15 Green's Functions for ODE

Green's functions for second order linear ODE are constructed explicitly. Symmetry of the Green's function can be demonstrated clearly.

Key words: δ-function. Green's operator, fundamental solution. Green's function, Sturm-Liouville problem.

Summary:
(1) Understand the method to construct a fundamental solution in **15.2**.
(2) If we can obtain a fundamental system of solutions, we can construct the Green's function for a regular Sturm-Liouville problem (**15.6-7**).

15.1 Fundamental solution exists for ODE. Let

$$\mathcal{L} \equiv \sum_{i=0}^{n} a_i(x) \left(\frac{d}{dx}\right)^{n-i}. \tag{15.1}$$

where $a_0(x) \neq 0$ and $a_0, \cdots, a_n$ are smooth functions. Then $\mathcal{L}u = 0$ has a fundamental solution ($\rightarrow$**14.2**). The difference of any two fundamental solutions is a solution to the homogeneous equation $\mathcal{L}u = 0$ ($\rightarrow$**11B.13**). □
We will demonstrate this for $n = 2$ below through explicitly constructing a fundamental solution.

15.2 Proof of 15.1 for $n = 2$. We wish to find a solution to

$$\left(a_0\frac{d^2}{dx^2} + a_1\frac{d}{dx} + a_2\right) w(x|y) = \delta(x - y). \tag{15.2}$$

Regard x as the time variable. and assume that $w(x|y) \rightarrow 0$ as $x \rightarrow -\infty$. Then. causality implies that $w(x|y) = 0$ for $x < y$. d^2w/dx^2 cannot have a singularity worse than $\delta(x - y)$, so that dw/dx is at worst discontinuous at $x = y$. and w is continuous at $x = y$ ($\rightarrow$**14.15**). Hence. we may assume $w(y + 0|y) = 0$. Since (15.2) is a second order ODE. we can construct its solution uniquely with one more condition. Integrate (15.2) from $x = y - \epsilon$ to $y + \epsilon$ for infinitesimal $\epsilon > 0$. We get ($w^{(1)} \equiv dw/dx$)

$$a_0(y)[w^{(1)}(y + 0|y) - w^{(1)}(y - 0|y)] = 1. \tag{15.3}$$

Here we have used the continuity of a_0 and w. We have already assumed that w is zero for $x < y$, so that this equation implies

$$a_0(y)w^{(1)}(y+0|y) = 1. \tag{15.4}$$

This is the needed second condition. In this way, we can construct a solution to (15.2).

15.3 Example: damped oscillator under an impact. Find a fundamental solution to[219]

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = \delta(t-s). \tag{15.5}$$

The fundamental solution constructed in **15.2** reads for this case

$$w(t|s) = (\omega^2 - k^2)^{-1/2} e^{-k(t-s)} \sin\left[\sqrt{\omega^2 - k^2}(t-s)\right] \Theta(t-s). \tag{15.6}$$

15.4 Regular Sturm-Liouville problem.

$$\mathcal{L}_{ST} u \equiv \left[\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\right] u = 0 \tag{15.7}$$

under the following boundary condition is called a *regular Sturm-Liouville problem* (cf. **21A.7**), if p is of constant sign:

$$B_a[u] \equiv Ap(a)u'(a) - Bu(a) = 0, \tag{15.8}$$

$$B_b[u] \equiv Cp(b)u'(b) - Du(b) = 0. \tag{15.9}$$

where A, B, C and D are constants (cf. **21A.7**).

15.5 Theorem. The Green's function for a regular Sturm-Liouville problem

$$\mathcal{L}_{ST} u = \delta(x-y) \tag{15.10}$$

under the above boundary condition exists, if the operator does not possess zero eigenvalue. The Green's function, when it exists, is a symmetric function of x and y. □

Exercise.
Under what condition does the following operator with the boundary condition: $u(0)$ bounded and $u(a) = 0$, not have the Green's function?

$$Lu = u'' + \frac{1}{x}u' + \left(k^2 - \frac{16}{x^2}\right)u. \tag{15.11}$$

[219] The best way to solve this under the condition $x = 0$ for $t < s$ and $x'(s+0) = 1$ (this corresponds to (15.4)) is to use the Laplace transformation (→**33**).

The symmetry of the Green's function is proved by explicitly constructing the required Green's function as follows:

15.6 Explicit form of Green's function. The Green's function for a regular Sturm-Liouville problem in **15.4** is given by

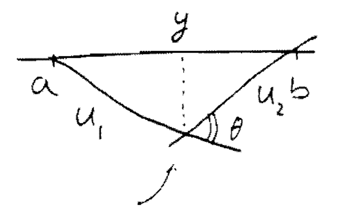
$$G(x|y) = \begin{cases} Ku_1(x)u_2(y) & \text{for } x < y, \\ Ku_2(x)u_1(y) & \text{for } x > y, \end{cases} \tag{15.12}$$

where $K^{-1} = p(x)(u_1u_2' - u_2u_1')$ (which is actually a constant); u_1 is a nontrivial solution to $\mathcal{L}_{ST}u = 0$ with $B_a[u] = 0$, and u_2 is a nontrivial solution to $\mathcal{L}_{ST}u = 0$ with $B_b[u] = 0$.□
Indeed, $G(x|y) = G(y|x)$ (→**20.28**, cf. **16A.20**). As we will see soon, $\{u_1, u_2\}$ is a fundamental system of solutions (→**24A.11**) for $\mathcal{L}_{ST}u = 0$.

15.7 Construction of Green's function. From $\mathcal{L}_{ST}G = \delta(x - y)$, we see that $G(y + 0|y) = G(y - 0|y)$, and

$$p(y)\left[\frac{\partial}{\partial x}G(x|y)\,|_{x=y+0} - \frac{\partial}{\partial x}G(x|y)\,|_{x=y-0}\right] = 1. \tag{15.13}$$

See (15.3) in **15.2**. We can always construct u_1 and u_2 as stated above. Let us construct G in the following form:



$$G(x|y) = \begin{cases} c_1(y)u_1(x) & \text{for } x < y, \\ c_2(y)u_2(x) & \text{for } x > y. \end{cases} \tag{15.14}$$

• 'crossing' is guaranteed by $W(u_1, u_2) \neq 0$.
• The angle θ can be controlled by uniform scaling by K.

To satisfy the conditions at $x = y$, we get

$$c_1(y)u_1(y) = c_2(y)u_2(y), \tag{15.15}$$
$$c_1(y)u_1'(y) - c_2(y)u_2'(y) = -1/p(y). \tag{15.16}$$

We can solve this for c_1 and c_2 only if $u_1u_2' - u_1'u_2 \neq 0$ (that is, the Wronskian (→**24A.6**) of u_1 and u_2 is nonzero), but this is guaranteed.[220] Since u_1 and u_2 satisfy $\mathcal{L}_{ST}u = 0$,

$$\frac{d}{dx}[p(x)(u_1u_2' - u_1'u_2)] = u_1\mathcal{L}_{ST}u_2 - u_2\mathcal{L}_{ST}u_1 = 0. \tag{15.17}$$

Hence

$$p(x)(u_1u_2' - u_2u_1') \equiv K^{-1} \tag{15.18}$$

[220] Notice that this condition is the condition that the Sturm-Liouville eigenvalue problem (→**35.1**) does not have zero eigenvalue. $u_1u_2' - u_1'u_2 = 0$ implies that $d(u_1/u_2)/dx = 0$ or $u_1 \propto u_2$. That is, u_1 satisfies $\mathcal{L}_{ST}u_1 = 0$ and $B_a[u_1] = B_b[u_1] = 0$, and $u_1 \not\equiv 0$. Hence, u_1 is an eigenfunction belonging to 0.

is a nonzero constant. Using this constant, we can solve as $c_1 = Ku_2$ and $c_2 = Ku_1$.

15.8 Remark. If we know a fundamental solution $w(x|y)$ to $\mathcal{L}u = \delta(x-y)$, then the general solution to this inhomogeneous equation can be written as (→**11B.13**)

$$G(x|y) = w(x|y) + A(y)u_1(x) + B(y)u_2(x). \tag{15.19}$$

A and B can be determined to satisfy the boundary conditions (they can depend on y).

15.9 Examples. The following examples can be solved either by the method of **15.7** or **15.8**.
(1) $u'' = 0$ with the boundary conditions $B_0[u] = u'(0) - u(0) = 0$ and $B_1[u] \equiv u'(1) = 0$. The Green's function for this is $G(x|y) = (x-y)\Theta(x-y) - (x+1)$.[221]
(2) $(d^2/dx^2 + k^2)u = 0$ with the boundary condition $B_0[u] \equiv u(0) = 0$ and $B_1[u] \equiv u(1) = 0$ (assume $\sin k \neq 0$). The Green's function for this is

$$G(x|y) = \begin{cases} \sin kx \sin k(y-1)/k\sin k & \text{for } x < y. \\ \sin ky \sin k(x-1)/k \sin k & \text{for } x > y. \end{cases} \tag{15.20}$$

Exercise.
(A) Obtain the Green's function with a Dirichlet condition of the equation

$$\sqrt{x}\frac{d}{dx}\left(\sqrt{x}\frac{du}{dx}\right) + a^2 u = 0 \tag{15.21}$$

on $[0, L]$, knowing that the general solution to this equation is given by

$$u(x) = A\sin(2a\sqrt{x}) + B\cos(2a\sqrt{x}). \tag{15.22}$$

(Calculation of K is messy, so you may forget about it.)
(B) Determine the Green's function for

$$L = \frac{d}{dx}x\frac{d}{dx} - \frac{1}{x} \tag{15.23}$$

with the homogeneous boundary conditions $u(0) = u(1) = 0$.[222]
(C) Consider the following 1-Schrödinger problem

$$(-\Delta + V)\psi = E\psi, \tag{15.24}$$

[221] The definition of $\Theta(x-y)$ at $x = y$ does not matter. That is, we may interpret Θ as a generalized function (→**14.4**).
[222] Hint: The equation is equidimensional (→**11B.14**).

where V vanishes at infinity. If this equation has a bound state (i.e., a solution in L_2-space →**20.19**. in otherwise normalizable as a wave function), it cannot be degenerate. In particular, the lowest energy bound state (ground state) cannot be degenerate. Prove this showing or answering the following:
(1) Degeneracy implies that there are two independent solutions for a given energy. What must be their Wronskian?
(2) The Wronskian for localized state is zero.

(D) Show that the Green's function for the following operator

$$\left[\frac{d}{dx}(1-x^2)\frac{d}{dx} - \frac{n^2}{1-x^2}\right] \tag{15.25}$$

with the boundary condition that the solution is bounded at $x = \pm 1$, where $n \in \boldsymbol{N}$, is given by

$$G(x|y) = \frac{1}{2n}\left(\frac{(1+x)(1-y)}{(1-x)(1+y)}\right)^{n/2} \tag{15.26}$$

for $x \le y$.

15.10 Theorem [Inhomogeneous boundary condition]. The solution to the following inhomogeneous boundary value problem:

$$\mathcal{L}u(x) = \varphi(x). \tag{15.27}$$

$$B_a[u] = \alpha, \quad B_b[u] = \beta. \tag{15.28}$$

where $\mathcal{L}$, B_a and B_b are the same as in **15.4**, and $BD \neq 0$, is given by

$$u(x) = \int_a^b dy G(x|y)\varphi(y) + p(a)B^{-1}\alpha\left(\frac{\partial G}{\partial y}\right)_{y=a} - p(b)D^{-1}\beta\left(\frac{\partial G}{\partial y}\right)_{y=b}. \tag{15.29}$$

[Demo] First, we note an analogue of Green's formula (→**16A.19**, cf. **2C.15**)

$$\int_a^b dx\, u\mathcal{L}v - \int_a^b dx\, v\mathcal{L}u = p(uv' - u'v)|_a^b. \tag{15.30}$$

Let $v(x) \equiv G(x|y)$, and u be the solution to the problem. Then, (15.30) implies

$$u(y) = \int_a^b dx G(x|y)\varphi(x) + \left\{p(x)[u(x)\frac{\partial G}{\partial x} - Gu'(x)]\right\}_{x=a}^{x=b}. \tag{15.31}$$

Exchanging x and y in this formula, and using the symmetry of the Green's function (→**15.6**, **20.28**), we get (note $B_b[u] = \beta$ and $B_b[G] = 0$)

$$Du(b)\partial_y G(x|y)|_{y=b} - Du'(b)G(x|b) = -\beta\partial_y G(x|b) \tag{15.32}$$

An analogous formula holds at the other end of the region. These relations allow us to rewrite the second term of (15.31) as desired.

Exercise.
Use the Green's function to solve

$$\left(\frac{d^2}{dx^2}+k^2\right)u=\sin kx \tag{15.33}$$

on $[0,1]$ with the boundary condition $u(0)=u(1)=1$.

15.11 Another method to solve inhomogeneous case. Practically, the following (usual) splitting method is also very useful: Separate the problem (15.27) + (15.28) as
(I) $\mathcal{L}u_1=0$ with the inhomogeneous boundary condition $B_a[u_1]=\alpha$, $B_b[u_1]=\beta$.
(II) $\mathcal{L}u_2=\varphi$ with the homogeneous boundary condition $B_a[u_2]=0$, $B_b[u_2]=0$.
The solution we want is given by u_1+u_2. (I) can be solved as usual (→**11B**), and (II) can be solved with the aid of the Green's function as $u_2=\int dyG(x|y)\varphi(y)$.