12 Constant Coefficient Linear ODE

A practical method is outlined to solve constant coefficient linear ODE explicitly (constructively). A method to analyze the stability of a fixed point is also explained. A useful theorem to locate eigenvalues of a matrix is Gershgorin's theorem.

Key words: Exponential of matrix, stability, hyperbolic fixed point, Hartman-Grobman theorem, Gershgorin's theorem.

Summary
(1) Practice calculating e^A when A is not diagonalizable (**12.2**, **12.5**).
(2) Linear stability analysis: the stability around a hyperbolic fixed point is completely determined by the linearized equation (**12.8-10**).
(3) There is a useful theorem to restrict the locations of eigenvalues of a (complex) square matrix on the complex plane (Gershgorin's theorem) (**12.10**).

12.1 The general form. n-th order ODE with constant coefficients can always be written in the form (→**11A.4-6**)

$$\frac{d\boldsymbol{u}}{dx} = A\boldsymbol{u}, \tag{12.1}$$

where A is a $n \times n$ constant matrix, and $\boldsymbol{u}$ consists of u, $u_1 \equiv du/dx$, $u_2 \equiv d^2u/dx^2, \cdots, u_{n-1} \equiv d^{n-1}u/dx^{n-1}$. We have only to solve the constant coefficient first order equation (12.1). For non-constant coefficient cases, see **24**.

12.2 Exponential function of matrix. Consider the following formal series

$$f(t) = 1 + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots + \frac{1}{n!}A^nt^n + \cdots, \tag{12.2}$$

where 1 is the $n \times n$ unit matrix. If this series is truncated at some finite order, the result should be an $n \times n$ matrix. We say the series converges if $f(t)$ applied to any finite vector $\boldsymbol{v}$ converges.[185] We define the *norm* of the matrix by

$$\|A\| \equiv \sup_{\boldsymbol{v}} |A\boldsymbol{v}|/|\boldsymbol{v}|. \tag{12.3}$$

[185] This is equivalent to the componentwise convergence of the matrix series.

iterative replacement should be tried

We can obtain

$$\|f(t)\| \le \exp(\|A\|t). \tag{12.4}$$

Hence. if the components of A are finite. then the series is absolutely convergent and consequently $f(t)$ is well defined.[186] The series is also uniformly (in t) convergent. Therefore.s we may termwisely differentiate it to get

$$\frac{df(t)}{dt} = Af(t). \tag{12.5}$$

Hence. $f(t)$ is written as $f(t) = e^{tA}$.

12.3 General solution to (12.1). The general solution to (12.1) is

$$\boldsymbol{u}(t) = e^{tA}\boldsymbol{u}_0. \tag{12.6}$$

where $\boldsymbol{u}_0$ is a constant n-vector (the initial condition vector). For an orthonormal basis $\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n\}$. $\{e^{At}\boldsymbol{e}_1, \cdots . e^{At}\boldsymbol{e}_n\}$ is a fundamental system of solutions of (12.1). Since e^{At} is nonsingular for any A, the dimension of the space spanned by the initial data and that of the solutions at any time t are identical. That is, $\boldsymbol{u}(0)$ and $\boldsymbol{u}(t)$ are one-to-one correspondent. Theoretically. the formal solution may be enough. but we must be able to calculate the matrix e^{tA} explicitly.

12.4 Diagonalizable cases. Since our equation is linear. *complexification* is always helpful. That is, we interpret the equation to be on $\boldsymbol{C}^n$ instead of $\boldsymbol{R}^n$. and take the real part of the solution to obtain the real solution to the original problem. If the matrix A is normal (i.e.. $A^*A = AA^*$). then A is diagonalizable by a similarity transformation.[187] In this case there is a unitary matrix U such that $U^*AU = \Lambda$. which is a diagonal matrix $\lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_n$. It is easy to demonstrate (return to the definitioin **12.2**) that

$$U^*e^{tA}U = e^{\Lambda t}. \tag{12.7}$$

Therefore. the general solution[188] to (12.1) reads

$$\boldsymbol{u}(t) = c_1\boldsymbol{p}_1e^{\lambda_1 t} + c_2\boldsymbol{p}_2e^{\lambda_2 t} + \cdots + c_n\boldsymbol{p}_ne^{\lambda_n t}, \tag{12.8}$$

where c_i are arbitrary constants and $\boldsymbol{p}_i$ is an eigenvector belonging to the eigenvalue λ_i (here all the eigenvalues are multiply taken into

[186] If we interpret $|\boldsymbol{v}|$ to be the ordinary Euclidean length, then the norm defined here is equal to the maximum of the square root of the eigenvalues of A^*A.

[187] This is only true in general when the vector space is considered on the field $\boldsymbol{C}$. This is why we need complexification.

[188] Here. 'general' means that a solution from any initial data can be obtained.

account according to their multiplicity). This should be obvious from (12.6). (12.7) and the structure of the unitary matrix $U = (\boldsymbol{p}_1, \boldsymbol{p}_2, \cdots, \boldsymbol{p}_n)$. if we interpret $\boldsymbol{p}_i$ to be column vectors.
If the matrix cannot be diagonalized by a similarity transformation, then polynomials of t appears in place of constants. All the cases including this nondiagonalizable case can be solved constructively[189] as follows:

12.5 Practical procedure.
(A) In the above the most general approach is described to solve (12.1). To solve a constant coefficient n-th order linear ODE

$$a_n \frac{d^n u}{dt^n} + a_{n-1}\frac{d^{n-1}u}{dt^{n-1}} + \cdots + a_1 \frac{du}{dt} + a_0 u = 0. \tag{12.9}$$

we need not consider the general matrix. but a very special form which can be guessed from **11A.5**. Let its characteristic roots. i.e.. the roots of

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0. \tag{12.10}$$

be $\lambda_1, \cdots, \lambda_r$ with the multiplicity $m_1, \cdots, m_r$. respectively. Then. the general solution for (12.9) is given by a linear combination of

$$\{e^{\lambda_1 t}, te^{\lambda_1 t}, \cdots, t^{m_1-1}e^{\lambda_1 t}, e^{\lambda_2 t}, \cdots, t^{m_2-1}e^{\lambda_2 t}, \cdots, t^{m_r-1}e^{\lambda_r t}\}. \tag{12.11}$$

A set of solutions which can span the totality of the solution space of an ODE is called its *fundamental system of solutions*.
(B) A general procedure to compute e^{tA} is as follows:
(1) Find the characteristic polynomial $f(x) = det(xI - A)$. and eigenvalues (the zeros of f). Let

Method using D is not explained Let us go directly to Laplace transformation

$$f(x) = (x - \lambda_1)^{\mu_1}(x - \lambda_2)^{\mu_2}\cdots(x - \lambda_k)^{\mu_k} \tag{12.12}$$

(2) Compute the partial fraction expansion

$$\frac{1}{f(x)} = \frac{g_1(x)}{(x-\lambda_1)^{\mu_1}} + \frac{g_2(x)}{(x-\lambda_2)^{\mu_2}} + \cdots + \frac{g_k(x)}{(x-\lambda_k)^{\mu_k}}. \tag{12.13}$$

(3) Compute

$$f_j(x) \equiv f(x)/(x - \lambda_j)^{\mu_j}. \tag{12.14}$$

Then make the following matrix (this is a projection operator →**20.19**)

$$P_j = f_j(A)g_j(A). \tag{12.15}$$

[189]'Constructive' means that an explicit procedure to obtain a solution is given.

(4) e^{At} is given by

$$e^{At} = e^{At}(P_1 + P_2 + \cdots + P_k). \tag{12.16}$$

Each term can be computed as follows:

$$\begin{aligned} e^{At}P_j &= e^{\lambda_j t}e^{(A-\lambda_j I)t}P_j, & (12.17) \\ &= e^{\lambda_j t}\sum_{m=0}^{\nu_j - 1}\frac{t^m}{m!}(A-\lambda_j I)^m P_j. & (12.18) \end{aligned}$$

In this calculation, we need not actually know what ν_j are. Simply calculate (12.18) until one gets the vanishing factor. Notice that ν_j does not exceed the multiplicity μ_j.[190]

A theoretical explanation why this procedure works is given in Appendix **a12**.

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Exercise.
(A) Solve the following linear ODEs:
(1)

$$\frac{du}{dt} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} u. \tag{12.19}$$

(2)

$$\frac{du}{dt} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} u. \tag{12.20}$$

(3)

$$\frac{du}{dt} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & -5 \\ 0 & 1 & 2 \end{pmatrix} u. \tag{12.21}$$

In this case the matrix can be diagonalized, but still the general method is useful.
(B) Construct the projection operators for eigenspaces of the following matrices

$$A = \begin{pmatrix} 1 & 9 \\ 1 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{12.22}$$

12.6 Inhomogeneous case. The general solution to the following inhomogeneous equation

$$\frac{du}{dx} = Au + f \tag{12.23}$$

[190] μ_j is the usual multiplicity (=algebraic multiplicity) of the eigenvalue λ_j. The number of eigenvectors (i.e., the dimension of the eigenspace for λ_j) need not be the same as μ_j. This dimension is the number ν_j.

is given by (use the method of variation of constants →**11.B.5**, **11B.13**)

$$\boldsymbol{u}(x) = e^{xA}\boldsymbol{u}_0 + \int_0^x e^{(x-y)A}\boldsymbol{f}(y)dy. \tag{12.24}$$

This has the usual form (→**11B.13**): sum of the general solution to the homogeneous equation (the first term) and a special solution for the inhomogeneous equation (the second term).

12.7 Stability question of a fixed point. Suppose we have a vector ODE

The motoin of charged particles

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$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v}(\boldsymbol{x}) \tag{12.25}$$

for which $\boldsymbol{x} = 0$ is a fixed point (i.e., $\boldsymbol{v}(0) = 0$). An important question is whether this solution is stable or not. That is, if we perturb the solution slightly $0 \to \delta\boldsymbol{x}$, does $|\delta\boldsymbol{x}|$ grow in time? If yes, then the solution cannot be stable. On the other hand, if this quantity goes to zero eventually for any small displacement, we may conclude that the fixed point is stable. The following theorem is a fundamental theorem (stated for the present case):

12.8 Theorem [Hartman-Grobman]. If 0 is a *hyperbolic fixed point*, that is, $d\boldsymbol{v}/d\boldsymbol{x}$ at $\boldsymbol{x} = 0$ does not have any pure imaginary eigenvalue, then for sufficiently small neighborhood of 0 the orbits of (12.25) and those of

$$\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x}. \tag{12.26}$$

where $A = d\boldsymbol{v}/d\boldsymbol{x}|_{\boldsymbol{x}=0}$, can be related one to one.[191] In particular, the stability (or instability) of 0 for (12.25) is equivalent to the stability (or instability) of 0 for (12.26).

12.9 Stability analysis of fixed point. **12.8** tells us that the stability of the fixed point of (12.25) is completely determined by the eigenvalues of the derivative $d\boldsymbol{v}/d\boldsymbol{x}$ evaluated at the fixed point, (if the fixed point is hyperbolic: if not, we must pay attention to the higher order terms: that is, linearization is not enough). If there is no eigenvalue whose real part is non-negative, then the fixed point is *linearly stable*. Thus the linear stability problem boils down to the eigenvalue problem. Sometimes the following theorem **12.10** is useful, which can locate the eigenvalues on the complex plain.

need an explanation

Discussion[Logical sloppiness].

[191] More precisely, the orbits are homeomorphic. That is, there is a continuous map which maps any orbit of (12.25) to that of (12.26) one to one continuously in both ways.

replace this with 入5 p110 Th4.9

Physicists often argue as follows. "Linearize the equation around the point of interest and make an equation for the small displacement δx. Since we find it shrinks to zero, we conclude that the point is stable." This argument is logically flawed. The lecturer is afraid that many working theoretical physicists do not feel any special problem in this argument until they are told that it does not show anything. Think. There are many such arguments in physics, and in most cases the conclusions are right. Can we empirically, then, ignore logic?

State that the ordinary discussion is wrong.

Exercise.
(1) Find the fixed point (equilibrium point) of

$$\begin{aligned} \frac{dx}{dt} &= x - xy, & (12.27)\\ \frac{dy}{dt} &= -y + xy. & (12.28) \end{aligned}$$

Show that the fixed point is not hyperbolic. Change the local coordinates around the fixed point to the polar coordinates, and demonstrate that the point is actually stable (i.e., the perturbation does not grow indefinitely).
(2) Study the stability of the origin pf the following Lorenz equation.[192]

$$\begin{aligned} \dot{x} &= -10(x-y), & (12.29)\\ \dot{y} &= rx - y - xz, & (12.30)\\ \dot{z} &= -\frac{8}{3}z + xy. & (12.31) \end{aligned}$$

Here r is a positive bifurcation parameter which controls the behavior of the system.
(3) Demonstrate that $x = 0$ is a stable solution (stable fixed point) of

$$\frac{dx}{dt} = Ax, \tag{12.32}$$

where

$$A = \begin{pmatrix} -1 & 0 & 1/2 & -1/3 \\ 1/4 & -1/2 & 1/5 & 0 \\ 1/4 & 0 & -1 & 1/2 \\ 1/4 & 1/3 & 4 & -5 \end{pmatrix}. \tag{12.33}$$

12.10 Gershgorin's theorem. Let $A = Matr\{a_{ij}\}$ be an $n \times n$ complex matrix. Its eigenvalues are all in the union $D = \cup_{i=1}^n C_i$, where C_i are discs called Gershgorin's disks:

$$C_i \equiv \{z \in \boldsymbol{C} \,|\, |z - a_{ii}| \le \sum_{j \ne i} |a_{ij}|\} \tag{12.34}$$

for $i = 1, \cdots, n$ (here no summation convention). The number of eigenvalues contained in each connected component of D is equal to the

[192] See, for example, E. A. Jackson, *Perspective of Nonlinear Dynamics*, vol.2 Sections 7.3-5.

number of disks making each connected component. □

[Demo] Let λ be an eigenvalue of A and $\boldsymbol{x} = (x_1, \cdots, x_n)^T$ a corresponding eigenvector. We have

$$\sum_{j=1}^{n} a_{ij} x_j = \lambda x_i, \quad (i = 1, \cdots, n). \tag{12.35}$$

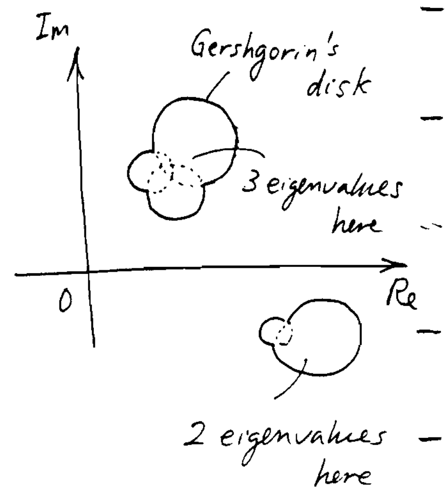
Since $\boldsymbol{x} \neq 0$, there must be x_k such that $|x_k| = \max_i |x_i| \neq 0$. For $i = k$ (12.35) reads

$$(\lambda - a_{kk}) x_k = \sum_{j \neq k} a_{kj} x_j \tag{12.36}$$

In other words,

$$|\lambda - a_{kk}| \leq \sum_{j \neq k} |a_{kj}| \frac{|x_j|}{|x_k|} \leq \sum_{j \neq k} |a_{kj}| = r_k. \tag{12.37}$$

you must choose the column or the raw to use this ineq.



This implies that $\lambda \in C_k$ which is obviously in D.

To prove the last part, we note the fact that the eigenvalues are continuously dependent on the matrix components. Let us split A into its diagonal part A_D and the off-diagonal part A_O: $A = A_D + A_O$. We make $A(t) = A_D + tA_O$. The second part of the theorem is trivially true for $A(0)$. The Gershgorin disks $C_i(t)$ for $A(t)$ depends on t continuously. The eigenvalues of $A(t)$ is also continuous functions of t. Hence, for any t (particularly for $t = 1$) the theorem must be true.

Discussion.

Study the trajectories of the eigenvalues of the following matrix $A(t)$ for $t \in [0, 1]$, and discuss their relation with the Gershgorin disks:[193]

$$A(t) = \begin{pmatrix} 0 & 3t \\ -7t & 8 \end{pmatrix}. \tag{12.38}$$

Notice that the eigenvalues do not move under the similarity transformation, but the matrix elements are altered, so that the estimate can be made better or worse with an application of a similarity transformation before applying the theorem. See the next example.

12.11 Application of Gershgorin's theorem.[194] Find the location of the eigenvalues of A.

$$A = \begin{pmatrix} 1 & \epsilon & 0 \\ \epsilon & 2 & \epsilon \\ 0 & \epsilon & 3 \end{pmatrix}. \tag{12.39}$$

If we apply the similarity transformation $A \to D^{-1}AD$, where

$$D = \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix}. \tag{12.40}$$

[193] Iri 1995

[194] From M. Iri, *Linear Algebra II* (Iwanami, 1994) p218. This is the best linear algebra textbook currently available, but in Japanese.

then the eigenvalue close to 1 can be located within the order of ϵ^2 instead of ϵ. This demonstrates the usefulness of similarity transformations applied before the estimation. It is not hard to find similar transformations allowing us to estimate the other eigenvalues with the same order of accuracy.

Difference euqations 入5 p45

Complex differential equations see Wasaw

APPENDIX a12 Decomposition of e^{At}

A theoretical basis of the practical method **12.5** is outlined here. Conventionally, the Jordan canonical form is used to compute e^{At}, but to make the Jordan canonical form may not be very easy.[195]

(1) Let $f(x)$ be the characteristic polynomial: $f(x) = det(xI - A)$. If

$$f(x) = (x - \lambda_1)^{\mu_1}(x - \lambda_2)^{\mu_2}\cdots(x - \lambda_k)^{\mu_k} \tag{12.41}$$

λ_j is an eigenvalue and μ_j is called its *multiplicity*.

(2) The lowest order polynomial $\varphi(x)$ satisfying $\varphi(A) = 0$ is called the *minimal polynomial* of A. φ must divide f and has the following form:

$$\varphi(x) = (x - \lambda_1)^{\nu_1}(x - \lambda_2)^{\nu_2}\cdots(x - \lambda_k)^{\nu_k}. \tag{12.42}$$

$0 < \nu_j \le \mu_j$. A necessary and sufficient condition for A to be diagonalizable is $\nu_j = 1$ for all j.

(3) **Theorem [Frobenius]**. Let $g(x)$ be the largest (highest order) common divisor of all the $(n-1)$-subdeterminant minors of $xI - A$. Then the minimal polynomial φ is given by $\varphi = f/g$, where f is the characteristic polynomial.

(4) $W_j \equiv ker(\lambda_j I - A)$ (i.e., all the vectors satisfying $A\boldsymbol{p} = \lambda_j \boldsymbol{p}$) is called the *eigenspace* of A belonging to λ_j. $\tilde{W}_j \equiv ker(\lambda_j I - A)^{\nu_j}$ (i.e., all the vectors satisfying $(\lambda_j I - A)^{\nu_j}\boldsymbol{p} = 0$) is called the *generalized eigenspace* of A belonging to λ_j. If A is diagonalizable, then $W_j = \tilde{W}_j$ for all j.

(5) $\tilde{W}_1 \oplus \tilde{W}_2 \oplus \cdots \oplus \tilde{W}_k = C^n$. That is, the vector space on which A is acting is decomposed into the direct sum of generalized eigenspaces.

(6) The projection operator P_j for the generalized eigenspace $\tilde{W}_j$ can be constructed as follows: Let f be the characteristic polynomial. Compute the partial fraction expansion

$$\frac{1}{f(x)} = \frac{g_1(x)}{(x-\lambda_1)^{\mu_1}} + \frac{g_2(x)}{(x-\lambda_2)^{\mu_2}} + \cdots + \frac{g_k(x)}{(x-\lambda_k)^{\mu_k}}. \tag{12.43}$$

Here $g_j(x)$ is a polynomial of order not larger than $\mu_j - 1$. Then

$$P_j = f_j(A)g_j(A), \tag{12.44}$$

where

$$f_j(x) \equiv f(x)/(x - \lambda_j)^{\mu_j}. \tag{12.45}$$

(7) $(A - \lambda_j I)^q P_j = 0$ for $q \ge \nu_j$.

(8) Now we can decompose e^{At} as follows: $e^{At}(P_1 + P_2 + \cdots + P_k)$. Here

$$e^{At}P_j = e^{\lambda_j t}e^{(A-\lambda_j I)t}P_j, \tag{12.46}$$

$$= e^{\lambda_j t}\sum_{m=0}^{\nu_j - 1}\frac{t^m}{m!}(A - \lambda_j I)^m P_j. \tag{12.47}$$

where we have used (7) after expanding the exponential function.

[195]For this approach see M. W. Hirsch and S. Smale, *Differential Equations, Dynamical Systems, and Linear Algebra* (Academic Press 1974).