# 10 Conformality and Holomorphy

In 2-space any holomorphic function whose derivative does not vanish defines a conformal map: a map which preserves the angles between any two lines. Harmonicity is preserved by conformal transformations. These facts imply that in 2-space conformal maps are very powerful tools to study the Laplace equation. Any 'reasonable' region can be conformally mapped to the unit disk (the Riemann mapping theorem). so that. in principle. if we can solve the Laplace equation in the disk. we can solve all the boundary value problems of the Laplace equation in 2-space. Some applications of Green's functions are given in **16D**.

**Key words**: conformality. Möbius transformation. Riemann mapping theorem.

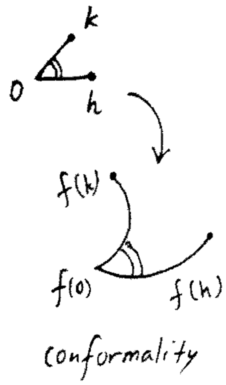
**Summary**:
(1) Understand the definition of conformality **10.1**. It is important to recognize that holomorphy and conformality are (almost) equivalent (**10.2**. **10.4**).
(2) Most regions can be conformally mapped to the disk (Riemann) **10.10**. A map which conformally maps a unit disk to a unit disk (both centered at the origin) must be a Möbius map **10.12**.[165]
(3) There is an algorithm to construct a conformal map which maps the upper half plane to the inside of a given polygon (Schwarz-Christoffel formula) **10.14**.
(4) Harmonicity is conformal-invariant **10.16**.

**10.1 Conformality**: Let $f$ be a map such that $f(c) = a$. and $\gamma_1$ and $\gamma_2$ be two curves crossing at $c$. $f$ is called *conformal* at $c$ if $f(\gamma_1)$ and $f(\gamma_2)$ crosses at $a$. and the angle from $\gamma_1$ to $\gamma_2$ is identical to the angle from $f(\gamma_1)$ to $f(\gamma_2)$. A map $f$ is called a *conformal map*. if it is conformal everywhere in its domain.

**Exercise**.
Confirm the conformality of

[165]There is a reference with computer-aided visualization: V I Ivanov and M K Trubetskov. *Handbook of Conformal Mapping with Computer-Aided Visualization* (CRC Press. 1995). containing a diskette (for PC). It says 'Handbook.' but detailed introduction is given. so it is a self-contained reference for those who know rudiments of complex analysis (of the level of these notes).

(1) $\sin z$ around $z = i + 1$.
(2) $\tan z$ around $z = i$.

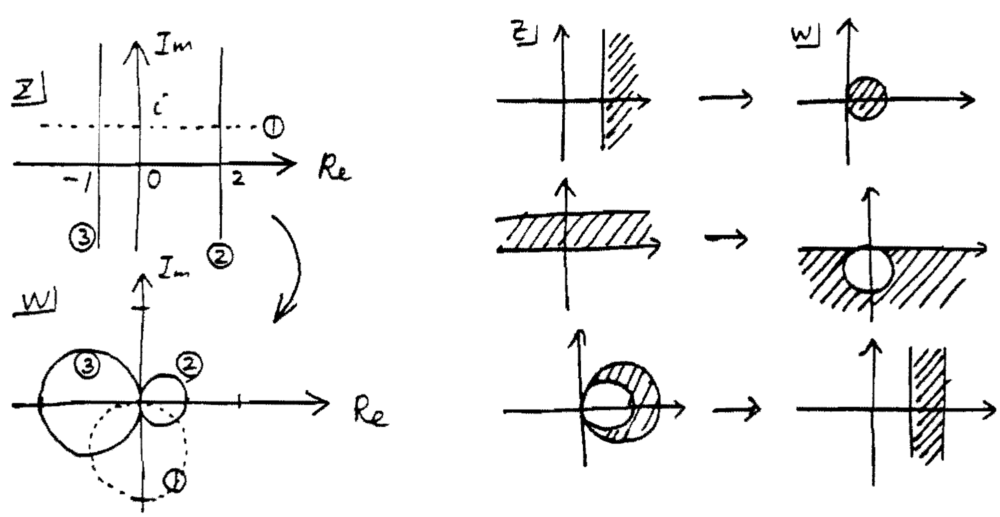



**10.2 Holomorphy implies conformality**. Take infinitesimal complex numbers $h$ and $k$, and consider an infinitesimal triangle $(0, h, k)$. The image of this triangle by a holomorphic function around the origin is given by $(f(0), f(h), f(k)) = (f(0), f(0) + f'(0)h, f(0) + f'(0)k)+$ higher order terms. Hence, the angle between $f(h)$ and $f(k)$ at $f(0)$ is identical to the angle between $h$ and $k$ at 0, if $f'(0) \neq 0$. In other words, a holomorphic function whose derivative is not equal to zero defines a conformal map.

**Exercise.**
Let $f$ and $g$ be a function on $\boldsymbol{C}$, and conformal.
(1) Is $f + g$ conformal? [What happens if $g = -f$?]
(2) Is $fg$ conformal?
(3) Is $f(g) = f \circ g$ conformal? (Assume that no complication due to the domains occur, for simplicity.)

**10.3 Convention: Conformal map from a region $D$ to another region $E$**: Henceforth, when we say $f$ is a conformal map from $D$ to $E$, we mean that $f$ is a *biholomorphic map* (i.e., the map and its inverse are both holomorphic) between $D$ and $E$.

**Exercise.**
(1) Illustrate the inversion $z \to w = 1/z$.



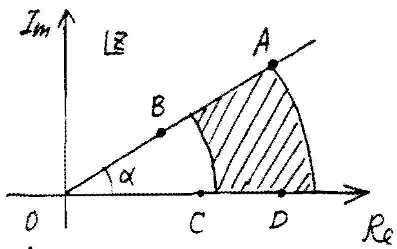

(2)

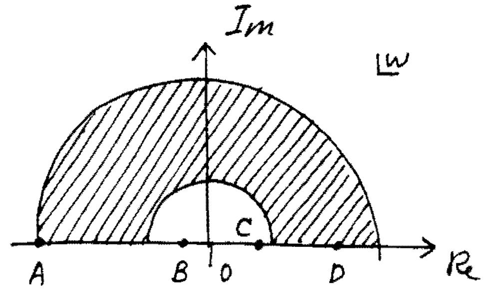
$$z \to w = \frac{z - z_0}{z - \overline{z_0}}. \tag{10.1}$$

Show that the real axis is mapped onto the unit circle around the origin. This is an example of the Möbius map (→**10.5**). Find the image of the upper half space.
(3)

$$z \to w = z^{\pi/\alpha}, \tag{10.2}$$

where $\alpha$ is a real number.



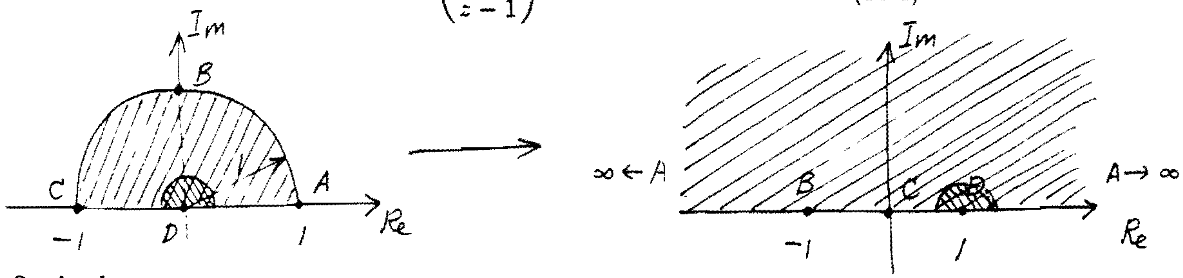




(4) Study the map:

$$z \to w = e^{\pi z/a}, \tag{10.3}$$

where $a$ is real.
(5) Study the map:

$$z \to w = \left(\frac{z+1}{z-1}\right)^2. \tag{10.4}$$




(6) Study the map:

$$z \to w = \left(\frac{1+z^{\pi/\alpha}}{1-z^{\pi/\alpha}}\right)^2. \tag{10.5}$$

(7) Study the map: $w = \tan z$.
(8) How is a square whose edges are parallel to the real and imaginary axes mapped by $e^z$?

**10.4 Conformality implies holomorphy**: Let $f : z \to w = f(z)$ be conformal in a region $D$. Then. $f$ is holomorphic in $D$ and $f'(z) \neq 0$.
□

[Demo] Let $\gamma_\theta(t) = z + te^{i\theta}$, and $\lambda_\theta(t) = f(\gamma_\theta(t))$.

$$\begin{aligned}\lambda'_\theta(0) &= u_x\cos\theta + iv_x\cos\theta + iu_y\sin\theta + iv_y\sin\theta \\ &= f_x\cos\theta + f_y\sin\theta = \frac{1}{2}[(f_x - if_y)e^{i\theta} + (f_x + if_y)e^{-i\theta}].\end{aligned} \tag{10.6}$$

From the figure we have

$$|\lambda'_\theta(0)| = e^{-i(\theta+\alpha)}\lambda'_\theta(0) = \frac{1}{2}[(f_x - if_y)e^{-i\alpha} + (f_x + if_y)e^{-i(2\theta+\alpha)}. \tag{10.7}$$

From this

$$\frac{d|\lambda_\theta'(0)|}{d\theta} = (f_x + if_y)(-2i)e^{-i(2\theta+\alpha)}, \tag{10.8}$$

but this is real for any $\theta$, so $f_x + if_y = 0$, which is the Cauchy-Riemann equation (→**5.3**). From (10.8), we get $f_x - if_y = 2f_x = 2f'(z) \neq 0$. Looman-Men'shov's theorem[166] tells us that if the real and imaginary parts of $f$ are partial differentiable, then the Cauchy-Riemann equation implies holomorphy.□

**Discussion.**
Let $\{f_n\}$ be a family of conformal maps, and it has a unique accumulation point $f = \lim_n f_n$.
(1) Is $f$ conformal?
(2) If not, make a counter example.
(3) Study a sufficient condition for $f$ to be conformal.
(4) Demonstrate that converging power series define conformal maps in their convergence disks. (We must assume $f \not\equiv 0$.)

## 10.5 Möbius transformation: The following map

$$w = \frac{az+b}{cz+d} \tag{10.9}$$

with $ad - bc \neq 0$ is called a *Möbius transformation* (or linear transformation or linear fractional transformation) (if $ad - bc = 0$, then $w$ is constant).

**Discussion.**
(A) Möbius transformations make a group: Let $G$ be a set on which a product $ab$ is defined (that is, if $a \in G$ and $b \in G$, then $ab \in G$). $G$ with this operation is called a *group*, if
(1) $a(bc) = (ab)c$, where $a, b, c \in G$ (*associativity*).
(2) There is an element $e$ (called the *unit*) such that $ea = a$ for any $a \in G$.[167],
(3) For any $a \in G$, there is an element (called the *inverse* of $a$) $a^{-1}$ such that $a^{-1}a = e$.[168]

The easiest way to demonstrate that Möbius transformations make a group is to realize the following relation: Let

$$w = \frac{az+b}{cz+d}, \quad w' = \frac{a'z=b'}{c'z+d'}. \tag{10.10}$$

Then,

$$w \circ w' = \frac{\alpha z+\beta}{\gamma z+\delta}. \tag{10.11}$$

[166]**Theorem [Looman-Men'shov].**
If $\partial f/\partial x$ and $\partial f/\partial y$ exist and satisfy the Cauchy-Riemann equation in a region $D$, then $f$ is holomorphic in $D$.

[167]Precisely speaking, this $e_L = e$ is called the left unit. The right unit is defined as $ae_R = a$ for any $a \in G$. Actually $e_L = e_R$ and is unique. Demonstrate this.

[168]Again, the left and right inverses can be defined, but they are identical for a given element.

where

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}. \tag{10.12}$$

(4) Notice that even if we multiply a common factor to $a, \cdots, d$, the result is the same map. Hence, we may demand that the matrices in the above have determinants normalized to unity. The group of $2 \times 2$ complex matrices whose determinants are unity is called $SL(2, C)$.[169]

(B) **Three points determine a Möbius transformation.** If we specify the destinations of three points by a Möbius transformation, then its form is fixed uniquely. This should not be hard to guess, if the reader consider the problem of fixing the coefficients in the Möbius map.

**Exercise.**
Let

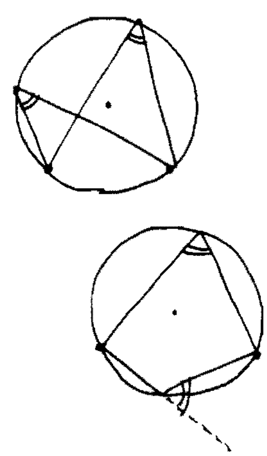
$$w(z) = \frac{z - i}{z + i}. \tag{10.13}$$

Then, $Im\, z > 0 \iff |w| < 1$. See **10.13**.

**10.6 Decomposition of Möbius transformation**: Any Möbius transformation is constructed by a consecutive applications of the following four elementary maps

$$z \rightarrow ze^{i\psi} \text{ rotation by angle } \psi. \tag{10.14}$$
$$z \rightarrow Rz \quad (R > 0) \text{ scaling.} \tag{10.15}$$
$$z \rightarrow z + \alpha \text{ translation .} \tag{10.16}$$
$$z \rightarrow 1/z \text{ inversion.} \tag{10.17}$$

**10.7 Circline**: Following Priestley[170], we will denote circles and lines collectively as *circlines*.



**10.8 Cocircline condition**: A necessary and sufficient condition for distinct four points $z, z_1, z_2, z_3 \in C \cap \{\infty\}$ to be on the same circline is that the *cross ratio* $(z, z_1, z_2, z_3)$:

$$(z, z_1, z_2, z_3) \equiv \frac{z - z_1}{z - z_3} \Big/ \frac{z_2 - z_1}{z_2 - z_3} \tag{10.18}$$

to be real. □

Note that this is equivalent to the elementary geometrical theorems illustrated here.

[169]Its important subgroup is $SU(2)$ consisting of unitary matrices, which is important in conjunction to the representation of the rotational group in 3-space.

[170]H.A. Priestley, *Introduction to Complex Analysis* Oxford UP, (revised edition 1990). This is a very nice introductory book.

**10.9 Möbius transformations map circlines to circlines.** This follows from the invariance of cross ratio under Möbius transformations. Let $f$ be a Möbius transformation. If $z_1, z_2, z_3$ are distinct, then

$$(f(z), f(z_1), f(z_2), f(z_3)) = (z, z_1, z_2, z_3). \tag{10.19}$$

This can be checked by an explicit computation. Immediately from this and **10.8**. we can verify the desired statement. □
We will see in **10.12** that only Möbius maps have this property.

**Exercise.**
Prove the relevant elementary geometrical theorem. and demonstrate that the cross ratio is invariant under Möbius transformations to complete the argument in the main notes.

**10.10 Riemann mapping theorem**: Let $D$ be a region in $\boldsymbol{C}$ such that $\partial D$ contains at least two points. Then, $D$ can be conformally mapped to the unit open disk. □
The proof of this theorem is beyond the scope of these lecture notes.[171] There is a constructive algorithm which asymptotically gives the desired conformal map (Koebe's method of image-domain). See Wilfe.

**10.11 There is a conformal map between two regions.** This theorem implies that if both regions $D$ and $D'$ have at least two points in their boundaries. then there is always a conformal map which maps one onto the other. Actually. the map can be fixed uniquely if we demand that a point $z \in D$ is mapped to $z' \in D'$, and a direction $\lambda$ at $z$ is mapped to a direction $\lambda'$ at $z'$. □
**Discussion [Joukowski transformation].**
F Let

$$z = \zeta + \frac{a^2}{\zeta}. \tag{10.20}$$

F where $a$ is a positive constant. The concentric circles $|\zeta| = b(\geq a)$ are mapped to confocal ellipses with the foci at $\pm 2a$. In particular, when $b = a$, the circle is mapped to the segment on the real axis connecting $\pm 2a$. For $b < a$ the circles are again mapped to the confocal ellipses.
(1) Demonstrate the above statements.
(2) Draw the images of circles on the $\zeta$-plane passing through $\zeta = a$. These images are generally called the *Joukowski wings*.

**10.12 'Unit disk → unit disk' must be Möbius**: Conformal transformations which map the unit disk onto itself must have the following

[171] For a complete proof. see for example. J.W. Dettman. *Applied Complex Variables* p257–. Chapter 6 of H.S. Wilfe. *Mathematics for the Physical Sciences* (Dover. 1962) is a self-contained exposition.

form

$$w = e^{i\theta}\frac{z-\alpha}{1-\overline{\alpha}z}, \tag{10.21}$$

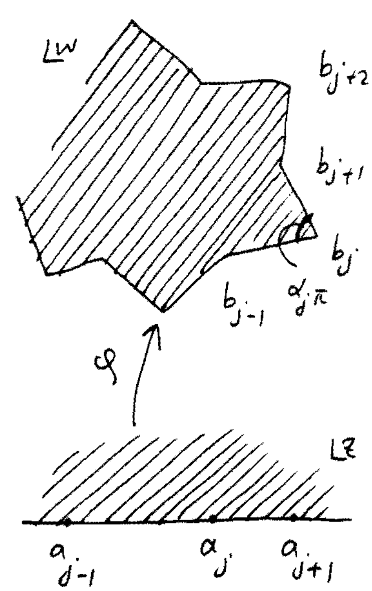
where $|\alpha| < 1$.□
Note that this is the map which maps the unit disk onto the unit disk, and the point $\alpha$ to 0.

**Discussion.**
The destination of three points on the unit circle fixes the map which maps the unit disk onto itself.

**10.13 'Upper half plane → the unit disk' must be Möbius**: Conformal maps which map the upper half plane onto the unit disk have the following form

$$w = e^{i\theta}\frac{z-\alpha}{z-\overline{\alpha}}. \tag{10.22}$$

**10.14 Schwarz-Christoffel formula**: The conformal map $w = \varphi(z)$ which maps the upper half plane ($\Im z > 0$) to the inside of the $n$-gon $\Pi$ in the figure whose vertices are at $b_1, \cdots, b_n$ (all assumed to be finite), and the interior angle at $b_j$ is $\pi\alpha_j$ $(0 < \alpha_j < 2)$ is given by



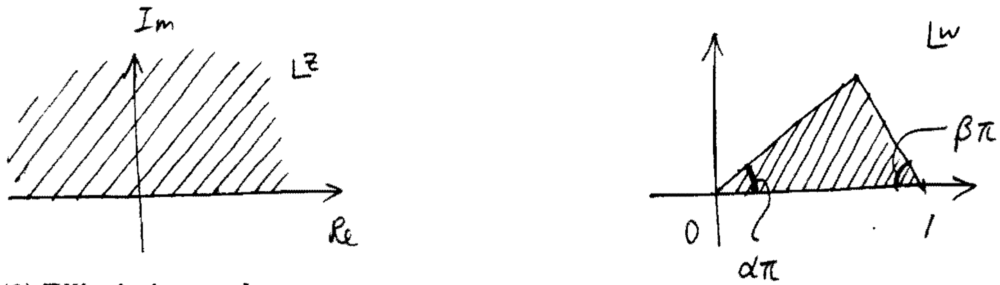

$$\varphi(z) = c\int_0^z \prod_{j=1}^n (z-a_j)^{\alpha_j - 1}dz + c'. \tag{10.23}$$

where $b_j = \varphi(a_j)$, and $c(\neq 0)$ and $c'$ are constants dependent on the position and the size of the polygon $\Pi$. If $a_n$ is at infinity, then drop the corresponding factor from (10.23). □
You need not remember the technicality. The point is that there is a method to construct a conformal map which maps a polygon to the upper half plane (hence to the unit disc, cf. **10.13**).
However, practically, it is wise to use a style book such as the one cited at the beginning of this section. Also, the lecturer has heard that now softwares are available for this formula.

**Discussion.**
Typical examples are
(1)

$$z \to w = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\int_0^z t^{\alpha-1}(1-t)^{\beta-1}dt, \tag{10.24}$$

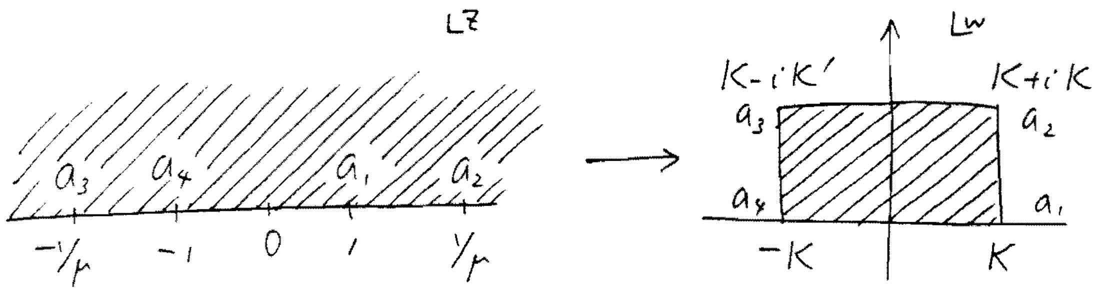
which maps the upper half plane to a rectangular triangle.




(2) **Elliptic integral:**

$$z \to w = c \int_0^z \frac{dz}{\sqrt{(1-z^2)(1-\mu^2 z^2)}}, \tag{10.25}$$

where $K$ and $K'$ are computed from $w(1)$ and $w(1/\mu)$.




**10.15 $\partial/\partial z$ and $\partial/\partial \overline{z}$.** It is often convenient to introduce the following operators:

$$\frac{\partial}{\partial z} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \overline{z}} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{10.26}$$

If we formally assume that $z$ and $\overline{z}$ are independent variables, then we get these relations. Furthermore, it is even true that

$$\frac{\partial z}{\partial z} = \frac{\partial \overline{z}}{\partial \overline{z}} = 1, \quad \frac{\partial \overline{z}}{\partial z} = \frac{\partial z}{\partial \overline{z}} = 0, \tag{10.27}$$

However, we should accept that (10.26) are definitions. Do not try to imagine the meaning of changing $z$ while keeping $\overline{z}$ constant. Notice that

$$\frac{\partial f}{\partial z} = f'(z). \tag{10.28}$$

$$\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} = \Delta. \tag{10.29}$$

and the Cauchy-Riemann equation (→**5.3**) reads

$$\partial f/\partial \overline{z} = 0. \tag{10.30}$$

Notice that the following is also true:

$$\overline{\left(\frac{\partial f}{\partial z}\right)} = \frac{\partial \overline{f}}{\partial \overline{z}}. \tag{10.31}$$

See also **16A.11-13**.

**Exercise.**
(1) Demonstrate the above statements.
(2) Let $f$ and $g$ be holomorphic in appropriate domains. Demonstrate

$$\frac{\partial g \circ f}{\partial z} = \frac{\partial g}{\partial w}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \overline{w}}\frac{\partial \overline{f}}{\partial z}, \tag{10.32}$$

$$\frac{\partial g \circ f}{\partial \overline{z}} = \frac{\partial g}{\partial w}\frac{\partial f}{\partial \overline{z}} + \frac{\partial g}{\partial \overline{w}}\frac{\partial \overline{f}}{\partial \overline{z}}. \tag{10.33}$$

(3) Let $P(z, \overline{z})$ be a polynomial of $z$ and $\overline{z}$. $P$ is holomorphic if and only if $P$ does not include $\overline{z}$.
(4) Let $f = u + iv$ be a holomorphic function on a region $D$. Then (→(10.29) ),

$$\Delta u = \Re\frac{\partial^2 f}{\partial z \partial \overline{z}}, \quad \Delta v = \Im\frac{\partial^2 f}{\partial z \partial \overline{z}}. \tag{10.34}$$

(5) Let $f$ be a holomorphic function, and $g$ be twice differentiable. Show

$$\Delta(g \circ f) = (\Delta g)|f'(z)|^2. \tag{10.35}$$

To show this note that

$$\frac{\partial^2 g \circ f}{\partial z \partial \overline{z}} = \frac{\partial^2 g}{\partial \zeta \partial \overline{\zeta}}\frac{\partial f}{\partial z}\overline{\frac{\partial f}{\partial z}}. \tag{10.36}$$

(6) Let $f$ be holomorphic in a region $D$. Since

$$0 = \frac{\partial}{\partial z}|f(z)|^2 = f(z)\frac{\partial}{\partial z}\overline{f(z)} = f(z)\overline{f'(z)}, \tag{10.37}$$

if $|f(z)|$ is constant on a region $D$, then $f$ must be a constant.

**10.16 Conformal Invariance of harmonicity**: Let $\Phi$ be harmonic (→**2C.11**, **5.6**) in a region $\Delta$ in the $w$-plane ($w = u + iv$), and $f$ be a conformal map $w = f(z)$ from the $z$-plane to the $w$-plane. Then, $\phi(x, y) = \Phi(u(x, y), v(x, y))$ (here $f(z) = u(x, y) + iv(x, y)$) is again harmonic in the domain $D$ which is the inverse image of $\Delta$ by $f$. In particular,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = |f'(z)|^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\Phi. \tag{10.38}$$

□

[Demo] The smartest demonstration may be: $\partial/\partial z = f'(z)\partial/\partial w$, and $\partial/\partial \overline{z} = \overline{f'(z)}\partial/\partial \overline{w}$, so that

$$\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z}\Phi = \frac{\partial}{\partial \overline{z}}f'(z)\frac{\partial}{\partial w}\Phi = f'(z)\overline{f'(z)}\frac{\partial^2}{\partial w \partial \overline{w}}\Phi. \tag{10.39}$$

Here we have used that $f'(z)$ is holomorphic (→**6.12**), so $\partial f'(z)/\partial \bar{z} = 0$ (the Cauchy-Riemann equation →**10.15**). □
See**16D** for applications.