13 Quasilinear First Order PDE

Quasilinear first order PDE has become increasingly important in recent years in physics in conjunction to renormalization group theory. Subsection A discusses how to solve general quasilinear first order PDE analytically in terms of characteristics. In Subsection B, as an application, we discuss homogeneous functions that are important in statistical mechanics and mechanics. In the last subsection C, the method outlined in A is applied to constant coefficient linear PDE including wave and diffusion equations.

Key words: quasilinearity, characteristic equation, characteristic curve, (generalized) homogeneous function

Summary:
(1) Quasilinear first order PDE can be solved with the aid of a system of ODE called characteristic equations, which can be written down easily (**13A.4**).
(2) Homogeneous functions (**13B.1**), their derivatives (**13B.3**) and the PDE they obey (**13B.2**) must be clearly understood.
(3) Constant coefficient cases may be solved by several standard tricks (**13C.3-5**).

13.A General Theory

13A.1 Quasilinear first order PDE. Let f_i $(i = 1, \cdots, n)$ and g be continuous functions of $x_1, \cdots, x_n$ and u.

$$\sum_{i=1}^{n} f_i \frac{\partial u}{\partial x_i} = g \tag{13.1}$$

is called a *quasilinear first order partial differential equation*. It is called linear, because it is a linear combination of partial derivatives. It is called 'quasilinear', because f_i and g are allowed to depend on u. It is clearly nonlinear in the physicists' sense, if f_i depends on u.

13A.2 Typical example. Suppose a flow field (i.e., the velocity field

$\boldsymbol{v}$) of an incompressible fluid is given. The continuity equation (the mass conservation) reads (→**a1E.5**)

$$\frac{\partial \rho}{\partial t} = -div\, \rho \boldsymbol{v} = -\boldsymbol{v} \cdot grad\, \rho. \tag{13.2}$$

This is a typical quasilinear first order PDE. From its meaning, if $\rho_{t=0}(\boldsymbol{r}) = f(\boldsymbol{r})$, then $\rho(t, \boldsymbol{r}) = f(\boldsymbol{r}(t))$, where $\boldsymbol{r}(t)$ is the particle trajectory starting from $\boldsymbol{r}$ at $t = 0$; that is, the solution to

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v} \tag{13.3}$$

with the initial condition $\boldsymbol{r}(0) = \boldsymbol{r}$. This is an example of the characteristic curve in the next entry.

13A.3 Two variable case. Consider

$$f(x, y, z)\frac{\partial z}{\partial x} + g(x, y, z)\frac{\partial z}{\partial y} = h(x, y, z), \tag{13.4}$$

where f, g and h are well-behaved functions[196] of x, y and z. To solve the equation is to find a relation among x, y and z so that (13.4) is true. We wish to find a 2-surface S given by $z = H(x, y)$ on which (13.4) holds. Suppose (x, y, z) and $(x + dx, y + dy, z + dz)$ are both on this surface. Then

$$\left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}, -1\right) \cdot (dx, dy, dz) = 0. \tag{13.5}$$

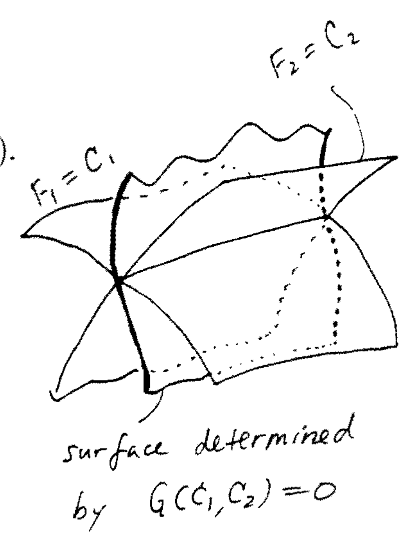
i.e., $\boldsymbol{n} = (\partial z/\partial x, \partial z/\partial y, -1)$ is a normal vector of the surface S. (13.4) implies that $\boldsymbol{n}$ and the vector (f, g, h) are orthogonal. That is, to solve (13.4) is to determine a surface $z = H(x, y)$ whose tangent vectors are (f, g, h). The equation of a curve whose tangent is (f, g, h) is given by

$$\frac{dx}{f} = \frac{dy}{g} = \frac{dz}{h}. \tag{13.6}$$

This is called the *characteristic differential equation* for (13.4) (cf. (13.3)). The solutions to (13.6) are called *characteristic curves*.

In the present case (13.6) is actually two ODE so that the general solution to (13.6) is given by two equations

$$F_1(x, y, z) = c_1, \quad F_2(x, y, z) = c_2, \tag{13.7}$$



[196]'Well-behaved' means that the relevant quantities are with sufficiently good properties, say smoothness, to allow us to ignore inessential technical details.

where c_1 and c_2 are integration constants. These equations describe surfaces. so their intersection is generically a curve we are looking for. It can be parametrized by these two parameters c_1 and c_2. If we change c_1 and c_2. the curve moves in space. If there is a functional relation between c_1 and c_2, then changing, say, c_1 (c_2 is slaved to c_1) produces a surface. Hence the general formula for the surface whose tangent vectors are given by (f, g, h) is given by

$$G(F_1. F_2) = 0, \tag{13.8}$$

where G is a (well-behaved) function which must be determined by auxiliary conditions. This is the general solution we have been looking for.

13A.4 How to solve quasilinear first ODE: method of characteristic equation. The characteristic equation for (13.1) is

$$\frac{dx_1}{f_1} = \frac{dx_2}{f_2} = \cdots = \frac{dx_n}{f_n} = \frac{du}{g}. \tag{13.9}$$

Solving this (actually n ordinary differential equations). we get n solutions corresponding to (13.7) (any convenient combinations can be chosen)

$$F_i(x_1. x_2. \cdots. x_n. u) = c_i \quad (i = 1. 2. \cdots. n). \tag{13.10}$$

from which we can get the general solution to (13.1) as

$$G(F_1. F_2. \cdots. F_n) = 0. \tag{13.11}$$

where G is a well-behaved function.

Historically. the method (and consequently the relation between the PDE and the ODE) was stated for the first time by Leibniz in his letter to l'Hospital in November. 1695.[197]

13A.5 Homogeneous case. If $g = 0$ in (13.1), then $du/g = du/0$ in (13.9) is interpreted as $u = const$. That is, one of the equations in (13.10) is u = const. In this case the general solution can be written as

$$u = G(F_1. F_2. \cdots, F_{n-1}). \tag{13.12}$$

where $F_1, \cdots, F_{n-1}$ are the remaining $n - 1$ relations of (13.10).

Discussion. [Complete integral].

[197] K. Okamoto. Butsuri. Jan. 1996.

A solution of a first order PDE is called a *complete integral*. if it has the same number of arbitrary constants as the number of independent variables.[198] If we have such a solution. we can make a solution which is dependent on a single arbitrary function w as follows: Let $a_n = w(a_1, \cdots, a_{n-1})$. and construct the envelope surface of $u(\boldsymbol{x}, a_1, \cdots, a_{n-1}, w(a_1, \cdots, a_{n-1}))$: that is, we make the following equations:

$$u_{a_1} + u_{a_n} w_{a_1} = 0, \tag{13.13}$$

$$\cdots \quad . \tag{13.14}$$

$$u_{a_{n-1}} + u_{a_n} w_{a_{n-1}} = 0. \tag{13.15}$$

From these equations we solve $n-1$ parameters as a function of $\boldsymbol{x}$. Then put these solutions into u. This is the desired solution. However. there is NO guarantee that the method can exhaust all the solutions constructed by the characteristic curve method.
(2) For example.

$$u = ax + by + \sqrt{1 - a^2 - b^2}z + c \tag{13.16}$$

is a complete integral of $(grad\, u)^2 = 1$.

13A.6 Examples.

(1) For (13.2). the characteristic equation reads

$$\frac{dt}{1} = \frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z} = \frac{d\rho}{0}. \tag{13.17}$$

or (13.3) and $d\rho = 0$ (→**13A.5**). Hence. $\rho(t, \boldsymbol{r}) = f(\boldsymbol{r}(t))$ in **13A.2** is justified.
(2)

$$(bx - ay)\frac{\partial f}{\partial x} + (ax + by - 1)\frac{\partial f}{\partial y} = 0. \tag{13.18}$$

The characteristic equation (→**13A.4**) is

$$\frac{dx}{bx - ay} = \frac{dy}{ax + by - 1} = \frac{df}{0}. \tag{13.19}$$

Solving this (→**11B.4**). the general solution is given by (cf. **13A.5**)

$$f(x, y) = G\left(\frac{b}{a}\arctan\frac{y-\beta}{x-\alpha} - \frac{1}{2}\log\left[1 + \left(\frac{y-\beta}{x-\alpha}\right)^2\right] - \log(x-\alpha)\right) \tag{13.20}$$

with $\alpha \equiv a/(a^2 + b^2)$ and $\beta \equiv b/(a^2 + b^2)$.
(3)

$$\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = (x - y)f. \tag{13.21}$$

[198] More precisely. the matrix $\partial^2 u/\partial_{x_i}\partial_{a_j}$ must be non-singular. where $u(x_i, a_j)$ is a complete solution.

Its general solution is given by

$$f = e^{-xy}G(x+y). \tag{13.22}$$

Exercise.
(A) Solve the following quasilinear first order PDE.
(1)

$$(y^2+z^2-x^2)\frac{\partial z}{\partial x} - 2xy\frac{\partial z}{\partial y} + 2xz = 0. \tag{13.23}$$

(2)

$$(bz-cy)\frac{\partial z}{\partial x} + (cx-az)\frac{\partial z}{\partial y} = ay - bx. \tag{13.24}$$

(3)[199]

$$\left[L\frac{\partial}{\partial L} + \frac{u}{\pi^2}\left(\frac{\epsilon\pi^2}{2} - u\right)\frac{\partial}{\partial u} + \frac{u}{(2\pi)^2}N\frac{\partial}{\partial N}\right] f = 0. \tag{13.25}$$

(4) Demonstrate that the solution to

$$-y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0 \tag{13.26}$$

is rotationary symmetric.
(B) Find the solution of $z(\partial z/\partial x) + \partial z/\partial y = 1$ passing through the curves $y = 2z$ and $x = z^2$.
(C) Solve

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = z. \tag{13.27}$$

and

$$x^2\frac{\partial z}{\partial x} - y^2\frac{\partial z}{\partial y} = y^2 - x^2. \tag{13.28}$$

Find the particular solution to the above equations going through $x = y = z$.

13.B Homogeneous Functions

13B.1 Homogeneous function of degree *p*. Let u be a well behaved real-valued function defined on a region in $\boldsymbol{R}^n$. If

$$u(\lambda x_1, \lambda x_2, \cdots, \lambda x_n) = \lambda^p u(x_1, x_2, \cdots, x_n) \tag{13.29}$$

[199] This is the renormalization group equation for the mean square end-to-end distance of a self-avoiding walk calculated by the ϵ-expansion method. What do you expect to happen in the $N \to \infty$ limit? Here, N is the number of steps.

for any real λ. u is called a *homogeneous function of degree p*, where p can be any real number. Since λ can be any number, $\lambda = x_1^{-1}$ is admissible, for example. This implies that a homogeneous function of degree p can be rewritten, for example, as

$$u(x_1, \cdots, x_n) = x_1^p f\left(\frac{x_2}{x_1}, \cdots, \frac{x_n}{x_1}\right). \tag{13.30}$$

Discussion.
More generally, a functional $f(\boldsymbol{r})$ is called a homogeneous function, if

$$f(\lambda \boldsymbol{r}) = g(\lambda) f(\boldsymbol{r}), \tag{13.31}$$

where g is a function of λ only. Show that

$$g(\lambda\mu) = g(\lambda) g(\mu). \tag{13.32}$$

If g is continuous at a point, then the following form is the only nontrivial solution to this functional equation: $g(x) = x^p$. Its proof is not easy.

13B.2 Theorem. A necessary and sufficient condition for u to be a once-differentiable homogeneous function of degree p is

$$\sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i} = pu. \tag{13.33}$$

[Demo] Necessity follows easily from the chain rule. To prove sufficiency, construct the general solution of (13.33) (→**13A.4**) and explicitly demonstrate that it is indeed homogeneous of degree p. Actually, we can easily get the form like (13.30).

13B.3 Theorem. Let u be a differentiable homogeneous function of degree $p \in \boldsymbol{R}$. Then, for any x_i, $\partial u / \partial x_i$ is a homogeneous function of degree $p - 1$. □
This follows trivially from the definition of homogeneous functions.

13B.4 Example from thermodynamics. *Extensive quantities* in thermodynamics such as the Gibbs free energy, magnetization, entropy are homogeneous functions of degree 1 of the masses of the constituents (chemical species) of the system. **13B.3** implies that *intensive quantities* such as temperature, chemical potential, pressure are homogeneous functions of degree 0. From, for example, $dE = TdS - pdV + \mu dN$ $E = TS - pV + \mu N$ follows according to Theorem **13B.2**.[200]

Exercise.
Let x be the extensive quantity X per unit mass. Show

$$de = Tds - pdv + \mu dn. \tag{13.34}$$

[200] The best thermodynamics textbook (introductory) for physicists is: H. B. Callen, *Thermodynamics* (Wiley, 1960).

13B.5 Kepler's third law. Consider an n-body conservative system with the potential energy given by $U(\boldsymbol{r}_1, \cdots, \boldsymbol{r}_2)$ which is a homogeneous function of degree p. This implies that the force is a homogeneous (vector-valued) function of degree $p-1$ (→**13B.3**). Newton's equation of motion

$$m_i \frac{d^2 \boldsymbol{r}_i}{dt^2} = -\frac{\partial U}{\partial \boldsymbol{r}_i}, \tag{13.35}$$

where m_i is the math of the i-th body. has the following scaling property: Scaling $\boldsymbol{r}_i \to \lambda \boldsymbol{r}_i$ and $t \to \mu t$ gives

$$m_i \frac{d^2 \boldsymbol{r}_i}{dt^2} = -\lambda^{p-2} \mu^2 \frac{\partial U}{\partial \boldsymbol{r}_i}. \tag{13.36}$$

Therefore. if $\mu = \lambda^{1-p/2}$. the equation of motion is invariant. For gravity. $p = -1$. so that this implies Kepler's third law ($T^2 = a^3$).[201]

13B.6 Generalized homogeneous function. If a function f satisfies

$$f(\lambda^a x. \lambda^b y) = \lambda^p f(x. y) \tag{13.37}$$

for any real λ and for some real numbers a. b and p. f is called a *generalized homogeneous function.* This is important in understanding critical phenomena. For example. the static scaling hypothesis (due to Widom) asserts that the Helmholtz free energy $F(T - T_c. H)$ of a magnet. where T_c is the critical temperature. and H the magnetic field. is a generalized homogeneous function for $\tau \equiv |T - T_c| \simeq 0$ and $H \simeq 0$:[202]

$$F = \tau^{2-\alpha} f(\tau / H^{1/\beta\delta}), \tag{13.38}$$

where $\alpha. \beta$ and δ are called *critical exponents.* These exponents and the functional form of f are universal for a class of materials.[203]

[201] For other examples. see J. M. Smith. *Mathematical Ideas in Biology* (Cambridge UP).

[202] See. for example. H. E. Stanley. *Introduction to Phase Transition and Critical Phenomena* (Oxford UP. 1971). Chapter 11. 12 and 15.

[203] See N. Goldenfeld. *Lectures on Phase Transitions and the Renormalization Group* (Addison Wesley. 1992).

13.C Application to Constant Coefficient Linear PDE

13C.1 Constant coefficient linear PDE. Introduce the notation $\partial_i \equiv \partial/\partial x_i$, and write collectively $\{\partial_i\}$. Let $P(\{x_i\})$ be a constant coefficient polynomial.

$$P(\{\partial_i\})u = g, \tag{13.39}$$

where g is a function of $\{x_i\}$, is called a constant coefficient linear partial differential equation. The general solution to (13.39) is the sum of the general solution[204] to the homogeneous problem $Pu = 0$ and a solution for $Pu = g$.

13C.2 Theorem [Malgrange-Ehrenpreis]. If g is C^∞ in a region $D \subset \boldsymbol{R}^n$, then (13.39) has a C^∞ solution in D. □[205]

13C.3 Factorization 'theorem'. If P is factorized into two mutually prime factors as $P = P_1P_2$, then the general solution to $Pu = 0$ is the sum of the general solutions to $P_1u = 0$ and $P_2u = 0$. □[206]

This should be obvious from $P(f_1 + f_2) = P_2P_1f_1 + P_1P_2f_2$. Here we assume the function f is sufficiently smooth.

Since P is a polynomial of many variables, there is no guarantee that we can factorize this into distinct first order factors of the form $\sum a_i\partial_i$. If we can, we can exploit our knowledge of first order linear PDE (→**13A.4**). If we cannot factorize P into first order operators, there is no general way to solve the PDE (however, see **13C.8**).

From now on we study only two independent variable cases.

13C.4 How to solve inhomogeneous equation. As is stated in **13C.1** we have only to find one solution to $Pu = g$ by whatever means we can use. Useful observations are:
(1) If $P = P_1P_2$, then $Pu = g$ can be solved step by step. First find u_1 such that $P_1u_1 = g$, and then solve $P_2u = u_1$.
(2) $P(\partial_x, \partial_y)e^{ax+by}u = e^{ax+by}P(\partial_x + a, \partial_y + b)u$. [This can easily be seen from, e.g., $\partial_x^n e^{ax+by}u = \partial_x^{n-1}e^{ax+by}(\partial_x + a)u$.]

[204] By "general solution" we mean a solution containing m arbitrary functions for a m-th order PDE in the linear case.

[205] For a proof, see G. B. Folland, *Introduction to Partial Differential Equation*, p84-7.

[206] Practically, the 'theorem' is very useful as we see below, but precisely speaking, the theorem cannot be true, because the smoothness of the solution to a lower order PDE need not be as large as the original higher order PDE. Hence, the 'theorem' is useful only when we look for sufficiently smooth solutions.

13C.5 Lemma. The general solution to

$$(a\partial_x + b\partial_y + c)^n u = 0 \tag{13.40}$$

is

$$u = e^{-cx/a}\sum_{i=0}^{n-1} x^i \phi_i(bx - ay), \tag{13.41}$$

where ϕ_i are arbitrary functions. (If $a = 0$, then replace $e^{-cx/a}$ with $e^{-cy/b}$ and x^i with y^i). □

To demonstrate this use (1) and (2) of **13C.4**. Also it is useful to remember the following standard trick. Let L be a linear first order differential operator and we wish to solve $L^2u = 0$. If we know the solution to $Lv = 0$, then introduce w as $u = wv$. The equation for w is usually easier to solve.

13C.6 Examples. Find the general solutions (Review **2B.4**).
(1) 1-space wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}. \tag{13.42}$$

For this equation

$$P(\partial_t, \partial_x) = \partial_t^2 - c^2\partial_x^2 = (\partial_t - c\partial_x)(\partial_t + c\partial_x). \tag{13.43}$$

Hence, the factorization theorem **13C.3** implies that the general solution to the wave equation is the sum of the general solution to $(\partial_t - c\partial_x)u = 0$ and $(\partial_t + c\partial_x)u = 0$. These can be solved easily by the standard method **13A.4**, so that the general solution to (13.42) is

$$u(t, x) = F(x - ct) + G(x + ct), \tag{13.44}$$

where F and G are arbitrary twice differentiable functions.[207] That is, the general solution is a superposition of right and left propagating waves as we have already seen in **2B.2**.
(2) 2-space Laplace equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{13.45}$$

This is the case with $c = i$ of (13.42). Hence, its general solution can be written as $F(x + iy) + G(x - iy)$. We are looking for real solutions.

[207] See the footnote of **13C.3**.

If $F(z) + G(\bar{z})$ is real. then it must be a real part of some analytic function.[208] Hence. the general solution is a real part of any analytic function (→**5.6**).

(3)

$$\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial y} = 0. \tag{13.46}$$

Its general solution reads

$$u(x, y) = F(2x - y) + e^y G(x), \tag{13.47}$$

where F and G are twice differentiable functions.

13C.7 Examples of inhomogeneous equations. Find general solutions:

(1)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x. \tag{13.48}$$

Use **13C.4**(1). or by inspection $u = x^3/6$ is a solution. Thus the general solution to this equation reads (→**13C.6**)

$$u(x.y) = F(x + iy) + G(x - iy) + \frac{x^3}{6}. \tag{13.49}$$

(2)

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \sin(x + at) \quad (a \neq \pm 1). \tag{13.50}$$

We use **13C.4**(2). Use the linearity of the equation and the fact that $Im\, e^{i(x+at)} = \sin(x+at)$. That is. get a solution to $(\partial_t^2 - \partial_x^2)u = e^{i(x+at)}$:

$$u = \frac{1}{2i(1 - a^2)} e^{i(x+at)} \tag{13.51}$$

Its imaginary part is the desired solution. Hence. the general solution to (13.50) is (→**13C.6**(1))

$$u = F(x - t) + G(x + t) + \frac{1}{a^2 - 1}\sin(x + at). \tag{13.52}$$

If $a = \pm 1$ (resonant case). then introduce v as $u = e^{i(x+at)}v$. and make the equation for v. This is a standard method to solve resonant

[208] Notice first that $G(\bar{z})$ may be considered as the complex conjugate of some analytic function $H(z)$. Hence. $F(z) + \overline{H(z)}$ is real. We know $F(z) + \overline{F(z)}$ is also real for all z. so that $F(z) - H(z)$ must be real for all z. However. such an analytic function must be a real constant. Hence, we may identify H and F.

problems.[209]

Exercise.
(1) Find the solution of $z(\partial z/\partial x) + \partial z/\partial y = 1$ passing through the curves $y = 2z$ and $x = z^2$.
(2) Consider the following telegrapher's equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} + (a+b)\frac{\partial u}{\partial t} + abu = 0, \tag{13.53}$$

where a and b are constants. The standard way to remove the first order derivative term is to introduce

$$v = e^{(a+b)t/2}u. \tag{13.54}$$

We have

$$\frac{\partial^2 v}{\partial t^2} - \left(\frac{a-b}{2}\right)^2 - c^2\frac{\partial^2 v}{\partial x^2} = 0. \tag{13.55}$$

If $a = b$, then the cable can propagate a wave without distorting the wave form, although the signal strength decays.

13C.8 Application to diffusion equation. For a diffusion equation in 1-space, the differential operator $P = (\partial_t - D\partial_x^2)$, so that we cannot factorize this into first order factors. However, if $P(a, b) = 0$, then $\exp(at + bx)$ is a solution. Hence, for example

$$e^{ikx - Dk^2t} \tag{13.56}$$

is a solution. Since the equation is linear, if the integral

$$u(x,t) = \int_{-\infty}^{\infty} dk f(k) e^{ikx - Dk^2t} \tag{13.57}$$

converges, then this turns out to be the general solution, where f is an appropriate function of k. For example, if f is in L_2 (→**20.5(2)**), the integral converges, and $u(x,t)$ is at least meaningful as a weak solution (a solution in the generalized function sense →**14**).

13C.4(2) is also useful to find a special solution to inhomogeneous diffusion equations. For example, consider

$$\partial_x^2 u - \partial_t u = \sin(ax + bt). \tag{13.58}$$

$$\tilde{u} = \frac{1}{a^4 + b^2}\left[-a^2\sin(ax+bt) + b\cos(ax+bt)\right] \tag{13.59}$$

is a solution.

[209] This is the general trick we considered in **13C.5**.