy the boundary conditions is nontrivial in general. However, if the domain enjoys nice symmetries, there is a clever way – method of image sources. Basically, we tessellate the space by the copy of the domain with appropriate sign change of the source terms (called *image sources*).

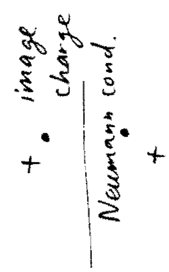
16A.7 Method of images I. Half space. The Green's function

$G(x, y, z|x', y', z')$ for 3-Laplace equation is the electrostatic potential at (x, y, z) due to a point charge at (x', y', z') with a suitable potential values specified on the boundary of the region. The Green's function for the Laplace equation on the $x > 0$ half space with a (homogeneous) Dirichlet boundary condition is given by



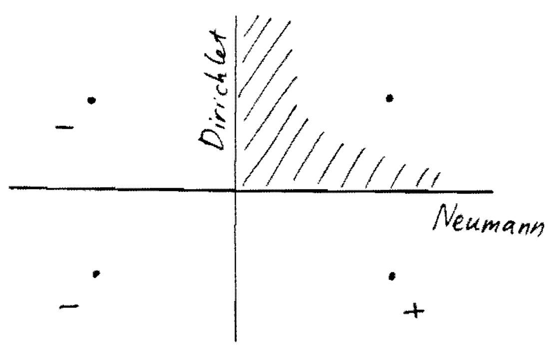
$$G_D(x, y, z; x', y', z') = \frac{1}{4\pi}\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x+x')^2+(y-y')^2+(z-z')^2}}\right] \quad (16.21)$$

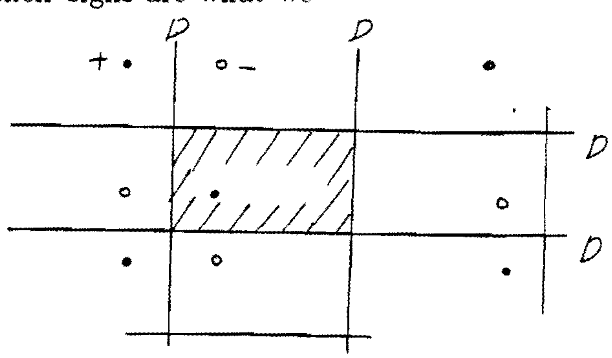
Here the source term is at (x', y', z'), and its image source is at $(-x', y', z')$. To maintain the zero potential condition at $x = 0$, the effects of the both sources must cancel exactly on the yz-plane. Hence, the image source must be -1.
If the boundary condition is the homogeneous Neumann condition at $x = 0$, then to kill the gradient on the yz-plane, the image charge must be $+1$. Hence, the Neumann function (= Green's function with a Neumann condition) for semiinfinite space reads



$$G_N(x, y, z; x', y', z') = \frac{1}{4\pi}\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} + \frac{1}{\sqrt{(x+x')^2+(y-y')^2+(z-z')^2}}\right] \quad (16.22)$$

16A.8 Method of images II. More complicated cases. As we have seen the locations of image sources and their signs are what we need. Several examples are illustrated.



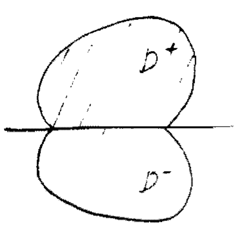


Warning. If the region under consideration is <u>bounded</u> (i.e., enclosed in a finite sphere whose center is located at the origin), then the Neumann condition Green's function (Neumann's function) for the Laplace equation requires an extra care (→**1.19(3)**), so we will NOT discuss this case here. See **36.7** and **37.9**.

Exercise.
(1) Find the Green's function for the Laplace equation on the infinite strip $(-\infty, +\infty) \times [0, \pi]$ with a homogeneous Dirichlet condition. [The reader must impose a further condition to single out the solution (→**1.19** Discussion (2)).
(2) Find the Green's function for the Laplace equation on the half 3-space defined by $x > a$, where a is a constant.
(3) Find the electric potential in 3-space with $x = 0$ and $y = 0$ maintained at zero potential and the charge Q placed at $(x', y', 0)$.

16A.9 Harmonicity and symmetry. Green's functions for the Laplace equation are harmonic (→**2C.11**) except at their singularities. If we study the method of images, the keys are
(1) harmonicity is preserved by reflection,
(2) charges are mirrored onto charges.
Hence, the essence of the method of images is that there are special symmetry operations preserving harmonicity.

16A.10 Reflection principle. Let D be a region such that if $(x_1, x_2, \cdots, x_{d-1}, x_d) \in D$, then $(x_1, x_2, \cdots, x_{d-1}, -x_d) \in D$. Write D^+ for the subset of D for $x_d > 0$ and $D^- = D \setminus D^+$. If u is harmonic (→**2C.11**) on D^+ and $u = 0$ for $x_d = 0$ (that is, on the boundary between D^+ and D^-), then the function g defined as



$$g(x_1, \cdots x_{d-1}, x_d) = u(x_1, \cdots x_{d-1}, x_d) \quad \text{on } D^+ \qquad (16.23)$$
$$= -u(x_1, \cdots x_{d-1}, -x_d) \quad \text{on } D^- \qquad (16.24)$$

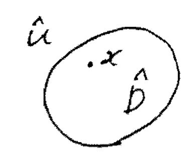
is a harmonic function on the whole D. □
[Demo] Inside D^- g is obviously harmonic. Therefore, we have only to take care of g near the boundary between D^+ and D^-. This is easy to show if we use the converse of the mean-value theorem **29.4**.

16A.11 Conformal mapping. A *conformal transform* (*Kelvin transform*) $\hat{u}$ of a function u is given in d-space by

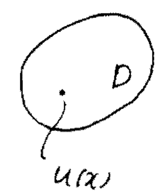
$$\hat{u}(\boldsymbol{x}) = |\boldsymbol{x} - \boldsymbol{a}|^{2-d} u((\boldsymbol{x} - \boldsymbol{a})/|\boldsymbol{x} - \boldsymbol{a}|^2). \qquad (16.25)$$

where $\boldsymbol{a}$ is a constant. This is the composition of translation $\boldsymbol{x} \to \boldsymbol{x} - \boldsymbol{a}$ and inversion $\boldsymbol{x} \to \boldsymbol{x}/|\boldsymbol{x}|^2$. Notice that this transformation makes the universe 'inside out': big scales become small and vice versa, and keeps the unit sphere centered at $\boldsymbol{a}$ intact.[223] That is, the Kelvin transformation makes the universe inside out.

16A.12 Harmonicity is conformal invariant. Let $\hat{D} \subset \boldsymbol{R}^d \setminus \{a\}$ and $D = \{\boldsymbol{x} \,|\, \boldsymbol{x} = (\boldsymbol{y} - \boldsymbol{a})/|\boldsymbol{y} - \boldsymbol{a}|^2, \boldsymbol{y} \in \hat{D}\}$. If u is harmonic on D, then $\hat{u}$ given by (16.25) is again harmonic on $\hat{D}$. □
Note that the Kelvin transformation transforms a harmonic function on a ball centered at $\boldsymbol{a}$ to a harmonic function defined on the domain outside the ball. Remember that a half space can be interpreted as a sphere with an infinite radius. For 2-space see **16D**.



16A.13 Demonstration of conformal invariance of harmonicity. To show **16A.12** we have only to demonstrate $\Delta\hat{u}(x) = 0$ honestly. However, a clever organization of calculation is desirable. We may set $\boldsymbol{a} = 0$ without any loss of generality. First, notice that



$$\frac{\partial^2}{\partial x_i^2} fg = \frac{\partial^2 f}{\partial x_i^2} g + 2\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i} + f\frac{\partial^2 g}{\partial x_i^2}. \tag{16.26}$$

and

$$\frac{\partial}{\partial x_i} u\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|^2}\right) = u_j\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|^2}\right)\left(\frac{\delta_{ij}}{|\boldsymbol{x}|^2} - 2\frac{x_i x_j}{|\boldsymbol{x}|^4}\right). \tag{16.27}$$

where $u_j = \partial u/\partial x_j$. We have only to show the formula for $\boldsymbol{x} \neq 0$. Note that $\partial_i^2 |\boldsymbol{x}|^{2-d} = 0$. The rest is left for the reader.

16A.14 Method of images III General case. The conformal invariance **16A.12** and the reflection principle **16A.11** provide a special method to solve Poisson's equation; actually, we have already used the reflection principle repeatedly (→**16A.7**. **16A.8**). It is often easier to solve a problem without boundary conditions at finite distance. Use **16A.11** and **16A.12** to extend the domain with boundary conditions to the whole space. An important point is that a singularity is conformally mapped to a singularity. That is. the images of charges must be charges (*image charge*).

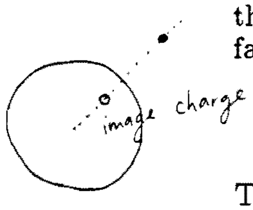
16A.15 Sphere, Dirichlet condition. Physically this is the problem of finding electric potential in the sphere surrounded by a grounded conducting sphere. A charge of q is at $(r, 0, 0)$ ($a > r > 0$, that is, inside the sphere) and a grounded sphere of radius a is centered at the origin. We use the conformal invariance of harmonicity (→**16A.12**).

[223] Actually, the definition of conformal maps should be more general, but if n is not even, then the combination of the Kelvin transformation and affine transformations exhaust the conformal transformation. For 2-space see **16D**.

Consider the conformal map which makes the sphere inside out:

$$\boldsymbol{r} \to a^2\boldsymbol{r}/r^2. \tag{16.28}$$

The image of $(r, 0, 0)$ due to this map is at $(a^2/r, 0, 0)$. Therefore the mapped field must have a singularity at this point. This means that there is a charge (image charge) q' there. This is determined by the fact that at $(a, 0, 0)$ the field must be zero:



$$\frac{q}{a^2/r - a} + \frac{q'}{a - r} = 0. \tag{16.29}$$

That is, $q' = -aq/r$. Thus, we have, inside the sphere

$$\phi = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(x-r)^2+y^2+z^2}} - \frac{a}{r\sqrt{(x-a/r)^2+y^2+z^2}}\right). \tag{16.30}$$

From this the Green's function for the sphere under the Dirichlet condition is obtained.

Exercise.
Construct the Green's function for a disk with a Dirichlet condition. Then, compare the result with the one obtained with the aid of conformal maps (→**16D.6**, **16D.11**).

16A.16 Charge outside conducting sphere, not grounded. Suppose the charge is outside the sphere. In this case the net charge induced on the sphere must be zero due to charge conservation. If we put q' given in **16A.15** at a^2/r and $-q'$ at the origin, all the boundary conditions are satisfied (See also Jackson, Section 2.6).

16A.17 Method of images for dielectric materials. Method of images can be generalized to the cases with dielectric materials, but only for the cases with plane surfaces (not applicable to dielectric spheres). See Jackson.

16A.18 How to use Green's function (homogeneous Laplace case). If the boundary condition is homogeneous, then linear inhomogeneous PDE can be solved in terms of its Green's function as outlined in **1.8**. A typical problem is to solve Poisson's equation under homogeneous boundary condition

$$-\Delta\psi = f(\boldsymbol{x}), \tag{16.31}$$

which is solved as

$$\psi(\boldsymbol{x}) = \int_D d\boldsymbol{y} G(\boldsymbol{x}|\boldsymbol{y}) f(\boldsymbol{y}). \tag{16.32}$$

where D is the domain of the problem.

16A.19 Green's formula. Let $D \subset \boldsymbol{R}^d$ be a bounded region, and u and v be C^2-functions defined on the closure of D. Then,

$$\int_D (v\Delta u + grad\, u \cdot grad\, v)d\tau = \int_{\partial D} v\, grad\, u \cdot d\boldsymbol{S}, \qquad (16.33)$$

and

$$\int_D (v\Delta u - u\Delta v)d\tau = \int_{\partial D} (v\, grad\, u - u\, grad\, v) \cdot d\boldsymbol{S}. \qquad (16.34)$$

[Demo] (16.33) follows immediately from $div(u\, grad\, v) = grad\, u \cdot grad\, v + u\Delta v$, and Gauss' theorem (→**2C.13**). The second formula (16.34) is obvious from (16.33).

Exercise.
Let $D \subset \boldsymbol{R}^d$ be a region on which u is harmonic. Show

$$\int_{\partial D} grad\, u \cdot d\boldsymbol{S} = 0. \qquad (16.35)$$

16A.20 Symmetry of Green's function. (See **36.4** and **35.2** also) Let $G(\boldsymbol{x}|\boldsymbol{y})$ be the Green's function on the domain D with the homogeneous Dirichlet condition. Then,

$$G(\boldsymbol{x}|\boldsymbol{y}) = G(\boldsymbol{y}|\boldsymbol{x}). \qquad (16.36)$$

To demonstrate this set $u = G(\boldsymbol{z}|\boldsymbol{x})$ and $v = G(\boldsymbol{z}|\boldsymbol{y})$ in Green's formula (the integration is over

16A.21 Solution to Dirichlet problem in terms of Green's function. (See the warning in **16A.8**.) The solution to the following Dirichlet problem on an open region D

$$-\Delta u = \varphi, \quad u|_{\partial D} = f, \qquad (16.37)$$

where φ and f are integrable functions, is given by

$$u(x) = \int_D G(x|y)\varphi(y)dy - \int_{\partial D} f(y)\partial_{n(y)}G(x|y)d\sigma(y). \qquad (16.38)$$

Here $\partial_{n(y)}$ is the outward normal derivative at y, τ is the volume element, and σ is the surface volume element.

The formula easily follows from Green's formula in **16A.19** with u being the solution and v being the Green's function for the problem. In this way with the Green's functions we can solve inhomogeneous boundary condition problems.

Discussion.
(A) The second term in (16.38) is understood as the electric potential made by an electrical double layer.
(B) The surface integral in (16.38) can be written in the following remarkable form

$$\frac{1}{2\pi}\int_{\partial\Omega} d\omega f(y). \tag{16.39}$$

where ω is the solid angle of the surface element (at y) seen from x.[224]
(C) Derive the following *Kirchhoff's formula*: For $x \in D$

$$u(x) = \int_D G_0(x|y)\varphi(y)dy - \int_{\partial D} d\sigma(y)\left[u(y)\partial_{n(y)}G(x|y) - \partial_{n(y)}u(y)G_0(x|y)\right], \tag{16.40}$$

where G_0 is the free space Green's function given in **16A.4**. This formula is meaningful even when x is outside D. Show that this is zero if $x \notin D$. Because the first term in the formula is smooth, the discontinuity comes from the surface integral. The formula cannot be used to obtain the solution because we usually do not know u and its derivative on the surface simultaneously. Recall that the Cauchy problem of the Laplace equation is generally not well posed (→**28.3** for well-posedness).

16A.22 Green's function for more general domain. We will discuss this in **36** and **37**.

16.B Green's Function for Diffusion Equation

16B.1 Fundamental solution of diffusion equation. A fundamental solution (→**14.2**) of a diffusion equation is a solution to

$$\frac{\partial\psi}{\partial t} - D\Delta\psi = \delta(t-s)\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{16.41}$$

It is easy to check by explicit calculation that in d-space

$$G(\boldsymbol{x}, t|\boldsymbol{y}, s) = \left(\frac{1}{4\pi D(t-s)}\right)^{d/2} \exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4D(t-s)}\right) \tag{16.42}$$

is a solution. Hence, this is a fundamental solution of a diffusion equation (→**16B.4**). This is also the Green's function (→**14.2**) for the

[224] B130.

diffusion equation (16.41) under the condition that the solution vanishes at infinity. This is often called the *diffusion kernel.* We can demonstrate (cf. **14.13**) that

$$w\text{-}\lim_{t\to s} G(\boldsymbol{x},t|\boldsymbol{y},s) = \delta(\boldsymbol{x}-\boldsymbol{y}). \tag{16.43}$$

Discussion.
(1) It is important to use the formal solution, or the integral equation form of PDE in terms of the Green's function, because it allows us to apply various approximation methods (→**D16B.10**). The formal solution of

$$\frac{\partial u}{\partial t} = D\Delta u + f(x,t)u \tag{16.44}$$

with an appropriate homogeneous boundary condition can be written as

$$u(x,t) = \int d\boldsymbol{y} G(\boldsymbol{x},t|\boldsymbol{y},0)u_0(\boldsymbol{y}) + \int_0^t ds \int d\boldsymbol{y} G(\boldsymbol{x},t|\boldsymbol{y},s)u(\boldsymbol{y},s)f(\boldsymbol{y},s), \tag{16.45}$$

where u_0 is the initial conditon, and G is the Green's function.
(2) Find the partial differential equation governing u satisfying the following integral equation

$$u(\boldsymbol{x},t) = \int d\boldsymbol{y} e^{-At} G(\boldsymbol{x},t|\boldsymbol{y},0)f(\boldsymbol{y}) - \int_0^t ds \int d\boldsymbol{y} e^{-A(t-s)} G(\boldsymbol{x},t|\boldsymbol{y},s)\{u(\boldsymbol{y},s)\}^3, \tag{16.46}$$

where G is given by (16.38), and f is a continuous function on the whole space. A is a constant, and the spatial integration range is the whole 3-space.

16B.2 Scaling invariant solution of diffusion equation. Looking at the diffusion equation, we realize that the equation is invariant under the scaling transformation $(\boldsymbol{x},t) \to (\lambda\boldsymbol{x},\lambda^2 t)$.[225] If we demand that the solution keeps its total mass after scaling (we know the diffusion equation conserves the total mass →**a1B.2**)

$$\int \psi(\boldsymbol{x})d\boldsymbol{x} = 1, \tag{16.47}$$

then, we conclude in d-space

$$\psi(\boldsymbol{x},t) = \lambda^d \psi(\lambda\boldsymbol{x},\lambda^2 t). \tag{16.48}$$

16B.3 Dimensional analysis. Another way to obtain the scale invariant solution is to perform *dimensional analysis.* Dimensional analysis is a way to find combinations of variables that are invariant under change

[225] This is actually the idea of dimensional analysis. See the next entry.

of units (i.e.. change of scales). The dimension of a quantity Q is often denoted by $[Q]$. Let the dimension of length be L: $[\boldsymbol{x}] = L$, and that of time be T: $[t] = T$. Then $[D] = L^2/T$. Also from $\int dx\, u = 1$, we get $[u] = L^{-d}$. We can construct two dimensionless quantities (i.e., scale invariant quantities):

$$[\boldsymbol{x}/\sqrt{Dt}] = 1, \quad [(tD)^{d/2}u] = 1. \tag{16.49}$$

Therefore. $u(Dt)^{d/2}$ must be a function of $\boldsymbol{x}/\sqrt{Dt}$:

$$u(\boldsymbol{x}, t) = (Dt)^{-d/2} f(\boldsymbol{x}/\sqrt{Dt}). \tag{16.50}$$

16B.4 Scaling invariant spherically symmetric solution to diffusion equation. If we assume that the solution is spherically symmetric around $\boldsymbol{x} = 0$. then f in **16B.3** depends on $r \equiv |\boldsymbol{x}|$. That is, there is a function h such that

$$\psi(\boldsymbol{x}, t) = t^{-d/2} h(r/\sqrt{Dt}). \tag{16.51}$$

Putting this into the diffusion equation, we get an ODE for h as a function of $x = r/\sqrt{Dt}$:

$$h'' + \left(\frac{d-1}{x} + \frac{x}{2}\right) h' + \frac{d}{2} h = 0. \tag{16.52}$$

Since the solution must be smooth at the origin. actually h must be a well-behaved function of x^2: $h(x) = g(x^2)$. g obeys the following equation:

$$\frac{d}{2}(g + 4g') + x^2 \frac{d}{dx}(g + 4g') = 0. \tag{16.53}$$

If we demand the boundedness of the solution. $g + 4g' = 0$ is the only choice. That is.

$$\psi(\boldsymbol{x}, t) = \frac{C}{t^{-d/2}} e^{-x^2/4Dt} \tag{16.54}$$

C is a constant determined by the normalization condition. We see (16.42) is obtained after shifting the source position in space time with the aid of the translational symmetry of the equation.

16B.5 Initial trick for diffusion equation. Consider the following initial-boundary value problem for the diffusion equation on a region D:

$$\frac{\partial \psi}{\partial t} = D\Delta\psi, \quad \psi_{t=0} = \psi_0, \quad \psi|_{\partial D} = \varphi. \tag{16.55}$$

This problem can be converted to

$$\frac{\partial \psi}{\partial t} = D\Delta\psi + \delta(t)\psi_0, \quad \psi = 0 \text{ for } t \leq 0, \quad \psi|_{\partial D} = \varphi. \tag{16.56}$$

That is, the inhomogeneous initial condition is always converted to the source term (→**a1A.7**) of the equation. This can be demonstrated by integrating the both sides of the equation (16.56) from $t = -\epsilon$ to $t = +\epsilon$ (cf. **15.2**), where $\epsilon > 0$ is taken to be 0 after integration with the assumption of the smoothness of the solution.

Exercise.
(A) Consider a uniform rod of length l with the thermal diffusion constant D (placed along the x axis as $[0, l]$). The rod is thermally insulated except at the ends. The end temperatures are specified as

$$T(0, t) = g(t), \quad T(l, t) = h(t) \tag{16.57}$$

for $t > 0$.[226] and the initial condition is

$$T(x, 0) = f(x). \tag{16.58}$$

Here f, g and h are assumed to be C^1 for simplicity (cf. a trick in **14B.5**).
(B) Consider a uniform rod of length l as above, but now the rod is not insulated. Heat is lost according to Newton's radiation law (→**D1.18**) with the ambient temperature T_0. The end at $x = 0$ is maintained at the temperature A, and the other end is insulated. Let the initial temperature be uniform and A. The equation has the following form

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2} - c(T - T_0). \tag{16.59}$$

The standard trick to solve this is to introduce the new dependent variable $\tau = e^{-ct}(T - T_0)$.

16B.6 Method of descent. Analogously to **16A.5** we can obtain $(d-1)$-space Green's function from the d-space version. In the present case, this demonstration is easy with the aid of the Gaussian integral (→**19.19**).

16B.7 Markovian property of diffusion kernel. The diffusion kernel (16.42) enjoys the following remarkable property called the *Markovian property*:

$$G(\boldsymbol{x}, t|\boldsymbol{y}, s) = \int d\boldsymbol{z} G(\boldsymbol{x}, t|\boldsymbol{z}, s')G(\boldsymbol{z}, s'|\boldsymbol{y}, s) \tag{16.60}$$

for any $s' \in (s, t)$. Notice that there is NO integration over the intermediate time s'. This can be demonstrated by direct integration. This can

[226] These conditions are compatible with any initial condition so long as $t = 0$ is excluded.

be more elegantly shown with the aid of Fourier and Laplace transform as we will see later (→**38.10**). This is the key to the Feynman-Kac path integral formula (→**38.11-13**).

16B.8 Random walk and heat kernel. Consider a walker whose n-th step is a vector $\boldsymbol{a}_n$. After N-steps, the position of the walker starting from the origin is $\boldsymbol{R} = \sum_{n=1}^{N} \boldsymbol{a}_n$. Each step vector is $\boldsymbol{e}_i$ or $-\boldsymbol{e}_i$ with equal probability $1/2d$, where $\boldsymbol{e}_i$ is the i-th the unit vector parallel to the i-th coordinate. The trajectory of the walker is a stochastic process called the random walk (on the simple cubic lattice, in this case). Let us compute the distribution function of the end position $\boldsymbol{R}$ after N steps. The density distribution is given by

$$f(\boldsymbol{R}, N) = \left\langle \delta\left(\boldsymbol{R} - \sum_{n=1}^{N} \boldsymbol{a}_n\right)\right\rangle. \tag{16.61}$$

where $\langle\ \ \rangle$ is the average over all the possible choices of all the steps. The best way to compute this average is to use its Fourier transform (→**32C.8**). or the generating function of $\boldsymbol{R}$:

$$\int_{-\infty}^{+\infty} f(\boldsymbol{R}, N)e^{i\boldsymbol{R}\cdot\boldsymbol{k}}d^d\boldsymbol{R} = \left\langle \exp\left[\sum_{n=1}^{N} i\boldsymbol{k}\cdot\boldsymbol{a}_n\right]\right\rangle. \tag{16.62}$$

$$= \langle \exp\left[i\boldsymbol{k}\cdot\boldsymbol{a}_1\right]\rangle^N. \tag{16.63}$$

Here the fact that all the steps obey the identical probability law has been used. From now on a physicist's approach is used.[227] We are interested in the large scale distribution. so we have only to study the above integral for small $\boldsymbol{k}$ only. We can approximate as

$$\int_{-\infty}^{+\infty} f(\boldsymbol{R}, N)e^{i\boldsymbol{R}\cdot\boldsymbol{k}}d^d\boldsymbol{R} = \left(1 - \frac{1}{2}k^2 + \cdots\right)^N \simeq e^{-Nk^2/2}. \tag{16.64}$$

Inverting this Fourier transform. we get

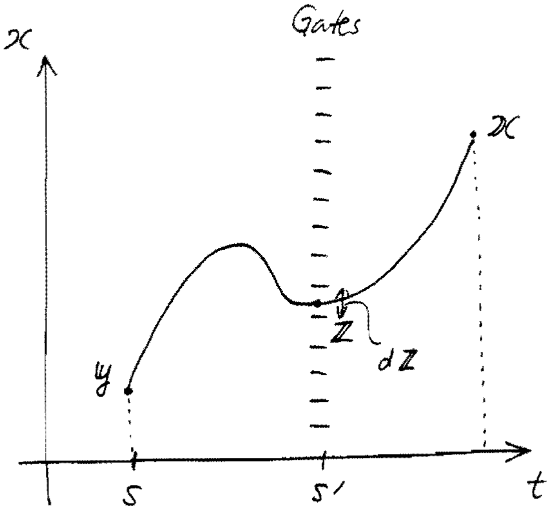
$$f(\boldsymbol{R}, N) = \sqrt{\frac{d}{2\pi N}}e^{-dR^2/2N}. \tag{16.65}$$

This is essentially the heat kernel. This implies that the diffusion equation describes the average behavior of the random walk. or the behavior of the ensemble of random walkers.

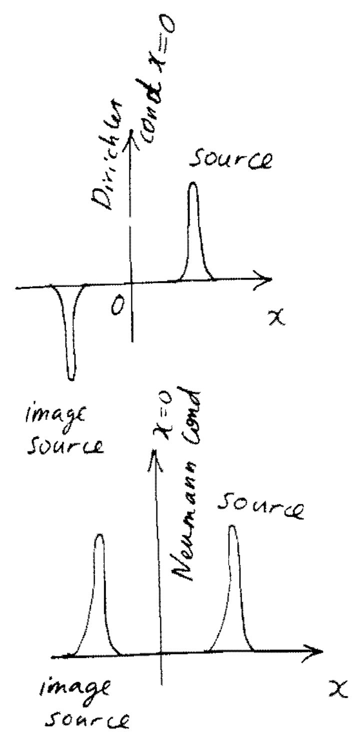
The Markovian property **16B.7** can be interpreted as the sum of

[227] For a more respectable approach, see W Feller. *Introduction to Probability Theory and Its Applications* (Academic Press). for example.

all the gate probabilities as shown in the figure.



16B.9 Method of images for diffusion equation – image sources. We know the Green's function (16.42) for the diffusion equation in the infinite space $\boldsymbol{R}^3$ (→**16B.1**). Now, consider the equation on the half space $x > 0$ with the boundary condition that $u = 0$ on the yz-plane.
(1) **Dirichlet case**. The unit impulsive source is placed at time $t = 0$ at $x = x' > 0$. Let G^- be the Green's function whose unit impulsive source at time $t = 0$ is placed at $x = -x'$. Then. $H \equiv G - G^-$ satisfies all the conditions of the problem. That is, H is the desired Green's function for the half space with a homogeneous Dirichlet condition at $x = 0$. This means that H is the solution to the whole space problem with +source at x' and −source at $-x'$. The latter is the *image source* for the current problem.
(2) **Neumann case**. If the boundary condition at the origin is a homogeneous Neumann condition. then $G+G^-$ should be the desired Green's function in the half space. That is. +source at $-x_0$ is the needed image source to make the problem a whole space problem. More complicated cases discussed in **16A.8** can be treated analogously. In the case of diffusion equation. there is no difficulty for the Neumann problem on a bounded region.



Exercise.
(A) Find the solution of the diffusion equation on [0, 1] with a homogeneous Dirichlet condition at $x = 0$ and a homogeneous Neumann condition at $x = 1$ with a unit impulsive source placed at $x = x_0$ at time $t = 0$.
(B) **Diffusion equation to defend God?** Kelvin accepted organic evolution advocated by Darwin. but he could not swallow the logical consequence of Darwinism: no design or in this case no divine intervention at the beginning of life. He used heat conduction to destroy Darwinism:
The temperature gradient in the Earth near its surface is roughly $v = 0.035$K/m at the present time. He assumed that the Earth was a homogeneous sphere of radius $R \simeq 6400$km. The evolution of the temperature $T(\boldsymbol{r}, t)$ at position $\boldsymbol{r}$ at time t obeys Fourier's law

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T.$$

At time $t = 0$ he assumed that the Earth was at its melting temperature which was $T_0 = 3000$K above the surface temperature for $|\boldsymbol{r}| < R$. Its surface temperature

must have been close to the present temperature for all $t > 0$ to allow life. Let us choose this to be the zero point of temperature for all $t > 0$.
(1) Using the numbers v, T_0 and R, give an argument that the thickness of the transition layer over which the temperature differs significantly from T_0 is much smaller than the Earth's radius at the present time.
(2) Hence, the full sphere problem simplifies to the 1-d problem:

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$$

under the condition that $T(x,0) = T_0$ for all $x > 0$, and $T(0,t) = 0$ for all $t > 0$. Find the solution.
(3) Using the value of v, compute the age (in years) of the Earth, assuming that the thermal diffusivity is $\kappa = 0.7 \times 10^{-6}\mathrm{m}^2/\mathrm{s}$.[228]
(4) Read the following to be a bit wiser as a physicist:
C. Darwin. *The Origin of Species* (Sixth edition Jan. 1872) Chapter X. " Sir W Thompson concludes that the consolidation of the crust can hardly have occurred less than 20 or more than 400 millions years ago. but probably not less than 98 or more than 200 millions years."
Ibid.. Chapter XV. "$\cdots$ and this objection, as urged by Sir William Thompson, is probably one of the gravest as yet advanced. I can only say firstly, that we do not know at what rate species change as measured by years, and secondly, that many philosophers are not as yet willing to admit that we know enough of the constitution of the universe $\cdots$"

Now we know Darwin was perfectly right. Thompson did not know the radioactivity. In a certain sense, in retrospect at least, Darwin pointed out the existence of unknown physics.

Later, Huxley commented: Mathematics may be compared to a mill of exquisite workmanship, which grinds your stuff of any degree of fineness; but nevertheless, what you get out depends what you put in: and as the grandest mill in the world will not extract wheat-flour from peascods, so pages of formulae will not get a definite result out of loose data. However, in this case the defect of the theory was much more serious. In any case Darwin did not have much respect of mathematics: Boltzmann was strongly influenced by Darwin, and he suggested that the 19th century may be called the century of Darwin.

16B.10 How to use Green's function: homogeneous boundary problems. In the case of the diffusion equation (with a source term →**a1B.4**),

$$\frac{\partial \psi}{\partial t} - D\Delta\psi = \sigma(\boldsymbol{x}, t), \tag{16.66}$$

[228] The number obtained here is ridiculously short (although much longer than some beliefs based on the misunderstanding of the Bible).

even if the boundary condition is homogeneous, we must take into account the initial condition (→**1.18**):

$$\psi(\boldsymbol{x},0) = f(\boldsymbol{x}). \tag{16.67}$$

We already know that the nitial condition can be absorbed into the source term (→**16B.5**). so that

$$\frac{\partial\psi}{\partial t} - D\Delta\psi = \sigma(\boldsymbol{x}.t) + f(\boldsymbol{x})\delta(t). \tag{16.68}$$

Thus the solution to (16.66) + (16.67) with a homogeneous boundary condition can be written in terms of the Green's function as

$$\psi(\boldsymbol{x}.t) = \int_D d\boldsymbol{y}\int_0^t dsG(\boldsymbol{x}.t|\boldsymbol{y}.s)\sigma(\boldsymbol{y}.s) + \int_D d\boldsymbol{y}G(\boldsymbol{x}.t|\boldsymbol{y},0)f(\boldsymbol{y}). \tag{16.69}$$

Here D is the domain of the problem (cf. **38.4**).

To solve inhomogeneous boundary value problems. we can use **16B.11**. but there is a clever trick. See **18.5**.

Discussion.
Solve the following semilinear parabolic equation to order ϵ in free 1-space:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \epsilon u^3 \tag{16.70}$$

with the initial condition

$$u(0) = \frac{1}{\sqrt{2\pi\delta}}e^{-x^2/2\delta}. \tag{16.71}$$

Here δ is a small positive constant. Demonstrate that the order ϵ term is asymptotically (for $t/\delta \gg 1$) proportional to $\ln(t/\delta)$.[229]

16B.11 Analogue of Green's formula for diffusion equation. We have a formula analogous to Green's formula (→**16A.19**) for diffusion equation. This generalization will be postponed to **38**.

[229] If the reader is familiar with the renormalization group theory, it is immediate from this observation that the problem has a renormalization group structure governing the long-time behavior.

16.C Green's Function for Wave Equation

16C.1 Free-space Green's function for 3-wave equation. We wish to solve

$$\left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)\psi = \delta(t)\delta(\boldsymbol{x}). \tag{16.72}$$

It is not hard to guess (from physics) a spherical symmetric solution as

$$\psi(\boldsymbol{x},t) = \frac{\delta(|\boldsymbol{x}| - ct)}{4\pi c|\boldsymbol{x}|}\Theta(t), \tag{16.73}$$

where the step function Θ is put to satisfy the causality. Check that this is indeed the solution. Now with the aid of space-time translational symmetry, we obtain the Green's function for the free space as

$$G(\boldsymbol{x},t|\boldsymbol{y},s) = \frac{\delta(|\boldsymbol{x}-\boldsymbol{y}| - c(t-s))}{4\pi c|\boldsymbol{x}-\boldsymbol{y}|}\Theta(t-s), \tag{16.74}$$

(16.74) is called the *retarded Green's function.*

Exercise.
(1) One way to obtain the Green's function for the wave equation is to use its temporal Fourier transformation (the Helmholtz equation):

$$(k^2 - \Delta)v_\omega(x) = \delta(\boldsymbol{x}). \tag{16.75}$$

Here $k = \omega/c$. Obtain

$$v_\omega(\boldsymbol{x}) = \frac{i}{2k}e^{ik|\boldsymbol{x}|}. \tag{16.76}$$

(2) There is a point source of wave at the origin. Describe the wave radiated from this source (respecting causality). That is, solve

$$(c^{-2}\partial_t^2 - \Delta)u(x,t) = Q\cos\omega t\,\delta(x) \tag{16.77}$$

in 3-space, respecting the radiation condition (i.e., there is no incoming wave).
(3) The same as (2) but with a point (oscillating) dipole at the origin. That is, solve the wave equation with the source

$$\rho(x,t) = -\boldsymbol{p}\cdot\nabla\delta(x)\cos\omega t, \tag{16.78}$$

where $\boldsymbol{p}$ is the dipole strength. Again respect causality.

16C.2 Retarded and advanced Green's functions. Since the wave equation is time reversal symmetric (→**1.14**),

$$\psi(\boldsymbol{x},t) = \frac{\delta(|\boldsymbol{x}| + ct)}{4\pi c|\boldsymbol{x}|}\Theta(-t), \tag{16.79}$$

must also be a solution. This is a strange 'anti-causal' solution, and is called the *advanced Green's function* in contrast to (16.74).

16C.3 Method of descent for wave equation. Applying this method explained in **16A.5** to the retarded Green's function (16.74), we can construct the retarded Green's function for 2-space as

$$G(\boldsymbol{x}|\boldsymbol{y}) = \frac{\Theta(ct - |\boldsymbol{x}-\boldsymbol{y}|)}{2\pi c\sqrt{c^2t - |\boldsymbol{x}-\boldsymbol{y}|^2}}\Theta(t). \tag{16.80}$$

Exercise.
Apply the method of descent to the Green's function (16.76) of the Helmholtz equation.

16C.4 Afterglow. Notice that the 2-space Green's function for the wave equation is not zero for $|\boldsymbol{x}-\boldsymbol{y}| < ct$. This implies that for an observer in 2-space a flash of a lamp at a distance brightens up the world slightly even after the first pulse arrived to the observer (*afterglow effect*) (see also **40.8**). We will see this is a feature of even dimensional space (→**32D.10**).

The difference between odd and even dimensional spaces also appears in the spherical wave as follows:

$$\phi(r,t) = u(r)e^{-i\omega t}. \tag{16.81}$$

Then, u obeys the Helmholtz equation (→**27A.24**, **39**)

$$\frac{\partial^2 u}{\partial r^2} + \frac{d-1}{r}\frac{\partial u}{\partial r} + k^2 u = 0, \tag{16.82}$$

where $k = \omega/c$. The general solution for this can be written in terms of

$$u(r) = \frac{1}{r^{d/2-1}}J_{\pm(d/2-1)}(kr). \tag{16.83}$$

The Bessel functions with half odd orders are written in terms of trigonometric functions, but not elementary otherwise (→**27**, **27A.19**).

16C.5 Method of images for wave equation. This is almost a repetition of what we have seen in **16A.7** and **16B.9**. If we assume that the boundary condition is Dirichlet, then the corresponding Green's function reads

$$G(\boldsymbol{x},t|\boldsymbol{y},s) = \frac{\delta(|\boldsymbol{x}-\boldsymbol{y}| - c(t-s))}{4\pi c|\boldsymbol{x}-\boldsymbol{y}|}\Theta(t-s) - \frac{\delta(|\boldsymbol{x}-\boldsymbol{y}'| - c(t-s))}{4\pi c|\boldsymbol{x}-\boldsymbol{y}'|}\Theta(t-s). \tag{16.84}$$

where $\boldsymbol{y}'$ is the position mirror symmetric to $\boldsymbol{y}$ with respect to the yz-plane. This is an idealized reflection from a hard wall. Consider the Neumann case.

Exercise.
With the aid of the method of images, write down the solution to the 1 dimensional wave equation

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u = 0 \tag{16.85}$$

on the half line $[0, +\infty)$ with the fixed end condition at $x = 0$ and the initial condition

$$u|_{t=0} = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \tag{16.86}$$

where $f(0) = g(0) = 0$. [This is NOT a Green's function problem.]

16.D Laplace Equation in 2-Space

16D.1 What do we know from complex analysis?
(1) The real and imaginary parts of a holomorphic function (→**2A.8**) are harmonic functions (→**5.6**). $\log|f|$ is harmonic on the region where f is holomorphic (→**5.7**) and nonzero.
(2) Let u be a harmonic function on a simply connected region D. Then. there is a holomorphic function ψ on D such that its real part is u. ψ is unique up to a pure imaginary additive constant (→**5.8**).
The uniqueness follows from the Cauchy-Riemann equation (→**5.3**); suppose there are two holomorphic functions f_1 and f_2 such that $\Re f_1 = \Re f_2 = u$ and$\Im f_1 = v_1$ and $\Re f_2 = v_2$. Then partial derivatives of $v_1 - v_2$ vanish. so it must be a constant. But since real part of f has no freedom of choice. the constant must be real.
(3) Harmonicity is conformal invariant (→**10.16**). □

16D.2 Neumann problem can be reduced to Dirichlet problem: The Neumann problem:

$$\Delta V = 0 \text{ (in } D). \quad \frac{\partial V}{\partial n}(s) = g(s) \text{ (on } \partial D) \tag{16.87}$$

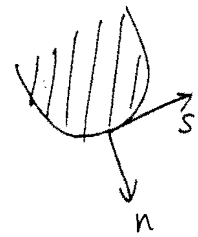
is converted to the problem to solve its conjugate harmonic function U (→**5.6**). which is the solution to the Dirichlet problem:

$$\Delta U = 0 \text{ (in } D). \quad U(s) = -\int_a^s g(s)ds \equiv h(s) \text{ (on } \partial D), \tag{16.88}$$

where a is any point on ∂D. □

Notice that the Neumann problem (16.87) is meaningful only when (→**1.19(3)**, **36.7**)

$$\int_{\partial D} g(s)ds = 0. \tag{16.89}$$



[Demo] Let U and V be conjugate harmonic functions. Then, we have

$$\frac{\partial U}{\partial n} = \frac{\partial V}{\partial s}, \quad \frac{\partial U}{\partial s} = -\frac{\partial V}{\partial n}, \tag{16.90}$$

where n is the outward normal and s is the arc length parameter (the positive direction = orientation of s in the standard way (→**6.4**)). They are disguised Cauchy-Riemann equations (→**5.3**).

From the second equation in (16.90), we get the condition for the Dirichlet problem (16.88). Thanks to (16.89) $h(s)$ is a univalent function on the boundary. □

16D.3 Solving Dirichlet problem by conformal map:
(1) Find a conformal map which maps the domain D onto a unit disk or the upper half plane (→**10.11**).
(2) Solve the transformed problem on the simplified domain. This can be done with the aid of Poisson's formula **16D.8**.
(3) Transform back to the original variable using the inverse conformal transform of the one used in (1).
However, the use of Green's function unifies everything we need.

16D.4 Green's function: $G(z, z_0)$ is called the *Green's function* for the region D with the pole at z_0, if
(1) $G(z, z_0)$ is harmonic (→**2C.11**) w.r.t. z on $D \setminus \{z_0\}$.
(2) In some nbh of z_0 $G(z, z_0) + (1/2\pi)\log|z - z_0|$ is harmonic.[230]
(3) For any $\zeta \in \partial D$ $G(z, z_0) \to 0$ when z approaches ζ from inside of D.

16D.5 Agreement with the previous definition. Let us compute the Laplacian of the Green's function explicitly. The Green's function for a region D can be written as

$$G(z, z_0) = h - \frac{1}{2\pi}\log|z - z_0| = h - \frac{1}{2\pi}\log|\boldsymbol{r} - \boldsymbol{r}_0|. \tag{16.91}$$

where h is a harmonic function without any singularity and $z = x + iy$, $\boldsymbol{r} = (x, y)$, etc. Let us integrate ΔG over a small disk D_r of radius r

[230] In many standard complex function theory books, $1/2\pi$ in front of log is not included. Then the Green's function is the solution to the equation with the source term $2\pi\delta$ instead of δ customary to the PDE theory.

centered at z_0:

$$\lim_{r\to 0}\int_{D_r} dxdy\Delta G = -\lim_{r\to 0}\frac{1}{2\pi}\int_{\partial D_r} d\ell\frac{\boldsymbol{r}-\boldsymbol{r}_0}{|\boldsymbol{r}-\boldsymbol{r}_0|^2} = -1, \tag{16.92}$$

where ℓ is the length element. Thus, the definition in **16D.4** is in full agreement with the definition of Green's functions in **16A.1**. See also **16A.4**.

16D.6 Example: Let z_0 be a point in the open unit disk. Then, the Green's function for the unit disk with the pole (source) at z_0 is given by

$$G(z,z_0) = \frac{1}{2\pi}\log\left|\frac{1-\overline{z_0}z}{z-z_0}\right| \tag{16.93}$$

□

[Demo] Except for z_0 $(1-\overline{z_0}z)/(z-z_0)$ is holomorphic on the unit disk, so (16.93) is harmonic except at z_0 due to **16D.4**(1). Obviously, $G+(1/2\pi)\log|z-z_0|$ is harmonic near z_0. G vanishes on the unit circle due to **10.12**. Hence **16D.4**(1)-(3) are all satisfied. □

Exercise.
Express (16.75) in terms of the polar coordinates. That is, set $z = re^{i\theta}$, and $z_0 = r_0 e^{i\theta_0}$. The result reads

$$G(z,z_0) = -\frac{1}{4\pi}\ln\frac{r^2+r_0^2-2rr_0\cos(\theta-\theta_0)}{1+(rr_0)^2-2rr_0\cos(\theta-\theta_0)}. \tag{16.94}$$

Next, with the aid of **16D.11**, find the Green's function for a disk of radius a. Check the conservation of heat flow at the rim. That is, integrate the outward heat flux and show that the total flow leaving the disk is equal to unity, if the temperature distribution is given by (16.75).

16D.7 Green's function solves Dirichlet problem: Let D be a region with ∂D being sufficiently smooth. Let G be the Green's function for this region, and u be a harmonic function on D and continuous on the closure of D. Then, for $z \in D$

$$u(z) = \int_{\partial D} u(\zeta)\frac{\partial G(\zeta,z)}{\partial n}ds. \tag{16.95}$$

Here $\partial/\partial n$ is the outward normal derivative at ζ, and s is the contour length coordinate of ζ along the boundary curve. □
This is a familiar formula ($\to$**16A.21**).

16D.8 Poisson's formula: If u is harmonic on $|z| < R$ and continuous on $|z| \le R$, then on $|z| \le R$

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} u(R,\phi)\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}d\phi. \tag{16.96}$$

□

This is obvious from **16D.6** and **16D.7** (cf. **18.6**).

16D.9 Solution to Dirichlet problem on disk: Schwarz' theorem: Let $f(\phi)$ $(0 \le \phi < 2\pi)$ be integrable.[231] Then

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}d\phi \tag{16.97}$$

is harmonic on $|z| < R$. □

The first half is essentially **16D.8**. but explicitly we can apply (→**19.17**) the Laplacian to (16.97):

$$\Delta u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} d\phi f(\phi)\Delta\Re\left(\frac{Re^{i\phi} + z}{Re^{i\phi} - z}\right) = 0. \tag{16.98}$$

16D.10 Fourier expansion of harmonic function on the disk: Under the same assumptions in **16D.8**, u can be Fourier-expanded in $|z| < R$ as

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta)\left(\frac{r}{R}\right)^n. \tag{16.99}$$

where the coefficients are given by

$$a_n = \frac{1}{\pi}\int_0^{2\pi} u(R,\theta)\cos n\theta d\theta, \text{ (for } n = 0, 1, \cdots) \tag{16.100}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} u(R,\theta)\sin n\theta d\theta, \text{ (for } n = 1, 2, \cdots). \tag{16.101}$$

□

[Demo] The integral kernel in the Poisson formula can be Fourier-expanded as

$$\frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} = 1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\cos n(\theta - \phi). \tag{16.102}$$

This follows easily from $(\zeta + z)/(\zeta - z) = 1 + 2\sum(z/\zeta)^n$ with $\zeta = Re^{i\phi}$ and $z = re^{i\theta}$. This is uniformly convergent, so (→**A5.15**) we may integrate the expansion of the integrand of the Poisson integral formula termwisely. The result is the one we wanted. □

The 3-space version of this formula is the spherical harmonics expansion.

[231] i.e., Lebesgue integrable (→**19.81**), so this theorem is distinct from Poisson's formula.

16D.11 Conformal mapping and Green's function: Let Δ be a region on the w-plane whose Green's function is $G_\Delta(w, w_0)$. If a conformal map (→**10.1**) $w = f(z)$ maps a region D on the z-plane onto Δ, then

$$G_D(z, z_0) \equiv G_\Delta(f(z), f(z_0)) \tag{16.103}$$

is the Green's function for the region D with the pole at z_0 such that $w_0 = f(z_0)$. □

This theorem with **16D.9** and the Riemann mapping theorem (→**10.10**) together demonstrate that for any singly connected region, there is a Green's function.

[Demo] We have only to check **16D.5**. Thanks to **10.16** $G_D(z, z_0)$ is harmonic on D except at z_0. f is continuous on the closure of Δ.[232] so G_D vanishes when z approaches ∂D from inside. Now we have

$$2\pi G_D(z, z_0)+\log|z-z_0| = 2\pi G_\Delta(f(z), f(z_0))+\log|f(z)-f(z_0)|-\log\left|\frac{f(z)-f(z_0)}{z-z_0}\right|. \tag{16.104}$$

$G_\Delta(w, w_0) + (1/2\pi)\log|w - w_0|$ is harmonic near w_0 and f is holomorphic near z_0, so the last term is harmonic. □

16D.12 Green's function for a region D: Let $w = f(z)$ be a conformal map which maps the region D on the z-plane onto the unit disk. Then the Green's function of D with the pole at $z_0 \in D$ is given by (→**10.11**)

$$G(z, z_0) = \log\left|\frac{1-\overline{f(z_0)}f(z)}{f(z)-f(z_0)}\right|. \tag{16.105}$$

□

16D.13 Harmonic function on the half plane: Let $f(x)$ $(-\infty < x < \infty)$ be integrable. Then

$$u(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{yf(\eta)}{y^2+(x-\eta)^2}d\eta \tag{16.106}$$

is harmonic on the upper half plane. If x_0 is a continuous point of f, then $u(x, y) \to f(x_0)$ when z approaches to x_0 from above. If f is uniformly continuous on $x_1 \le x \le x_2$, then the convergence of $u(x, y) \to f(x)$ is uniform. □

[Demo] The integral converges uniformly for $y > 0$, so we may exchange the order of differentiation and integration. Notice that the imaginary part of $1/(z-\eta)$ is harmonic, so the integral is also harmonic. The convergence result will not be proved here. □

[232] We must demonstrate this – Carathèodory's theorem.