14 Generalized Function

The δ-function is not an ordinary function. and is meaningful only inside the integral. The theory of distribution in the sense of Sobolev and Schwartz rationalizes such objects like the δ-function. Rudiments of the theory is outlined from the practitioner's point of view. Calculation of Green's functions may be facilitated by the theory of generalized functions which justifies apparent abuses of classical analysis.

Key words: generalized function. distribution. test function. Schwartz class. regular distribution. convolution. δ-function. differentiation of δ-function. Heaviside step function. Cauchy principal part.

Summary:
(1) Generalized functions are defined by their outcome when they are applied to test functions (**14.4**. **14.8**). Whenever, the reader feels intuition is doubtful. use test functions.
(2) Understand the definition of differentiation of generalized functions (**14.14**). All the elementary rules of calculus survive for generalized functions: besides. the order of limit and differentiation can always be exchanged (**14.18**).
(3) Change of the variables in δ-function should not cause any problem (**14.12-14**).
(4) Be familiar with convolution (**14.22**).

14.1 Green's function and delta-function. The fundamental idea of Green was introduced at **1.8**. where we realized that it is very convenient to introduce an object δ_a which has weight 1 at point a but zero elsewhere. If we consider the mass density distribution $\rho(x)$ corresponding to this weight distribution. then we need $\rho(x)$ which is $+\infty$ at $x = a$ but zero elsewhere. The symbol $\delta(x-a)$ was introduced for such an object in **3.8** in conjunction to functional derivative. Already such an object was justified within complex analysis as an example of hyperfunction (→**8B.12**. **8B.13**). Basically. to implement Green's idea we have to solve. for example.

$$-\Delta\psi = \delta(x-a) \tag{14.1}$$

under an appropriate boundary condition. We need a systematic theory of such "density functions."

14.2 Green's function and fundamental solution. Let L be a linear differential operator.[210] Any solution to

$$L\psi(x) = \delta(x-y) \tag{14.2}$$

is called a *fundamental solution*. If it satisfies, further, the auxiliary conditions of the problem, we say the solution is the *Green's function* for the problem. As we will see later (→**16A.21** for an example), we have only to consider homogeneous auxiliary conditions, so when we say $G(x|y)$ is the Green's function of a problem which is described by the linear differential operator L and linear auxiliary conditions (→**1.5**) $A\psi = f$, we mean

$$LG(x|y) = \delta(x-y) \tag{14.3}$$

with the corresponding homogeneous auxiliary condition $AG(x|y) = 0$.

14.3 Motivation of the theory: Delta function as linear functional. Let $\mathcal{D}$ be a set of real-valued functions on $\boldsymbol{R}$. Let us define a map $T_\delta : \mathcal{D} \to \boldsymbol{R}$ as $T_\delta(f) = f(0)$. Recall that T_δ is exactly the 'integral' of f times δ over $\boldsymbol{R}$ in the original 'definition' of δ (→**3.8** or **8B.13**). The most obvious and important property of T_δ is its linearity:

$$T_\delta(af + bg) = aT_\delta(f) + bT_\delta(g), \tag{14.4}$$

where a and b are reals. Therefore, we are tempted to write T_δ in terms of integral (→**6.2**) with some integration kernel δ: $T_\delta(f) = \int dx\, \delta(x) f(x)$ as in the original 'definition.' However, for T_δ there is no ordinary function δ satisfying this equality, because its 'value' at 0 cannot be finite. Still, T_δ is well-defined. Therefore, we <u>define</u> δ through T_δ:

14.4 Generalized function. Let T_q be a linear functional defined on a set $\mathcal{D}$ of real-valued functions on $\boldsymbol{R}$. The formal symbol $q(x)$ such that for $f \in \mathcal{D}$

$$T_q(f) = \int dx\, q(x) f(x) \tag{14.5}$$

is called a *generalized function*. The following notation, reminding us of the inner product (→**20.3**), is also often used for convenience:

$$T_q(f) = \langle q, f \rangle. \tag{14.6}$$

[210] A linear operator (→**1.4**) L is called a ***differential operator***, if $Lf(x)$ depends on $f(x)$ and its derivatives at x. For example, $-d^2/dx^2 + V(x)$, where V is a function, is a linear differential operator.

The definition must include the following rule for changing the independent variable. The rule is exactly the same as in the case of ordinary functions:

$$\int dx q(x) f(x) = \int ds \varphi'(s) q(\varphi(s)) f(\varphi(s)), \tag{14.7}$$

where q is a generalized function. f is a test function, and $x = \varphi(s)$ defines the change of variable.

14.5 δ-function: an official definition.[211] The symbol $\delta(x)$ such that

$$T_\delta(f) = \int \delta(x) f(x) dx = f(0) \tag{14.8}$$

is called the *δ-function.* Since the variable x in δ transforms as the usual x in the ordinary functions. the symbol for the map $f(x) \to f(a)$ is written as $\delta(x-a)$ (as we already noted in **14.1**):

$$\int \delta(x-a) f(x) dx = \int \delta(y) f(y+a) dy = f(a). \tag{14.9}$$

Exercise.

(A)

Evaluate

(1)

$$\int_{-5}^{5} \cos x \, \delta(x) dx. \tag{14.10}$$

(2)

$$\int_{-5}^{10} \delta(x) \log \Gamma(x+5) dx. \tag{14.11}$$

(B) A mass M is located at $x = 0$ on an infinite string: that is the density of the string is $\rho(x) = \rho + M\delta(x)$. Write down the equation of motion for the string under uniform tension τ (→**a1D.11**). We wish to consider the effect of the point mass on the incident wave. The wave $F(t - x/c)$ is incident from $x = -\infty$. The displacement is written as

$$u(x,t) = \begin{cases} F(t-x/c) + R(t+x/c) \text{ for } x < 0, \\ T(t-x/c) \text{ for } x > 0. \end{cases} \tag{14.12}$$

Show that

$$T'' + \gamma T' = \gamma F. \tag{14.13}$$

where $\gamma = \tau/Mc$. Find T in terms of F.[212]

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[211] It is often called the Dirac δ-function. ignoring the fact that this type of functions have been used for well over a hundred years.

[212] See G L Lamb. Jr. *Introductory Applications of Parital Differential Equations* (Wiley. 1995).

14.6 Value of generalized function at each point is meaningless. The value of a generalized function at a point is totally meaningless, because changing the value of a function at a point does not affect its integral (→**19.3**, **19.7**). Therefore, although δ-function was originally 'defined' such that $\delta(x) = 0$ for $x \neq 0$, according to our official definition, we cannot mention anything about the value of $\delta(x)$ for any $x \in \boldsymbol{R}$. Consequently, the product of delta functions containing common variables is a very dangerous concept.

Discussion.
However,
(A) **Localization theorem of generalized functions**. The value of a generalized function g at each point does not make sense, but it is possible to make such a statement meaningful as $g = 0$ in a neighborhood of a point. To this end, we must define $g = 0$ on an open set U.

We say $g = 0$ on U, if $\langle \phi, g \rangle = 0$ for any $\phi \in C_0^\infty(U)$ (= the set of all the C^∞ functions whose support is in U, that is, $\phi = 0$ outside U). Two generalized functions f and g are equal on U if $f - g = 0$ on U.

Theorem [Localization theorem]. For any x if there is its neighborhood on which $g = 0$, then $g = 0$ in the sense of generalized functions.

Theorem[Localization of derivatives]. If $f = g$ on U, then their derivatives are identical on U.

About the local properties, Section 3.2 of R. D. Richtmeyer, *Principles of Advanced Mathematical Physics* vol.1 (Springer, 1978) may be accessible.

(B) Of course, $f(g)$ does not usually make any sense for generalized functions f and g.

14.7 Multidimensional delta funcion. The definition of generalized functions on multidimensional space should be obvious (→**20.257** for curvilinear coordinate cases).

Exercise.
Let $d\tau$ be the volume element in 3-space, and r be the radial coordinate of the spherical coordinates, $\boldsymbol{r}$ is the position vector. Evaluate

$$\int d\tau e^{-r^2} \delta(r). \tag{14.14}$$

Here r must be considered as a function of $\boldsymbol{r}$. What is its difference from

$$\int d\tau e^{-r^2} \delta(\boldsymbol{r})? \tag{14.15}$$

14.8 Test functions. Since we cannot evaluate generalized functions pointwisely, the only way to study the property of a generalized function is to apply it to functions in an appropriate function set $\mathcal{D}$. The set $\mathcal{D}$ is called the set of *test functions*.

Discussion.
The choice of the test function set $\mathcal{D}$ is a matter of convenience. but the set must satisfy some obvious conditions such as its closedness: if $f_n \to f$ and $f_n \in \mathcal{D}$, then f should also be in $\mathcal{D}$. If the set $\mathcal{D}$ is very poor. then many generalized functions become indistinguishable. On the other hand, if $\mathcal{D}$ is too large, then we must meticulously pay attention to minute details of the generalized functions. From the practitioner's point of view. we need not pay much attention to $\mathcal{D}$, but should remember that very often $\mathcal{D}$ is the set of all the C^∞-functions with compact domains (i.e.. C_0^∞) or the set of all the *functions of rapid decay* (or rapidly decreasing functions. *Schwartz-class functions*):

$$\mathcal{D} = \{f : \boldsymbol{R} \to \boldsymbol{C}.\ C^\infty \text{ such that } x^n f^{(r)}(x) \to 0 \text{ as } |x| \to \infty \text{ for } \forall n, r \in \boldsymbol{N}\}.$$

(In words. $\mathcal{D}$ consists of infinite times differentiable functions whose any derivative decays faster than any inverse power.) The generalized functions defined on this $\mathcal{D}$ is called *generalized functions of slow growth.*

14.9 Equality. Two generalized functions p and q are said to be equal.[213] if no test function can discriminate them:

$$T_p(f) = T_q(f) \text{ for all the test functions } f \iff p = q. \qquad (14.16)$$

14.10 Regular distribution. Let $\mathcal{D}$ be an appropriate test function set (see the footnote in **14.8**). If a generalized function q is equal to some ordinary function. we say q is a *regular distribution.*

14.11 $\delta(ax) = |a|^{-1}\delta(x)$. To demonstrate an equality of generalized functions. the surest way is to return to the definition of the equality **14.9.**

$$\int \delta(ax)f(x)d(x) = \int \delta(y)f(y/a)|a|^{-1}dy = |a|^{-1}f(0/a) = \int |a|^{-1}\delta(x)f(x)dx \qquad (14.17)$$

for any test function. so that we may conclude the desired relation.

14.12 $\delta(g(x))$. Let g be a differentiable function. If $g \neq 0$ at x, then a sufficiently small neighborhood of x does not contribute to $\int dx\delta(g)f$. If $g(x_0) = 0$. then $g(x) \simeq g'(x_0)(x - x_0)$ locally. so we may replace $\delta(g(x))$ with $|g'(x_0)|^{-1}\delta(x - x_0)$ locally ($\to$**14.11**). In this way we have the following general formula for differentiable g:

$$\delta(g(x)) = \sum_i |g'(x_i)|^{-1}\delta(x - x_i), \qquad (14.18)$$

where the summation is over all the real zeros $\{x_i\}$ of g.
Demonstrate this as a hyperfunction ($\to$**8B.13**).

[213] This equality is consistent with the equality discussed in **14.6** Discussion (A).

Exercise.
Evaluate
(1)

$$\int_{-5}^{5} \delta(3x) \cos x dx. \tag{14.19}$$

(2)

$$\int_{0}^{4} \delta(1-5x) \sin x dx. \tag{14.20}$$

(3)

$$\int_{-\infty}^{\infty} \delta(x^2-5x+6) x^2 dx. \tag{14.21}$$

(4)

$$\int_{-\infty}^{\infty} \delta(\sin 2\pi x) 2^{-|x|} dx \tag{14.22}$$

14.13 Convergence of generalized function. A sequence of generalized functions q_n is said to converge to q, if

$$\langle q_n, f\rangle \to \langle q, f\rangle \text{ for all } f \in \mathcal{D}, \tag{14.23}$$

and is written as $\lim_{n\to\infty} q_n = q$. If we use the integral form, we have

$$\lim_{n\to\infty} \int dx q_n(x) f(x) = \int dx \lim_{n\to\infty} q_n(x) f(x). \tag{14.24}$$

That is, limit and integration can be freely interchanged, if we interpret an ordinary function as a regular distribution (→**14.10**). Consequently, termwise integration of series can be performed freely. If we never take the result outside the integral symbol, then we need not worry whether the final result is again a regular distribution or not. Recall that Green's functions (→**1.8**, **14.1**, **16**) always appear inside the integral symbol in practice. Hence, calculus of generalized function becomes a powerful tool especially when we construct and use Green's functions.

Discussion.
We have learned that if the limit of a sequence $\{\varphi_n\}$ converges weakly, then the limit is a generalized function.
(A) Let $G(x,t|y,0)$ be the Green's function for the diffusion equation in the free space (→**16B.1**) Show that for a continuous f

$$\lim_{t\to 0} \int_{\mathbf{R}^3} dy G(x,t|y,0) f(y) = f(x). \tag{14.25}$$

We write this as

$$w\text{-}\lim_{t\to 0} G(x,t|y) = \delta(x-y). \tag{14.26}$$

where w-lim denotes the *weak limit* which is meaningful only inside the integration. This is a possible definition of Dirac's δ-function.
(B) This observation allows us to introduce generalized functions in a different way (due to Korevaar). We prepare a weakly convergent[214] sequence (called *D-fundamental series*[215]) of sufficiently smooth functions $\{\varphi_n\}$ and declare its weak limit to be a generalized function.[216]

Take a positive number sequence ϵ_n such that $\lim_n \epsilon_n = 0$. Consider an arbitrary sequence of non-negative continuous functions $\{\varphi_n\}$ such that $\varphi_n(x) = 0$ for $|x| \geq \epsilon_n$ and

$$\int_{-\infty}^{\infty} \varphi_n(x)dx = 1. \tag{14.27}$$

The D-fundamental sequence $\{\varphi_n\}$ defines the $\delta(x)$.
(C) Demonstrate that indeed the sequence in (B) weakly converges to the δ-function in the $n \to \infty$ limit.
(D) The following sequences are examples of D-fundamental sequences for the delta function:

$$\varphi_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2}. \tag{14.28}$$

$$\varphi_n(x) = \frac{n}{\pi}\frac{1}{1+n^2x^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-|k|/n}dk. \tag{14.29}$$

$$\varphi_n(x) = \frac{\sin nx}{\pi x} = \frac{1}{2\pi}\int_{-n}^{n} e^{ikx}dk = \frac{1}{\pi}\int_0^n \cos(xk)dk. \tag{14.30}$$

$$\varphi_n(x) = \frac{1}{2\pi}\frac{\sin[(n+1/2)x]}{\sin(x/2)} \quad \text{(the Dirichlet kernel).} \tag{14.31}$$

$$\varphi_n(x) = n[\Theta(x+1/2n) - \Theta(x-1/2n)]. \quad (\Theta \text{ is the Heaviside step function}), \tag{14.32}$$

$$\varphi_n(x) = 0 \text{ for } |x| \geq 1/n \text{ and } n - n^2|x|, \text{ otherwise .} \tag{14.33}$$

$$\varphi_n(x) = \frac{1}{2n\pi}\left[\frac{\sin(nx/2)}{\sin(x/2)}\right]^2 \quad (\text{due to Fejer} \to \mathbf{17.10}) \tag{14.34}$$

$$\varphi_n(x) = \sum_{j=-n}^{n} e^{2j\pi ix} \quad \text{for } x \in (-2\pi, 2\pi). \tag{14.35}$$

14.14 Differentiation of generalized functions. If q is an ordinary differentiable function, then for $f \in \mathcal{D}$

$$\int q'(x)f(x)dx = -\int q(x)f'(x)dx. \tag{14.36}$$

[214] w.r.t. a test function set
[215] D for distribution.
[216] A precise definition in the original paper: J. Korevaar, Indagationes Math., **17**, 1955, is much more elaborate, because he wished to construct a theory equivalent to the one due to Schwartz. Here, the lecturer only wishes to make a subclass, so only a grossly simplified version of the original is given.

The right hand side makes sense, if f is differentiable. Our test function (→**14.8**) is always infinite times differentiable, so that we may regard (14.36) as the definition of q': the derivative q' of a generalized function q is defined as the generalized function satisfying

$$\langle q', f\rangle = -\langle q, f'\rangle. \tag{14.37}$$

By definition, generalized functions are infinite times differentiable.

Discussion.
Derivatives of generalized functions are given by the derivatives of its D-fundamental sequence (if it is differentiable →**14.18**). As we can see in **14.13** some are differentiable many or infinite times but some are not. We may choose convenient differentiable sequences without any contradiction.

Exercise.
(A) Show

$$x\delta'(x) = -\delta(x). \tag{14.38}$$

(B) Let $\Theta(x, y)$ be 1 when $x > 0$ and $y > 0$, and 0, otherwise. Then,

$$\frac{\partial^2\Theta(x, y)}{\partial x\partial y} = \delta(x)\delta(y). \tag{14.39}$$

(C) Evaluate
(1)

$$\int_{-5}^{5} \delta'(5x)\cosh x dx. \tag{14.40}$$

(2)

$$\int_{-\infty}^{\infty} \delta'(x^2 - 1)\cos x dx. \tag{14.41}$$

14.15 Examples.

(1)

$$\frac{d|x|}{dx} = sgn(x) \equiv \begin{cases} 1 \text{ for } x \geq 0, \\ -1 \text{ for } x < 0. \end{cases} \tag{14.42}$$

The reader may conclude this by intuition.[217] A standard demonstration may be to start with, for a test function f,

$$\begin{aligned} -\int dx|x|f'(x) &= -\int_0^\infty dx\, xf'(x) + \int_{-\infty}^0 dx\, xf'(x) \\ &= \int_0^\infty dx f(x) - \int_{-\infty}^0 dx f(x) = \int sgn(x)f(x)dx. \end{aligned} \tag{14.43}$$

[217] The value at $x = 0$ does not matter.

(2)

$$\frac{dsgn(x)}{dx} = 2\delta(x). \tag{14.44}$$

(3) The *Heaviside step function* $\Theta(x)$ is defined by $\Theta(x) = (1+sgn(x))/2$.

$$\frac{d\Theta(x)}{dx} = \delta(x). \tag{14.45}$$

14.16 All the ordinary rules for differentiation survive. For example. the chain rule is applicable.
When the reader feels uneasy in some use or abuse of generalized functions. always return to the definition **14.4**: operate the generalized function to test functions. See the next example.

14.17 Cauchy principal value $P(1/x)$. $P(1/x)$ is defined by

$$\int_{-\infty}^{\infty} P\frac{1}{x}f(x)dx \equiv P\int_{-\infty}^{\infty}\frac{f(x)}{x}dx. \tag{14.46}$$

where P is defined as **8B.10**.
(1) We have

$$\frac{d\ln|x|}{dx} = P\frac{1}{x}. \tag{14.47}$$

(2) Note that

$$xf = 1 \Rightarrow f = P\frac{1}{x} + c\delta(x). \tag{14.48}$$

where c is a constant. A demonstration follows. Note that obviously $xP(1/x) = 1$. Let $\phi \in \mathcal{D}$.

$$\begin{aligned}
\int f(x)\phi(x)dx &= \int f(x)\left[P\frac{1}{x}(\phi(x)-\phi(0))x + \phi(0)\right]dx \\
&= \int \left[P\frac{1}{x}(\phi(x)-\phi(0)) + f(x)\phi(0)\right]dx \\
&= \int P\frac{1}{x}\phi(x)dx + \int\left(f(x) - P\frac{1}{x}\right)\phi(0)dx \\
&= P\int\frac{\phi}{x}dx + const. \times \phi(0). \qquad (14.49)
\end{aligned}$$

Exercise.
Compute

$$P\int_{-1}^{1}\operatorname{cosec} x\, dx. \tag{14.50}$$

14.18 Theorem [Differentiation and limit always commute].

$$q_n \to q \Rightarrow q_n' \to q'. \tag{14.51}$$

□

This is a remarkably simple result. Termwise differentiation of series is allowed. To demonstrate the theorem is easy: For $f \in \mathcal{D}$

$$\langle q_n', f\rangle = -\langle q_n, f'\rangle \to -\langle q, f'\rangle = \langle q', f\rangle. \tag{14.52}$$

Compare this with the situation of the ordinary calculus (→**18.1** footnote. **A5.16**): we need uniform convergence of termwisely differentiated series.

Exercise.
From

$$\tanh\left(\frac{x}{\epsilon}\right) \to \text{sgn}x \tag{14.53}$$

show

$$1/\left[\epsilon\cosh^2(x/\epsilon)\right] \to 2\delta(x). \tag{14.54}$$

14.19 Example.

$$\frac{n}{\pi(1+n^2x^2)} \to \delta(x) \Rightarrow -\frac{2n^3x}{\pi(1+n^2x^2)^2} \to \delta'(x). \tag{14.55}$$

14.20 Example: Coulomb potential. In 3-space we have

$$\Delta\frac{1}{|\boldsymbol{x}|} = -4\pi\delta(\boldsymbol{x}). \tag{14.56}$$

Let us take a test function (→**14.8**) f and compute

$$\left\langle \Delta\frac{1}{|\boldsymbol{x}|}, f\right\rangle = \left\langle \frac{1}{|\boldsymbol{x}|}, \Delta f\right\rangle = 4\pi\int_0^\infty dr\, r^2\frac{1}{r}\Delta\overline{f} = 4\pi\int_0^\infty r\Delta\overline{f}\,dr, \tag{14.57}$$

where overline implies the average over directions (over θ and φ). Since (→**14.9**) $\overline{f}$ is spherically symmetric, we have $\Delta\overline{f} = r^{-1}d^2(r\overline{f})/dr^2$ (→**2D.10**). Hence.

$$\left\langle \Delta\frac{1}{|\boldsymbol{x}|}, f\right\rangle = 4\pi\int_0^\infty \frac{d^2}{dr^2}r\overline{f}\,dr = 4\pi[r\overline{f}' + \overline{f}]_0^\infty = -4\pi f(0). \tag{14.58}$$

14.21 Integral of generalized functions. A generalized function F is an *integral* of f, if $F' = f$. That is, $\langle F, \phi\rangle = -\langle f, \Phi\rangle$, where $\Phi' = \phi \in \mathcal{D}$.[218] Just as the ordinary calculus, we have a

Theorem. The integral of a generalized function is unique up to an additive constant. □

In summary, all the ordinary calculus rules survive.

Discussion.

The following integrals are sometimes useful (→**32C** also):

$$|u| = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1-\cos ku}{k^2}dk, \tag{14.59}$$

$$\pi\delta(x) \pm iP\frac{1}{x} = \int_0^{\infty} e^{\pm ixk}dx, \tag{14.60}$$

$$\frac{\pi}{2}\min(a,b) = \int_0^{\infty} \frac{\sin ax \sin bx}{x^2}dx. \tag{14.61}$$

14.22 Convolution. The *convolution* $p*q$ of two generalized functions p and q is defined as

$$\langle p * q, f\rangle \equiv \int dx \int dy\, p(x)q(y)f(x+y). \tag{14.62}$$

We use the following notation as well which is consistent with the above formula.

$$p * q(x) = \int p(y)q(x-y)dy. \tag{14.63}$$

Notice that $*$ is commutative, that is, $p * q = q * p$, and associative, that is, $q_1*(q_2*q_3) = (q_1*q_2)*q_3$. Therefore, we may define $q_1*q_2*q_3*\cdots$.

Exercise.

Compute $\delta' * |x|$. See **14.23**(2).

14.23 Example.

(1) $\delta * q = q$. That is, the delta function serves as the unit element for $*$-product.

(2) $\delta' * q = q'$. More generally, $\delta^{(n)} * q = q^{(n)}$. For example, $(\Delta\delta)*q = \Delta q$.

(3) $(p*q)' = p' * q = p * q'$. This can be demonstrated easily with the aid of (2) and associativity of $*$-product.

[218] There is a technical difficulty in this definition, since Φ may not be in $\mathcal{D}$. This problem can be overcome. See, for example, D. H. Griffel, *Applied Functional Analysis* (Ellis Harwood LTD, 1981), p. 38 Theorem 2.2.

(4) The solution to Poisson's equation (→**1.2**)

$$\Delta\phi = -\frac{\rho}{\epsilon_0} \text{ with } \phi \to 0 \quad (|x| \to \infty) \tag{14.64}$$

is given by $\rho * (1/4\pi\epsilon_0|x|)$ (→**14.20**). This is an implementation of Green's idea (→**1.8**).