Calogero-Moser system and Lax pair 人 5 p103

11 Ordinary Differential Equation: General

The general theory of ordinary differential equations (ODE) is outlined with precise statements. In the second half of the section, elementary analytical techniques to solve ODE are summarized for convenience.

Key words: general solution, particular solution, singular solution, normal form, Cauchy-Peano's theorem, Lipshitz condition, Cauchy-Lipshitz' theorem, separation of variables, perfect differential equation, integrating factor, Bernoulli equation, Riccati equation, Lagrange's method.

Summary
(1) Any (normal form) ODE can be converted to a first order vector ODE (**11A.4-6**).
(2) For simple first order ODE, look up representative examples first. Some representative examples are in **11B**.
(3) For linear ODE, although a general theory will be given in the following sections (**12**, **24**), simple second order constant coefficient equations can be solved without any difficulty (**11B.11-13**).

11.A General Theory

11A.1 Practical advice. See, for example, Schaum's outline series *Differential Equations* by R. Bronson for elementary methods and practice. To learn the theoretical side, V. I. Arnold, *Ordinary differential equations* (MIT Press 1973: there is a new version from Springer) is highly recommended. E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, (McGraw-Hill, 1955) is a standard classic reference. I cannot recommend D. Zwillinger, *Handbook of Differential Equations* (Academic Press, 1989). This book may be useful, but the organization should be more intelligent.

Exercise. If you do not have any problem with the following ODE, then you can skip **11B**.
Find the general solutions of the following ODE.
(1) (→**11B.5**)

$$x\frac{dy}{dx} + 2y = \sin x. \tag{11.1}$$

(2) (→**11B.6**)

$$\frac{dy}{dx} + \frac{y}{x} = x^2y^3. \tag{11.2}$$

(3) (→**11B.7**. $y = x$ is a solution.)

$$\frac{dy}{dx} = y^2 - xy + 1. \tag{11.3}$$

(4) (→**11B.10-13**)

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = xe^{-2x}. \tag{11.4}$$

11A.2 Ordinary differential equation. Let y be a n-times differentiable function of $x \in \boldsymbol{R}$. A functional relation

$$f(x. y(x). y'(x), \cdots . y^{(n)}(x)) = 0 \tag{11.5}$$

among $x.y(x)$. $y'(x)$, $\cdots$. $y^{(n)}(x)$ is called an *ordinary differential equation* (ODE) for $y(x)$. and n is called its *order*. where the domain of f is assumed to be appropriate. Such $y(x)$ that satisfies $f = 0$ is called a *solution* to the ODE.

Discussion.
Which is more general (or more powerful as a descriptive means), (normal form→**11A.5**) ODE or (normal form) difference equations:

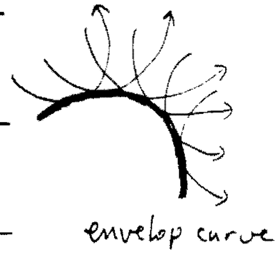
$$y_{k+n} = F(x. y_k. y_{k+12}, \cdots , y_{k+n-1}) \tag{11.6}$$

?[172]
[Hint: look up the following technical terms. suspension, and Poincaré section in a standard dynamical systems textbook.]

11A.3 General solution, particular solution, singular solution. The solution $y = \varphi(x. c_1. c_2. \cdots , c_n)$ to $f = 0$ in **11A.2** which contains n arbitrary constants $c_1. \cdots. c_n$ (which are called *integral constants*) is called the *general solution* of $f = 0$. A solution which can be obtained from this by specifying the arbitrary constants is called a *particular solution*. A solution which cannot be obtained as a particular solution is called a *singular solution*. For example, the envelope curve[173] (general solutions is a singular solution.

[172] This can alway be written in terms of differences $\Delta_1(k) \equiv y_{k+1} - y_k$, and higher order differences $\Delta_2(k) = \Delta_1(k+1) - \Delta_1(k)$, etc. Therefore, (11.6) may be considered as an n-th order difference equation. If the equation is linear with constant coefficients. then there is a general method to solve it (→**33**).

[173] The envelop curve of a smooth family of curves $\{F(x, \alpha) = 0\}$, where α is a parameter. is a curve tangent to all the members of the family, and is given by the conditions $F(x. \alpha) = 0$ and $\partial F(x. \alpha)/\partial\alpha = 0$.



Discussion.
(A) Consider the following equation called *Clairaut's equation*:

$$y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right). \tag{11.7}$$

(1) Show that its general solution is

$$y = Cx + f(C), \tag{11.8}$$

where C is a constant. [Hint. Differentiate (11.7) and factor out the second derivative. See **11B.16**.]
(2) The envelope curve of the family of lines defined by (11.8) is also a solution of (11.7). This is a singular solution.
(B) In **11A.11** $x \equiv 0$ is a singular solution to (11.26).

11A.4 Normal form. If the highest order derivative of y is explicitly solved as

$$y^{(n)}(x) = F(x, y, y', \cdots, y^{(n-1)}) \tag{11.9}$$

from $f = 0$, we say the ODE is in the *normal form*.[174]

11A.5 Normal form ODE is essentially first order. Let $y_j \equiv y^{(j-1)}$ $(j = 1, \cdots, n)$. Then (11.9) can be rewritten as

$$\frac{dy_1}{dx} = y_2, \tag{11.10}$$

$$\frac{dy_2}{dx} = y_3, \tag{11.11}$$

$$\cdots \tag{11.12}$$

$$\frac{dy_{n-1}}{dx} = y_n, \tag{11.13}$$

$$\frac{dy_n}{dx} = F(x, y_1, y_2, \cdots, y_n). \tag{11.14}$$

That is, (11.9) has been converted into a first order ODE for a vector $\boldsymbol{y} = (y_1, y_2, \cdots, y_n)^T$. Any normal form n-th order scalar ODE can be converted into the n-vector first order ODE of the form

$$\frac{d\boldsymbol{y}}{dx} = \boldsymbol{v}(x, \boldsymbol{y}). \tag{11.15}$$

Any solution $\boldsymbol{y}(x)$ can be understood as an orbit parametrize by 'time' x in the n-space (= phase space) in which $\boldsymbol{y}$ lives.

[174] Notice that not normal ODE's may have many pathological phenomena, but we will not pay any attention to the non-normal form case.

The inverse problem of 11A.5: When can we bring a first order ofrm into a normal form?
When linear: iff condition known R W Brokett Finite dimensional linear system (Wiley 1970); W J Tirrol: AMM 199 Oct p705

11A.6 Nonautonomous equation is not special. In (11.9) if F does not depend on x explicitly, we say the ODE is *autonomous*. If not, it is called *nonautonomous*. Parallelly, if $\boldsymbol{v}$ does not depend on x explicitly, we say (11.15) is autonomous; otherwise, nonautonomous. If we introduce one more variable t such that $dx/dt = 1$, then the set of equations in **11A.5** becomes autonomous:

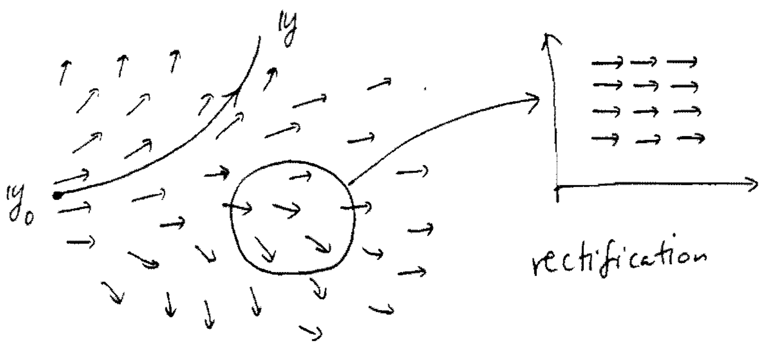
$$\frac{d\boldsymbol{y}}{dt} = \boldsymbol{v}(x, \boldsymbol{y}),$$
$$\frac{dx}{dt} = 1. \tag{11.16}$$

Hence, there is no fundamental difference as to the basic theory between autonomous and nonautonomous cases.[175] Thus to understand ODE, we have only to understand first order autonomous vector ODE.

11A.7 Initial value problem for first order ODE. To solve

$$\frac{d\boldsymbol{y}}{dx} = \boldsymbol{v}(\boldsymbol{y}) \tag{11.17}$$

under the condition that $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is called an *initial value problem*, where $\boldsymbol{y}(0)$ is called the *initial data*. The vector field $\boldsymbol{v}$ defining an ODE may be considered to be a flow velocity field on an n-space. Hence, the initial value problem is geometrically a problem to find an orbit passing through $\boldsymbol{y}_0$ at 'time' $x = 0$.



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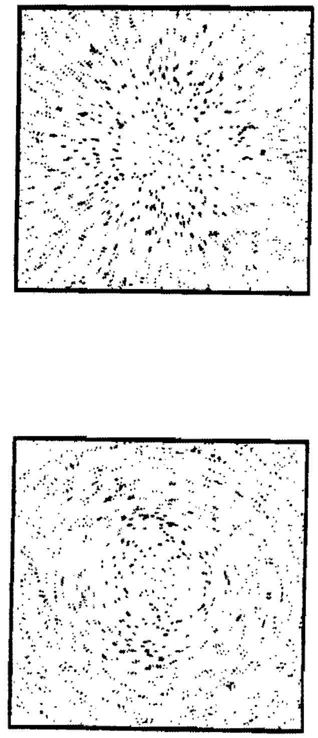
We summarize the standard theorems in the following. However, the general idea can be understood intuitively. A point where $\boldsymbol{v} = 0$ is called a *critical point*. Not near a critical point, the essence of the unique existence of the solution is given by the *rectification*. That is,

[175] Of course, the dimension of the phase space is increased by one, and this could cause a tremendous qualitative difference.

the flow can be transformed to a constant flow parallel to the first coordinate (by a one-to-one continuous map ≡ homeomorphism):

$$\frac{dy}{dx} = e_1. \tag{11.18}$$

This should be ituitively easy to understand through imagining the vector field being drawn on a rubber sheet.[176]



Glass patterns (from Kelso)

Discussion [Glass patterns]. An interesting method to make and visualize simple vector field is the Glass patterns. Make a random dot pattern more or less uniformly distributed on a sheet of paper. and make its transparency copy (it could be slightly scaled. or warped. so generating the points on computer may be advantageous). Then. superpose it on the original. If the displacements of the points are small. the reader will recognize a clear pattern. because her brain is a good detector of spatial correlation. The random dot moiré patterns are called *Glass patterns* after its discoverer L. Glass.[177] Applications of dynamical systems (= qualitative studies of differential equations) to cognitive psychology can be found in a recent book. J. A. Scott Kelso. *Dynamic Patterns. the self-organization of brain and behavior* (MIT Press. 1995).

11A.8 Theorem [Cauchy-Peano]. If for (11.17) $\boldsymbol{v}$ is continuous on a region $D \subset \boldsymbol{R}^n$. then for any $\boldsymbol{y}_0 \in D$ there is a solution $\boldsymbol{y}(x)$ of (11.17) passing through this point whose domain is an open interval $(\alpha.\omega)$ $(-\infty \le \alpha < \omega \le \infty)$. □

11A.9 Lipschitz condition. Let $\boldsymbol{v}$ be a continuous vector function whose domain is a region $D \subset \boldsymbol{R}^n$. For any compact[178] set $K \subset D$, if for any $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ both in K there is a positive constant L_K (which is usually dependent on K) such that

$$|\boldsymbol{v}(\boldsymbol{y}_1) - \boldsymbol{v}(\boldsymbol{y}_2)| \le L_K|\boldsymbol{y}_1 - \boldsymbol{y}_2|. \tag{11.19}$$

then $\boldsymbol{v}$ is said to satisfy a *Lipschitz condition* on D.
If $\boldsymbol{v}$ and $d\boldsymbol{v}/d\boldsymbol{y}$ are both continuous in D, then $\boldsymbol{v}$ satisfies a Lipschitz condition on D.

if C1 Lipshitz 入5 p18

Discussion.
(A) **[Hölder continuity].**
If a function f satisfies

$$|f(x) - f(y)| \le L|x - y|^\alpha \tag{11.20}$$

[176] Read the introductory part of the book review by P. Holmes, Bull. Amer. Math. Soc. **22**. 339 (1990).
[177] Nature. **223**. 578 (1960): L. Glass and R. Perez. Nature **246**, 3603 (1971).
[178] 'Compact' means in finite dimensional space 'closed and bounded' (→**A1.25**).

on its domain for constants L and $\alpha \in (0, 1)$, f is said to be ***Hölder continuous*** of order α. In particular, if $\alpha = 1$, f is said to be ***Lipschitz continuous***. A C^1 function is Lipschitz continuous due to the mean value theorem.
(B) **Cantor set and Cantor function (devil's staircase).** Let $x \in [0, 1]$ be written as

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}. \tag{11.21}$$

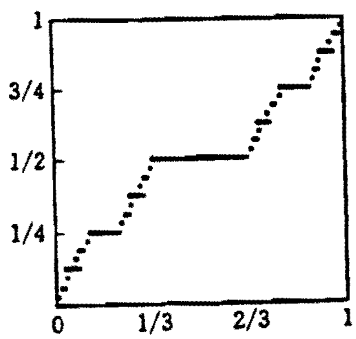
where $a_n \in \{0, 1, 2\}$. The function f is defined as follows:
(a) If $a_1, \cdots, a_{r-1}$ are not 1, but $a_r = 1$

$$f(x) = \sum_{n=1}^{r-1} \frac{a_n}{2^{n+1}} + \frac{1}{2^n}. \tag{11.22}$$

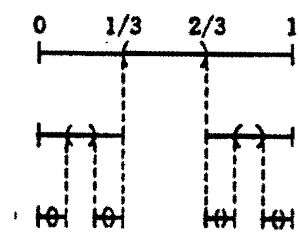
(b) Otherwise

$$f(x) = \sum_{n=1}^{\infty} \frac{a_n}{2^{n+1}}. \tag{11.23}$$

That is, $f(x)$ has the binary expansion $a_1/2 \cdots a_n/2$.
Sketch the function.



Devil's staircase



Cantor set

The function increases on the classical Cantor (→**17.19**) set:[179]

$$C \equiv \left\{ x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \middle| a_n \in \{0, 2\} \right\}. \tag{11.24}$$

(3) The Cantor function is Hölder continuous (see (A) above) of order $\log 2/\log 3$.
(4) What is the total length of C?
(5) Is C countable or uncountable? Is $[0, 1]\backslash C$ countable or uncountable? (→**17.18(4)**, **A1.16**).

11A.10 Theorem [Cauchy-Lipschitz uniqueness theorem]. For

[179] More generally, a perfect (that is, there is no isolated point) nowhere dense set is called a *Cantor set*.

(11.17). if $\boldsymbol{v}$ satisfies a Lipschitz condition on D, then if there is a solution passing through $\boldsymbol{y}_0 \in D$, it is unique. □

Discussion.
(A) **Why is the unique existence theorem important?** Physicists almost always ignore the existence theorem and the uniqueness theorem. However, they are very crucial even from the physics point of view. According to the Newton-Laplace determinacy (an empitrical fact), the motion of a point mass is completely determined by its initial position and velocity. Therefore, if the motion obeys a differential equation at all, it is easy to guess that the equation must be a second order equation. If we demand that there must be time reversal symmetry, we arrive at Newton's equation of motion (without the first order derivatives).[180] Is this guess really correct? If f is reasonable, yes. This affirmative answer is supplied by the unique existence theorem.
(B) Even if the Lipshitz condition is not satisfied: If the variables are separable as

$$\frac{dy}{dx} = \frac{Y(y)}{X(x)}, \tag{11.25}$$

and X and Y are continuous and not zero near (x_0, y_0), then the solution near this point is unique. However, the condition is important as we see in the next.

11A.11 Importance of being more than continuous. If the initial condition is given at the critical point of the vector field (i.e., where $v = 0$) , then the solution need not be unique. However, if the vector field is differentiable, then uniqueness still holds in this case. Consider for some positive integer n the following equation with the initial condition $x = 0$:

$$\frac{dx}{dt} = x^{1-1/n}. \tag{11.26}$$

$x \equiv 0$ is obviously a solution, but this is not the unique solution (Find the other). However, if we consider $dx/dt = x$, then $x \equiv 0$ is the only solution.

Exercise.
Find all the solutions such that $x = 0$ at $t = 0$ for (11.26).

Hint infinitely many

11A.12 Continuous dependence on the initial conditions. If the vector field is Lipschitz continuous (→**11A.9**), then the solution at time t depends on the initial condition continuously.

11A.13 Smooth dependence on parameter. If the vector field is smooth, then the solution at finite time is as smooth as the vector field. If the vector field is holomorphic (→**5.4**), then the solution is

[180] This is the way Arnold introduces Newton's equation of motion in his book, *Mathematical Methods of Classical Mechanics* (Springer, 1979).

Jacobi's elliptic functions 入5 p33

also holomorphic. Then, we can use perturbation theory to obtain the solution in powers of the parameter. This was the idea of Poincaré.

11.B Elementary Solution Methods

11B.1 Method of quadrature. To solve an ODE by a finite number of indefinite integrals is called the *method of quadrature*. Representative examples are given in this subsection. In practice, consult any elementary textbook of ODE or outline series.

11B.2 Separation of variables. The first order equation of the following form

$$\frac{dy}{dx} = p(x)q(y), \tag{11.27}$$

where p and q are continuous functions, is solvable by the separation of variables: Let $Q(y)$ be a primitive function of $1/q(y)$ and P that of p. Then $Q(y) = P(x) + C$ is the general solution, where C is the integration constant (→**11A.10** Discussion (A)).

Exercise.
(1) Show that

$$\frac{dy}{dx} = f(ax + by + c) \tag{11.28}$$

can be separated with the new dependent variable $u = ax + by + c$.
(2) Solve

$$\frac{dy}{dx} = x(x + y). \tag{11.29}$$

should be in 11B.5

11B.3 Perfect differential equation. Consider the first order ODE of the following form

$$\frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)}, \tag{11.30}$$

where $Q \neq 0$. If there is a function Φ such that $\Phi_x = P$ and $\Phi_y = Q$, then (11.30) is equivalent to

$$d\Phi = \Phi_x dx + \Phi_y dy = 0, \tag{11.31}$$

so that $\Phi(x,y) = C$, C being the integral constant, is the general solution.

Exercise.
(A) Show that the separable case (→**11B.2**) is a special case of perfect diferential

equations.
(B) Solve the following differential equation:

$$(x^2 + \log y)dx + \frac{x}{y}dy = 0. \tag{11.32}$$

11B.3a Integrating factor. Even if P and Q may not have such a 'potential' Φ. P and Q times some common function factor $I(x, y)$ called *integrating factor* may have a 'potential' Ψ:

$$d\Psi = IPdx + IQdy. \tag{11.33}$$

Then $\Psi = C$. C being the integral constant. is the general solution to (11.30).

It is generally not easy to find an integrating factor. However, we can easily check whether there is an integrating factor dependent on x alone or y alone. In such cases we can explicitly construct an integrating factor.

A necessary and sufficient condition for (11.30) to have an integrating factor dependent only on x is that

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) \tag{11.34}$$

is a function of x alone. An integrating factor can be obtained in this case as

$$I(x) = \exp\left(\int_{x_0}^{x} \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\right). \tag{11.35}$$

Exercise.
(1) Guess a necessary and sufficient condtion for (11.30) to have an integraing factor depedent only on y. and demonstrate your guess.
(2) Show that

$$I(x) = \exp\left(\int_{x_0}^{x} p(s)ds\right) \tag{11.36}$$

is an integrating factor for (11.30) in **11B.5**.

The existence problem of the integrating factor is crucial to thermodynamics. The second law. in essence. asserts that the heat form $\omega = dE - \sum x_i dX_i$. where E is the internal energy. X_i is an extensive variable. and x_i its conjugate intensive quantity. has an integrating factor called the absolute temperature (or its reciprocal).

Notice that if the number of independent variables (x and y in (11.30). E and X_i in thermodynamics) is two. then locally always intergrating factors do exist. See Discussioin (B) below.

This is perfect

Exercise.
Solve

$$\frac{dy}{dx} = \frac{2 + ye^{xy}}{2y - xe^{xy}}. \tag{11.37}$$

Discussion.
(A) If there is one integrating factor, then there are infinitely many. Suppose λ is an integrating factor of $Pdx + Qdy$ such that $du = \lambda(Pdx + Qdy)$. Show that any $\mu = \lambda v(u)$, where $v(u)$ is any differentiable function of u, is an integrating factor.
(B) **Incompleteness of elementary exposition of thermodynamics.** Born[181] pointed out that

$$dQ = Xdx + Ydy \tag{11.38}$$

always has an integrating factor. His argument is as follows. $dQ = 0$ means

$$\frac{dy}{dx} = -\frac{X}{Y} \tag{11.39}$$

so that it has (at least locally) a solution $\varphi(x, y) = C$ (Notice that this integration is generally impossible, if there are more than two variables). Hence,

equivalent to the statement that 2D space is conformally flat

$$\varphi_x dx + \varphi_y dy = \left(\varphi_x - \varphi_y \frac{X}{Y}\right) dx = 0. \tag{11.40}$$

for any dx. This implies that $\varphi_x/\varphi_y = X/Y$, so that there must be an integrating factor.

This observation has a grave consequence on elementary exposition of thermodynamics, because if a system is described in terms of E and V (as is customarily done in the Carnot cycle), then we do not need the second law to assert that there is an integrating factor for the heat form $\omega = dE + pdV$. The elementary introduction is, if not incorrect, grossly incomplete. This was first recognized by Born and motivated Caratheodory to study the mathematical foundation of thermodynamics.
(C) Demonstrate that

$$dQ = -ydx + zdy + kdz, \tag{11.41}$$

where k is a constant, has no integrating factor.

11B.4 Homogeneous equation. The following type of ODE is called a *homogeneous equation*:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right). \tag{11.42}$$

If we introduce $w = y/x$, this reduces to the separable case **11B.2**:

$$\frac{dw}{dx} = \frac{f(w) - w}{x}. \tag{11.43}$$

[181] Read M. Born, Physik Z. **22**, 218, 249, 282 (1922), if you can read German. The lecturer recommends this review article to every serious (statistical) physicist.

Exercise.
(1)

$$\frac{dy}{dx} = f\left(\frac{ax + by + c}{a'x + b'y + c'}\right) \tag{11.44}$$

can be converted to the homogeneous form ($a'b - ab' \neq 0$ is assumed). How can you do this? What happens if $a'b - ab' = 0$?
(2) Solve

$$\frac{dy}{dx} = \frac{y^4 + x^4}{xy^3}. \tag{11.45}$$

(3) Solve

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy} \tag{11.46}$$

11B.5 Linear first order equation, variation of constants. The first order equation

$$\frac{dy}{dx} = p(x)y + q(x) \tag{11.47}$$

is called a *linear equation*. The equation can be solved by the method of variation of constants. Let

$$y(x) = C(x)e^{\int^x p(s)ds}. \tag{11.48}$$

Then. the equation for C can be integrated easily. As we will see in **11B.13**. the method of variation of parameters *always* works for linear equations (Lagrange's method).

11B.6 Bernoulli equation. The first order equation of the following form is called a *Bernoulli equation*:

$$\frac{dy}{dx} = P(x)y + Q(x)y^n. \tag{11.49}$$

where n is a real number. Introducing the new variable $z(x) = y(x)^{1-n}$, we can reduce this equation to the case **11B.5** for $z(x)$.

Exercise.
Solve
(1)

$$\frac{dy}{dx} + xy - xy^2 = 0. \tag{11.50}$$

(2)

$$\frac{dy}{dx} + y - y^{-2} = 0. \tag{11.51}$$

The eqution that governs the tangent p(t) of aplane curve obeys a Riccati equation 入5 p15

11B.7 Riccati's equation. The first order equation of the following form is called a *Riccati's equation*:

$$\frac{dy}{dx} = R(x)y^2 + P(x)y + Q(x). \tag{11.52}$$

If $R = 0$, then it is linear (→**11B.5**): if $Q = 0$, then it is a Bernoulli equation. Otherwise, there is no general way to solve this equation by quadrature. However, if we know one solution $y = y_1(x)$ for this equation, the function $v(x) = y(x) - y_1(x)$ obeys the following Bernoulli equation.

$$\frac{dv}{dx} = [P(x) + 2R(x)y_1(x)]v(x) + R(x)u^2(x). \tag{11.53}$$

so we can obtain the general solution for (11.52) as $v + y_1$ in terms of the general solution v to this Bernoulli equation.

Discussion.[182]
Riccati discussed

$$\frac{dy}{dx} + ay^2 = bx^\alpha. \tag{11.54}$$

where a, b and α are constant. To avoid trivial cases, we assume all of them are nonzero. Liouville demonstrated that this equation can be solved in terms of elementray functions (trigonometric, exponetial, algebraic functions and their elementary combinations) only in the following cases:
(i) $\alpha = -2$.
(ii) $\alpha = -4n/(2n-1)$ for $n = 1, 2, \cdots$.
(iii) $\alpha = -4n/(2n+1)$ for $n = 1, 2, \cdots$.

11B.8 Second order ODE. This has the following form[183]

$$\frac{d^2y}{dx^2} = f\left(x, \frac{dy}{dx}\right). \tag{11.55}$$

If it is autonomous (→11A.6), it can be reduced to a first order PDE by introducing $p = dy/dx$ as the new unknown function, and y as the independent variable:

$$p\frac{dp}{dy} = f(y, p). \tag{11.56}$$

[182] K Yosida, *Solution Methods for Differential Equations*, second ed. (Iwanami, 1978) p20–.
[183] We consider only the normal forms (→**11A.4**).

If f does not depend on p, then this is **11B.2**, so p can be obtained. The resultant solution is interpreted as the first order ODE for y

$$\frac{1}{2}\left(\frac{dy}{dx}\right)^2 = \int^y dz\, f(z) + const., \tag{11.57}$$

which is again separable. This is a well-known method to solve 1D autonomous classical mechanical system.

11B.9 Method of reducing the order

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = 0 \tag{11.58}$$

can be converted to Riccati's equation by introducing

$$z = \frac{1}{y}\frac{dy}{dx}. \tag{11.59}$$

The result is

$$\frac{dz}{dx} + z^2 + a(x)z + b(x) = 0. \tag{11.60}$$

reducing

This method is due to d'Alembert (→**2B.7**) and is called the *method of lowering the order*. (11.59) is called *d'Alembert's transformation*. This technique allows us to reduce n-th order linear ODE to $n-1$-th order (generally nonlinear) ODE in general.

11B.10 The standard form of linear second order ODE. If $a = 0$ in (11.58), the equation is said to be in the *standard form*. If $a \neq 0$, then we introduce

$$z = y \exp\left(\frac{1}{2}\int^x a(x')dx'\right). \tag{11.61}$$

We have

$$\frac{d^2z}{dx^2} = -\left(b - \frac{a^2}{4} - \frac{a'(x)}{2}\right)z. \tag{11.62}$$

This form is a useful starting point for approximate solutions.

Discussion.
The following equation:

$$\mathcal{L}_{ST}\,u \equiv \left[\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\right]u = 0 \tag{11.63}$$

is called a Sturm-Liouville equation (→**15.4**), and

$$\frac{1}{w(x)}\mathcal{L}_{ST}u = \lambda u \tag{11.64}$$

with appropriate boundary conditions is called a Sturm-Liouville eigenvalue problem (→**35**). Any second order linear ODE can be converted to the Sturm-Liouville form.
(1) Demonstrate that

$$p_2(x)\frac{d^2u}{dx^2} + p_1(x)\frac{du}{dx} + p_0(x)u - \lambda u = 0 \tag{11.65}$$

can be converted to the Sturm-Liouville form with the following relations

$$w(x) = \frac{1}{p_2(x)}\exp\left[\int^x \frac{p_1(t)}{p_2(t)}dt\right]. \tag{11.66}$$

$$p(x) = w(x)p_2(x). \tag{11.67}$$

$$q(x) = w(x)p_0(x). \tag{11.68}$$

(2) Convert Bessel's equation (→**27A.1**) to the Sturm-Liouville form:

$$\frac{d}{dx}\left(x\frac{du}{dx}\right) + \left(1 - \frac{m^2}{x^2}\right)u = 0. \tag{11.69}$$

(3) By the following *Liouville transformation*

$$u(x) = v(t)[p(x)w(x)]^{-1/4} \tag{11.70}$$

with

$$t = \int^x \sqrt{\frac{w(s)}{p(s)}}ds \tag{11.71}$$

the above Sturm-Liouville equation can be converted to the Schrödinger form:

$$-\frac{d^2v}{dt^2} + V(t)v = \lambda v. \tag{11.72}$$

where the potential is given by[184]

$$V(t) = \frac{q(x)}{w(x)} + [p(x)w(x)]^{-1/4}\frac{d^2}{dt^2}[p(x)w(x)]^{1/4}. \tag{11.73}$$

In this formula x is understood as the function of t as defined by (11.71). This form is a good starting point to study asymptotic behaviors of the solutions.
(4) Convert Bessel's equation into the Schrödinger form:

$$\frac{d^2v}{dt^2} + \left[k^2 - \frac{m^2 - 1/4}{t^2}\right]v = 0. \tag{11.74}$$

[184] d/dt in the following formula acts only in $V(t)$: it is NOT an operator acting even outside the formula.

Compare this result with (11.62).

11B.11 Linear second order ODE with constant coefficients. Consider

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0, \tag{11.75}$$

where a and b are constants.

$$P(\lambda) = \lambda^2 + a\lambda + b \tag{11.76}$$

is called its *characteristic polynomial*. and its roots are called *characteristic roots*. We will discuss the general theory in the next section (→**12.5**). but in this simple second order case the general conclusion is the following:

11B.12 Theorem [General solution to (11.75)]. If the characteristic roots of (11.75) are α and β ($\neq \alpha$). then its general solution is the linear combination of $\varphi_1(x) = e^{\alpha x}$ and $\varphi_2(x) = e^{\beta x}$. If $\alpha = \beta$. then the general solution is the linear combination of $\varphi_1(x) = e^{\alpha x}$ and $\varphi_2(x) = xe^{\alpha x}$ (the characteristic roots need not be real.) □
$\varphi_1(x)$ and $\varphi_2(x)$ are called *fundamental solutions* and $\{\varphi_1(x), \varphi_2(x)\}$ is called a *system of fundamental solutions* for (11.75). A set of solutions is a fundamental system. if it spans (is a basis set of) the totality of the solution set of the ODE. See **24A** for more general statements.

Exercise.
Study the qualitative behavior of the following equation when the (bifurcation parameter) ϵ changes its sign:

$$\frac{d^2x}{dt^2} + 2\epsilon\frac{dx}{dt} + \omega^2 x = 0. \tag{11.77}$$

11B.13 Inhomogeneous equation, Lagrange's method of variation of constants. An ODE

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x) \tag{11.78}$$

with nonzero f is called an *inhomogeneous* ODE (the one without f is called a *homogeneous* equation). The general solution is given by the sum of the general solution to the corresponding homogeneous equation and one particular solution to the inhomogeneous problem. A method to find a particular solution to (11.78) is *Lagrange's method of variation of constants*. Let $\varphi_i(x)$ be the fundamental solutions. We determine the functions $C_i(x)$ to satisfy (11.78):

$$u(x) = C_1(x)\varphi_1(x) + C_2(x)\varphi_2(x). \tag{11.79}$$

One solution can be obtained from

$$\frac{dC_1}{dx} = -\frac{f(x)\varphi_2(x)}{W(x)}, \quad \frac{dC_2}{dx} = \frac{f(x)\varphi_1(x)}{W(x)}, \tag{11.80}$$

where $W(x) = \varphi_1(x)\varphi_2'(x) - \varphi_2(x)\varphi_1'(x)$, the Wronskian (→**24A.6**) of the fundamental system $\{\varphi_1, \varphi_2\}$. □
If the two characteristic roots α and β are distinct, then such a u is given by

$$u(x) = \frac{1}{\alpha - \beta}\left(\int_0^t ds f(s) e^{\alpha(t-s)} - \int_0^t ds f(s) e^{\beta(t-s)}\right). \tag{11.81}$$

Lagrange's method can be generalized to n-th order linear ODEs.

Discussion
Consider the relation of Lagrange's method and Green's function (→**15**). Riemann introduced Green's functions to solve linear ODE , so it is often called Riemann's function as well.

can be applied to difference equations as well.

Exercise.
Solve
(1)

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = \sin t. \tag{11.82}$$

(2)

$$\frac{d^2x}{dt^2} + 4x = \cos^2 2t. \tag{11.83}$$

(3)

$$\frac{d^2x}{dt^2} + x = \sin t. \tag{11.84}$$

11B.14 Equidimensional equation: invariance under scaling. If an ODE is invariant under the scaling of independent variable $x \to ax$, then we call the equation an *equidimensional equation* (in x). Explicit appearance of x in the equation can be removed by introducing $t = \ln x$ as the new independent variable. That is, the equation becomes an autonomous equation (→**11A.6**) in t (→**11B.15** Discussion).

If the equation is linear, then the general solution is given by the linear combination of the power of x, whose exponent can be determined by introducing x^μ (this is understood as $\log x$ if $\mu = 0$) into the equation. For example, the general solution to

$$\frac{d^2}{dr^2} rR = \frac{R}{r}\ell(1+\ell) \tag{11.85}$$

is given by $R(r) = Ar^\ell + Br^{-\ell-1}$. The equation appears when we separate the variable of the Laplace equation in the spherical coordinates (→**18.6**).

Exercise.
Show that the following Euler's differential equation

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = 0 \tag{11.86}$$

can be transformed to a constant coefficient linear ODE with the introduction of new independent variable t as $x = e^t$.

11B.15 Scale invariant equation. If an ODE is invariant under the scaling $x \to ax$ and $y \to a^p y$ for some p. we call the equation *scale invariant*. In this case. $v \equiv y/x^p$ obeys an equidimensional ODE. so that we can use the trick in **11B.14**.

Discussion.
The following equation is called the *Thomas-Fermi equation*

$$\frac{d^2\varphi}{dx^2} = \frac{1}{\sqrt{x}}\varphi^{3/2}. \tag{11.87}$$

This is a scale invariant equation under $x \to ax$ and $\varphi \to a^{-3}\varphi$. In the physical situation. the equation is solved under the boundary condition $\varphi(0) = 1$ and $\lim_{x\to\infty} \varphi(x) = 0$. so that such a simple scaling invariance does not hold. Still. if one wishes to get an asymptotic form for large x. this should be a good strategy.
(1) Show that in this asymptotic limit

$$\varphi(x) \simeq \frac{144}{x^3} \tag{11.88}$$

is a reasonable approximation.
(2) To obtain the correction to this solution. let us write

$$\varphi(x) = \frac{144}{x^3} + \psi. \tag{11.89}$$

and solve the equation to first order in ψ. The equation for ψ to this order becomes

$$\frac{d^2\psi}{dx^2} = \frac{18}{x^2}\psi. \tag{11.90}$$

(3) This is an equidimensional equation (→**11B.14**). so we can obtain the solution in the power form. The result is

$$\psi \propto x^{-\beta} \tag{11.91}$$

with $\beta = (-1+\sqrt{73})/2 \simeq 3.77$. If we obey the instruction above. we introduce $t = \ln x$ to convert the equation into

$$\left(\frac{d^2}{dt^2} - \frac{d}{dt}\right)\psi = 18\psi. \tag{11.92}$$

This is easy to solve (→**12.5**(A))).
(4) In this case the following amazing solution can be constructed. Using both the asymptotic solution and the correction we computed. we can construct

$$\varphi = \frac{144}{x^3(1+Cx^{-0.77})^n}. \tag{11.93}$$

where C and n are adjustable parameter. To make $\varphi(0)$ finite, we must choose $3 - 0.77n = 0$ or $n = 3.9$. Now. we can impose the boundary condition at 0. $C = 144^{0.77/3}$. Hence.

$$\varphi(x) = \left[1 + \left(\frac{x}{12^{2/3}}\right)^{0.77}\right]^{-3.90}. \tag{11.94}$$

According to Migdal. this solution agrees well with the numerical result.

11B.16 Clairaut's differential equation. Th following differential equation is called *Clairaut's differential equation*

$$y = px + f(p). \tag{11.95}$$

where $p = dy/dx$ and f is a C^1-function. Its general solution is

$$y = Cx + f(C). \tag{11.96}$$

where C is a constant. The equation has a singular solution (→**11A.3**). which is the envelop curve of (11.96).
Let us assume that (11.95) has a solution $y = y(x)$ which is not exhausted by (11.96). Put this in (11.95). and differentiate it with x. We obtain

$$\frac{dp}{dx}\left(x + \frac{df(p)}{dp}\right) = 0. \tag{11.97}$$

This implies $p = C$. or

$$x + \frac{df(p)}{dp} = 0. \tag{11.98}$$

This is the equation obtained from the derivative of (11.96) with respect to C and (11.95).

Exercise.
Solve

$$y = px + p - p^2. \tag{11.99}$$