1 Introduction

Representative linear partial differential equations (PDE) are introduced, and the main strategy to study them – the use of superposition principle – is outlined. This is an introductory section of the lectures, so the contents are not arranged in a logical order. For example, although we discuss PDE here, partial differentiation is reviewed in the next section. The Laplacian is introduced, but its more detailed discussion in conjunction to vector analysis also appears there. Therefore, the reader has only to try to get a general flavor.

Key words: Wave equation, diffusion equation, Laplace equation, linear boundary-value problems, linear operator, superposition principle, Fourier decomposition, Green's function, principal part, hyperbolic, parabolic, elliptic, Dirichlet, Neumann, Robin.

Summary:
(1) Superposition principle is the key to linear problems. Consequently, it dominates 'conventional mathematical methods for physicists' (**1.4-6**, **1.8**).
(2) The Laplacian describes the deviation from average (**1.13**, **1.14**).
(3) The nature of each type of second order linear PDE (**1.15**) can be understood with a representative physical example (**1.17-19**). Discretizing PDE is also a very good practical way to understand its nature (**1.15**, **1.18**(4)).
(4) Do not forget that PDE is not the only mathematical means to describe Nature (**1.12**).

The purpose of this introductory section is to give an overview of standard linear PDE, and a quick exposition of practical approaches to understand their properties. This is not an introduction for those who have never seen any PDE. A minimum prerequisite may be that the reader does not feel uncomfortable with elmentary derivations of the wave and the diffusion equations (in 1-space) in **a1D.11** and in **a1B.2**.

1.1 Spatially extended system. The main aim of the notes is to exhibit elementary applicable mathematics that may be helpful in understanding spatially *extended systems*, for which variations of quantities from place to place matter. Most systems we encounter in physics are such systems. There are important exceptions such as point mass systems. lumped circuits. etc.. but they are highly idealized systems, and are usually considered as limiting cases of extended systems.

Traditionally, spatially extended systems have been described in terms of partial differential equations (PDE →**1.16**), so that mathematical physics courses have mainly been devoted to the tools and concepts useful for PDE.

As is briefly stated in **1.12** we should be more flexible in modeling extended systems: PDE are not the only tools. However, many mathematical ideas surrounding PDE are still basic and also highly useful in other contexts as well, so that the course mainly discusses PDE.

Discussion.
For conventional topics. see Appendix **a1**.
Consider the following 'non-conventional' problems:
(1) On sand dunes. we can find fine ripples (aeolian ripples) on the surface. Make a (hopefully quantitative) model of these ripples. R A Bagnold. *The Physics of Blown Sand and Desert Dunes* (William Morrow. 1941. New York) is still a classic reference. The study was supported by the need of desert warfare. There is a recent attempt by Nishimori and Ouchi. Phys. Rev. Lett. **71**. 197 (1993). See also R S Anderson and K L Bunas. Nature **365**. 740 (1993). There will be a review article by Nishimori in Int. J. Mod. Phys.

We can also find sand ripples in shallow waters. Is the mechanism forming these underwater ripples the same as the ripples on the dunes?
(2) On mountain slopes without vegetation. sometimes ripple patterns made of pebbles can be found (B T Werner and B Hallet. Nature **361**. 142 (1993)). These are formed by frosts. Make a model of these patterns. cf. P A Mulheran. J. Phys. I France **4**. 1 (1994).
(3) Read H Meinhardt. *The Algorithmic Beauty of the Sea Shell* (Springer. 1995). which contains a disket for PC. A cellular automaton model comparable to this work has appeared recently: Kusch et al.. J. Theor. Biol. **178**. 333 (1996).
(4) For snow crystals. see E Yokoyama and T Kuroda. Phys. Rev. A **41**. 2038 (1990).
(5) Understanding developmental processes is of vital importance in understanding evolution and taxonomy of animals. There is a very serious attempt to model *Drosophila* development. The state of the art paper may be J. Reinitz and D. H. Sharp. "Mechanism of *eve* stripe formation," Mechanism of Development. **49**, 133 (1995).

1.2 Wave, Diffusion, and Laplace equations. The most typical partial differential equations appearing in classical physics are the following three equations **1.2a-c**. Their derivations in physical situations can be found in Appendix **a1**. but a general argument in **1.14** should tell the reader sufficiently convincingly why these are the representatives.

In the following, t is time. ψ denotes a (scalar) field (a function of space $\boldsymbol{x}$ and time), and the symbol Δ is the *Laplacian* ($\Delta = div\,grad$ →**2C.11**), which reads in the Cartesian coordinates as

$$\Delta \equiv \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}, \tag{1.1}$$

where d is the spatial dimensionality (cf. **16A.2**).

1.2a Introduction to wave equation. The *Wave equation* (examples: **a1D.9-11**, **a1F.8**) is given by

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \Delta \psi, \tag{1.2}$$

where a positive constant c is called the *wave speed* (→**2B.3**; also see the Discussion below).

1D: as a model of vibrating string by d'Almberg (1752); Euler extended it in 1759. D Brnoulli extended it to 2 and 3D See Brezis Adv Math 135, 76 (1998).

a1D.11 is the most elementary example of this equation, which can be understood without vector analysis.

Exercise.
Show in 1-space that $\sinh(x - ct)$ satisfies the wave equation. More generally, $f(x \pm ct)$ is a solution, if f is twice differentiable (→**2B.4**).

Discussion: Propagation of singularity.
Rewrite the 1D wave equation

$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2} \tag{1.3}$$

in the following form:

$$\frac{\partial v}{\partial t} = c\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial t} = c\frac{\partial v}{\partial x}. \tag{1.4}$$

Is there any curve $\phi(x, t) = 0$ on which u and v are continuous but their derivatives jump?

Assume that there is a curve $\phi(x, t) = 0$ along which the solution is not smooth, and denote by $[a]$ the jump in a across the curve. We must have

$$[v_t] - c[u_x] = 0, \quad [u_y] - c[v_x] = 0. \tag{1.5}$$

Notice that u and v are smooth along the curve, so that, for example,

$$[u_t] = [u_\phi]\phi_t, \tag{1.6}$$

where u_ϕ denotes the directional derivative along the normal direction to the curve. The slope of the curve is given by $dx/dt = -\phi_t/\phi_x$. Hence, (1.5) implies

$$[u_\phi]\frac{dx}{dt} + c[v_\phi] = 0, \quad [u_\phi] + c[v_\phi]\frac{dx}{dt} = 0. \tag{1.7}$$

If there is a curve we are seeking, there must be a nontrivial solution for $[u_\phi]$ and $[v_\phi]$, so that

$$\left(\frac{dx}{dt}\right)^2 - c^2 = 0, \tag{1.8}$$

or

$$\frac{dx}{dt} = \pm c. \tag{1.9}$$

That is, singularities can propagate along $x \pm ct$ = constant. These curves are called the *characteristic curves* (→**30.2**). To have such curves is a characteristic feature of hyperbolic equations (→**1.16**, **30**).

1.2b Introduction to diffusion equation. The *Diffusion equation* (examples: **a1B.2**, **a1C.1**, **a1F.17**) is given by

$$\frac{\partial \psi}{\partial t} = D\Delta\psi, \tag{1.10}$$

where a positive constant D is called the *diffusion constant.*

a1B.2 may be the simplest example if adapted to 1-space as follows:
Let j be the flux density of heat in the x-direction. Consider a small interval $[x, x+dx]$. At x $j(x)$ of heat flux is flowing into the interval in unit time, and at $x+dx$ $j(x+dx)$ is flowing out in unit time. Hence, in the small time span dt, the net 'income' of the interval is given by

$$j(x) - j(x+dx) = -\frac{\partial j}{\partial x}dx + o[dx]. \tag{1.11}$$

We assume Fourier's law $j = -\kappa \partial T/\partial x$ (T is temperature), and the heat capacity of the interval is Cdx. The energy conservation tells us

$$Cdx\frac{\partial T}{\partial t} = -\frac{\partial}{\partial x}\left(-\kappa\frac{\partial T}{\partial x}\right)dx. \tag{1.12}$$

If we assume κ is a constant, we obtain the 1D diffusion equation.

Discussion: Burgers equation.
The following equation is called the *Burgers equation*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \tag{1.13}$$

where ν is a positive constant (viscosity). The equation was proposed as a model equation of the Navier-Stokes equation (→**a2B.5**). This is a nonlinear equation in the sense that the equation is not invariant under the scaling of u. However, this is a disguised linear equation.
(1) With the aid of the famous *Cole-Hopf transformation:*[1]

$$u = -2\nu\frac{\partial \ln \varphi}{\partial x} \tag{1.14}$$

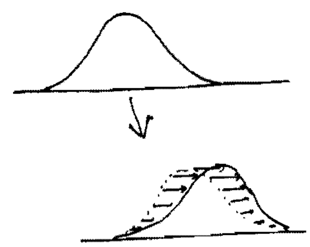
[1]Eberhard Hopf, 1902-1983.

demonstrate that (1.13) can be transformed to the diffusion equation:

$$\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial x^2}. \tag{1.15}$$

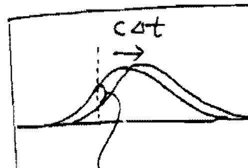
(2) If the reader solves this problem for the initial condition for $u(x,0) = \sin x$, she will easily realize that the shape of the wave is distorted and the one side of the wave becomes steeper. This tendency is enhanced if ν is small. Thus we can imagine that for

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0. \tag{1.16}$$



any non-increasing smooth nonconstant initial condition eventually gives rise to a non-smooth solution. This is the formation of *shocks*. Since shock is a discontinuity, it cannot be studied by classical solutions. A function which is sufficiently differentiable and satisfies a PDE is called a *classical* solution of the PDE. To describe a shock we need non-classical solutions. A wider class of solutions with less smoothness such as the shock solution in the current example are called *generalized solutions* or *weak solutions*.

(3) For (1.16) to have a C^1-solution[2] for all $t > 0$, the initial condition must be non-decreasing. This is intuitively easy to see, if we note that the solution to



$$\partial u/\partial t + c\partial u/\partial x = 0 \tag{1.17}$$

$$\Delta f = f(x) - f(x - c\Delta t) = -\frac{\partial f}{\partial x} c\Delta t$$

with the initial condition $u(x,0) = f(x)$ is given by

$$u(x,t) = f(x - ct). \tag{1.18}$$

If the initial condition is decreasing, then at some finite time, the solution becomes discontinuous. If f is not C1, this is a weak solution. Evans p18.

1.2c Introduction to Laplace and Poisson equation. The *Laplace equation* (examples: **a1B.3**, **a1D.10**, **a1F.6**, **a1E.9**)[3] is given by

$$0 = \Delta\psi. \tag{1.19}$$

This appears as the equation governing the stationary solution to the wave equation or the diffusion equation.

Exercise.
In 2-space $\ln(x^2 + y^2)$ satisfies the Laplace equation away from the origin ($\to$**5.7**).

The following equation

$$f = \Delta\psi \tag{1.20}$$

[2] C^p, where p is a nonnegative integer, means the class of functions which are p-times continuously differentiable.

[3] Pierre Simon Laplace, 1749-1827. See **33.3** for a brief biography. He introduced Δ in 1785 in his work on celestial mechanics.

is called *Poisson's equation* (examples: **a1E.10**, **a1F.6**).[4] where f is an appropriate[5] function.

1.3 Typical problem. A typical problem we wish to analyze is a *initial-boundary value problem.* For example, consider the wave equation on a region[6] D

$$\left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)\psi(\boldsymbol{r},t) = \rho(\boldsymbol{r},t), \tag{1.21}$$

with the boundary condition[7]

$$\psi(\boldsymbol{r},t) = \varphi(\boldsymbol{r},t) \quad \text{for } \boldsymbol{r} \in \partial D \ \text{ for all } t > 0, \tag{1.22}$$

and with the initial condition:

$$\psi(\boldsymbol{r},0) = f(\boldsymbol{r}) \ \text{ and } \partial_t\psi(\boldsymbol{r},0) = g(\boldsymbol{r}) \ \text{ on } D. \tag{1.23}$$

A general strategy to solve this type of problem is to decompose the problem into simpler problems with the aid of linearity (→**1.5**).

Exercise.
Set up the following problems mathematically:
(1) At time $t = 0$ the two rods have uniform temperatures T_1 and T_2, respectively, and they are brought into thermal contact with each other at their one ends. The remaining ends are insulated. The lengths, heat capacities per unit length, and thermal conductivities of the uniform rods labeled 1 and 2 are denoted as $\{L_1, L_2\}$, $\{c_1, c_2\}$ and $\{K_1, K_2\}$. Find the equations governing the temperature field along the rods with appropriate auxiliary conditions.
(2) A semi-infinite cylindrical rod of radius b has a heat capacity per unit volume C, and thermal conductivity K. The curved sides are insulated thermally and the end is subjected to a time-dependent flux of heat

$$F = A\cos 2\mu^2 t \quad \text{for } t > 0.$$

The temperature $T(r,\varphi,z)$ is initially zero throughout the material.
(3) A pair of infinite, insulating planes are placed parallel to the xy-plane. The first one is at $z = 0$ and the second at $z = a$. The plates are coated with electric charge so that the electrostatic potential is

$$\phi(x,y,0) = \varphi_0 \sin ky \quad \phi(x,y,a) = \varphi_0 \cos ky.$$

[4] Siméon Denis Poisson, 1781-1840.

[5] In mathematics such adjectives as 'appropriate', 'suitable,' etc., are used to avoid technical details. We wish to assume that f is sufficiently well-behaved, e.g., smooth, to avoid pathology.

[6] A connected set is called a *region*. It is understood to be an open set unless its closedness is mentioned explicitly. For a summary of elementary topology see **A1**.

[7] ∂A denotes the boundary of the set A. This is a standard notation.

where k is a positive constant. Find the electrostatic potential, and the surface charge density. [You must assume some extra condition on the asymptotic behavior of the solution at infinity. →**1.19** Discussion.]
(4) A semi-infinite string lying along the x-axis and a harmonic oscillator are coupled to each other at $x = 0$. The harmonic oscillator is only allowed to move in the y-direction. Also the string is allowed to make transversal displacement $\phi(x, t)$ in the y-direction. Ignore gravity. The oscillator is made of a mass m and a spring of Hooke's constant k.
(5) A very long rod of small cross-section (this is a 1-space problem) is periodically heated at one end as $T = T_0 + T_1 \cos\omega t$. The heat diffuses along the rod with thermal diffusivity D, and also radiate out from the rod into the surrounding medium held at constant temperature T_0 according to the Newton's law of cooling (cf. (1.57)). Find $T(x.t)$.
(6) There is a spherical cavity of radius a whose center is at the origin in an infinite medium of conductivity σ. Far from the cavity is a uniform current of density j_0 in the z-direction. Find the current field in the medium. the electric field in the cavity. and the charge density at the cavity surface.
(7) A spherical planet of radius R is formed at time $t = 0$. The initial temperature of the planet is $T = 0$. but the planet contains radioactive materials whose decay generates heat at a rate of h per unit volume per unit time. Assume h is uniform throughout the planet. The heat capacity of the planet is C per unit volume, and the heat conductivity is K. The surface of the planet is at temperature $T = 0$ for $t > 0$. Find the history of T. (cf. **16B.9** Exercise (B))
(8) A hollow nonconducting half spherical shell of radius R carries a uniform surface charge σ. Its center is at the origin and its disk-shaped boundary is on the xy-plane with the body below it. Suppose you put a particle of mass m. charge q at a point $(0.0.z_0)$ $(0 < z_0 < R)$. Find the σ such that the electric and gravitational forces balance. (→**26B.7**)

1.4 Linearity and superposition principle. An operator is a map which maps a function to another function (0r number. or another mathematical object).[8] For example.

$$d/dx : f \to df/dx \tag{1.24}$$

and

$$\mathrm{sq} : f \to f^2 \tag{1.25}$$

are operators. An operator L is a *linear operator*. if

$$L(\alpha\psi_1 + \beta\psi_2) = \alpha L\psi_1 + \beta L\psi_2. \tag{1.26}$$

[8] As we will see later in **34A**. an operator is not solely defined by its form (say. d/dx). Its domain must be very carefully specified. The form and the domain together define an operator. Therefore. the exposition here is informal.

where α and β are numerical constants. This relation is called the *superposition principle.* For example. d/dx. and $\partial_t^2 - c^2\Delta$ in **1.3** are linear operators. sq above is a nonlinear operator.

Discussion.
Alexander Bell's original motivation was to send several Morse code messages along the same telegraph line by using different frequencies, exploiting the superposition principle.

1.5 Linear decomposition of problem. In the problem **1.3**, notice that the auxiliary conditions are also written in terms of linear operators: to evaluate the values of ψ at the boundary ∂D, the map

$$A_1 : \psi \rightarrow \psi|_{\partial D} \tag{1.27}$$

may be interpreted as a linear map. Likewise. the evaluation of the initial value

$$A_2 : \psi \rightarrow \psi|_{t=0} \tag{1.28}$$

is also linear. Hence. the linear problem in **1.3** has the following general form

$$L\psi = \rho. \text{ with } A_1\psi = \varphi. \text{ and } A_2\psi = f. \tag{1.29}$$

The superposition principle allows us to decompose this into
(1)

$$L\psi_1 = 0 \text{ with } A_1\psi_1 = \varphi \text{ and } A_2\psi_1 = f, \tag{1.30}$$

and
(2)

$$L\psi_2 = \rho \text{ with } A_1\psi_2 = 0 \text{ and } A_2\psi_2 = 0. \tag{1.31}$$

(1) can further be decomposed into
(11) $L\psi_{11} = 0$ with $A_1\psi_{11} = \varphi$ and $A_2\psi_{11} = 0$ and
(12) $L\psi_{12} = 0$ with $A_1\psi_{12} = 0$ and $A_2\psi_{12} = f$. Such a further decomposition may or may not be useful (see separation of variable, e.g.. **18**).

In (2) if $\rho = \rho_1 + \rho_2$. we have only to solve $L\psi_i = \rho_i$. thanks to the superposition principle. but again whether such a further decomposition is useful or not depends on each problem.

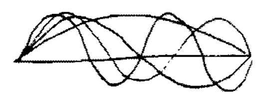
It is conceivable to use the superposition principle to decompose a given problem into standardized 'atomic' problems. This ultimate use of the principle was proposed by Fourier and by Green: they proposed two major strategies to exploit linearity.

1.6 Fourier's idea, ca1807. Daniel Bernoulli in 1735 asserted that the general solution to the 1D wave equation is given in the following form:

$$\psi(x.t) = \frac{1}{2}a_0(t) + \sum_{n=1}^{\infty}\{a_n(t)\cos nkx + b_n(t)\sin nkx\}. \tag{1.32}$$

where $a_i(t)$ and $b_i(t)$ are functions of time t. He claimed that a general function of t can be written in terms of the linear combination of trigonometric functions (see Fourier expansion **17**).

Discussion.
Daniel Bernoulli conceived his scheme through an attempt to solve the wave equation on an interval $[0, L]$ with a homogeneous Dirichlet boundary condition. Physically, he considered the wave equation as a limit of a harmonic chain, so it was very natural for him to guess that the solution must be a superposition of harmonic modes:



$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{L}. \tag{1.33}$$

For the n-th mode, its wavelength is $\lambda = 2L/n$. The wave speed is c, so that its angular frequency must be $\omega = 2\pi c/\lambda = nc\pi/L$. Hence,

$$\frac{d^2 a_n(t)}{dt^2} = -\left(\frac{nc\pi}{L}\right)^2 a_n(t). \tag{1.34}$$

However, Euler(→**4.4**), d'Alembert(→**2B.7**), and others did not agree, because they seemed to think that the condition that a function is constructed from trigonometric functions restricts the generality of the constructed functions. The work in the 18th century was forgotten (入6 p34)

Fourier, who came to the stage about 50 years later, wished to verify D. Bernoulli's idea mathematically. He gave the now famous formulas for the Fourier coefficients (→**17.1**), and claimed that Bernoulli's assertion was correct for any bounded function with the aid of termwise integration. His idea was not accepted by the leading French mathematicians of his day like Laplace(→**33.3**), Langrange(→**3.5**), and Poisson, although they recognized the importance. This explains well why Fourier's idea was a source and spur of modern mathematics, and has strongly influenced its foundation (→**17.18**).

1.7 Who was Fourier?[9] Joseph Fourier was born on March 21, 1768 in Auxerre, the ninth child of a master tailor. Although he became an orphan at the age of ten, his brilliance gained him a free place at the local Benedictine school. He was a teacher of the school when the Revolution began: he became the president of the revolutionary committee of Auxerre. He was arrested under Robespierre's regime. Only the fall of Robespierre saved his head. After much more turmoil (being arrested, released, rearrested, etc.) he eventually succeeded Lagrange in the Chair of Analysis and Mechanics at Ecole Polytechnique, when he was ordered to join Napoleon's invasion of Egypt. He occupied several important administrative and political posts there.

$\boxed{F}$

Fourier

[9] This is from Section 92-3 of T. W. Körner, *Fourier Analysis* (Cambridge, 1990), which is probably the best introductory book of Fourier analysis for those who can appreciate right mathematical taste.

He returned to France in 1801 when the French expedition surrendered. He was appointed the Prefect around Grenoble. During his 14 year tenure, he drained twenty thousand acres of swamp around Bourgoin, which resulted in major economic and health benefits, and made a new road across the Alps (present Route N91). Meanwhile he helped the making of the *Description of Egypt* and made an epoch in Egyptology. On a personal level he encouraged Champollion and as Prefect preserved his protegè from conscription.

While being Prefect, in 1804 he took up the heat conduction problem. In three remarkable years he found the diffusion equation (→**28**, **38**), developed new methods to solve them, and applied them to support his solutions.

In 1807 he submitted his work to the Academy, but was rejected (the committee consisted of Lagrange, Laplace, Monge, and Lacroix). In 1811 the Academy gave him a second chance. He submitted his earlier essay with some further results (the most notable among which was the introduction of Fourier transformation →**32**). Although Fourier received the grand prize, the accompanying report made it clear that Lagrange and Laplace had not withdrawn their objections. This episode should be taken more as a tribute to the originality of Fourier's methods than a reproach to mathematicians Fourier greatly respected (and, in Lagrange's case, admired).

Again, some political turmoil disrupted his life briefly, but eventually his essay was published in 1822 and he was elected permanent mathematical secretary of the Academy. He encouraged as a 'grand old man' younger talent such as Liouville (cf. **15.4**), Sturm (cf. **11B.7**, **15.4**), Dirichlet (cf. **17.18**(1), **1.18**(2)), and Navier (cf. **a1E.6**).

1.8 Green's idea, 1828. Green established Green's theorem (→**16A.19**) and applied it to the electric potential problem, and wrote down the fundamental integral formula for harmonic functions (→**a1B.3**, **2C.11**). In retrospect, his theory is another ultimate exploitation of superposition principle: a function can be decomposed into the sum of impulses. If we denote the impulse of unit strength concentrated at point x by δ_x,[10] any function f can be regarded as

$$f = \sum_x f(x)\delta_x. \tag{1.35}$$

Here we interpret the summation over x intuitively. Thus to solve $Qu = f$, we have only to solve the problem for $Qu = \delta_x$ thanks to the superposition principle. Its solution G_x was introduced by Green,

[10] In terms of the δ-function we will learn later (→**3.8**, **8B.12**, **8B.13**, **14.5**) that we may write '$\delta_x = \delta(x-y)dy$.' δ_x itself is a respectable mathematical object called the *atomic measure* concentrated at x.

which Riemann (→**7.15**) later called a *Green's function.* The solution to the original problem can be written as

$$u = \sum_x f(x)G_x. \tag{1.36}$$

where the interpretation of the summation symbol is the same as in (1.35).
Remark. As we will learn later (→**16**, **36**, **38-40**), we need Green's functions for homogeneous (i.e., zero) boundary conditions only. Hence, in these days, when we speak about Green's functions, they are understood with homogeneous boundary conditions.

Discussion
Read J. Schwinger, "The Greening of Quantum Field Theory: George and I," in *Julian Schwinger, The physicist, the teacher, and the man* (World Scientific, 1996) edited by Y. J. Ng.

1.9 Who was Green?[11] George Green was born in June 1793 in the village of Saxondale six miles from Nottingham. He became a pupil of a secondary school in 1801 where he studied until the summer of 1802. A 27 year old teacher, Robert Goodacre, was able to interest George in mathematics and natural science. However, his father's baker business flourished, so he had to be his father's assistant. Thus he had to educate himself: he learned Laplace's (→**33.3**) *Analytical Mechanics*, the work of Lagrange (→**3.5**), and also a complete collection of Proceedings of the Royal Society was available. He also learned Coulomb and Poisson (cf. **16D.8**, **32C.2**).

In 1826 a public subscription library was opened in Nottingham, which helped to publish his first, largest and most important scientific work "An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism" in 1828. The edition was small and most of them were scattered among homes of his fellow subscribers. Thomson (subsequently Lord Kelvin) managed to get three copies with much difficulty less than 20 years later.

In 1829, his father died, and in 1833 Green decided to enter Cambridge. In 1837 he passed brilliantly the tripos and became a fourth wrangler, and on October 31, 1839 he was elected Fellow of Gonville and Caius, his alma mater. However, his health failed, and he died of influenza on May 31, 1841 in Sneiton (another village close to Nottingham).

1.10 Conventional mathematical physics. To solve linear problems exploiting superposition principle (→**1.4**) is the sole topic of conventional mathematical physics. To this end (→**18**, **23**), we need Fourier expansion and its generalization in terms of special functions,

[11] Yu. A. Lyubimov, "George Green: his life and works (on the occasion of the bicentenary of his birthday)," Physics-Uspekhi **37**, 97-109 (1994).

which turn out to be eigenfunctions of ordinary differential equation eigenvalue problems (→**35**). Thus, we must learn how to solve (linear) ODEs (→**24**). Complex analysis (→**4-10**) is a prerequisite for studying special functions. These topics perhaps exhaust conventional mathematical physics.

1.11 What is (should be) modern mathematical physics? Analytically exactly solvable problems are very limited. Most problems we encounter in real life are not solvable by the standard conventional tools. so that we often use computers. Consequently, it is very important to know representative numerical methods.

Even if the reader could produce numbers (flood of numbers) with the aid of a computer. how is she sure about her numerical result? It is of course important to know various problems and difficulties in numerical analysis (→**31**). but it is probably more important to have sound mathematical intuition about the qualitative features of the equation. Hence. we should clearly know relevant theorems on qualitative aspects of the problem (→**28-30**).

Furthermore. if we could construct an analytical approximation. our understanding of the problem is often greatly enhanced. Therefore. the lecturer believes that the modern applicable analysis must have three pillars:
(1) Analytic exact methods and relevant basic theorems (eg.. existence and uniqueness) [This is the conventional part.]
(2) Computational methods. and
(3) Approximation methods and qualitative approaches (with relevant theorems).

Due to the limited time available. really important topics (2) and (3) must be excluded from the course. However. the lecturer wishes to distribute bits of related topics throughout the course.

1.12 PDE vs. other modeling tools. In these notes we mainly discuss PDEs. That is. we discuss the models of Nature in terms of continuum mathematics. PDEs are often used to describe macroscopic features of Nature (→**a1A.1**). Hence. whether Nature is actually continuous or discrete at very small scales should be an irrelevant question. In other words, lattice or discrete models and continuum models should be indistinguishable.[12] when PDE modeling is useful. On the other hand. in order to use computers, discrete descriptions are much more convenient than continuum ones. Consequently, it is often computationally advantageous to use discrete models: cellular automata, coupled maps, cell-dynamical systems. etc. It is not a wise attitude to claim that continuum description is more fundamental or less so than the discrete description. In practice. we should be able to go between

[12]This is. of course. the main idea behind lattice models of field theories. and numerical schemes. Riemann (→**7.15**) considered this possibility, too. in his famous lecture (→**2D.2**) delivered in the presence of Gauss.

these descriptions as easily as possible. This attitude is especially advantageous in using computers productively as a tool to study Nature.

Discussion.
(A)[13] Let $\{\psi_t(n)\}$ be the set of values (say, between 1.5 and -1.5) on a torus (i.e., the lattice with a periodic boundary condition) whose point is denoted by n at time $t \in \boldsymbol{N}$. Assume that

$$\psi_{t+1}(n) = c\psi_t(n) + \mathcal{I}_t(n) - \langle\langle \mathcal{I}_t(n) \rangle\rangle, \tag{1.37}$$

where $c \in (0, 1]$, $\langle\langle \cdot \rangle\rangle$ is a local average (→**1.13**), and

$$\mathcal{I}_t(n) \equiv A \tanh \psi_t(n) - \psi_t(n) + 0.5[\langle\langle \psi_t(n) \rangle\rangle - \psi_t(n)]. \tag{1.38}$$

(1) For $c = 1$ show that the total sum of $\psi_t(n)$ over the lattice is time independent.
(2) If $c = 1$ and $A \in (0, 1)$, then, $\psi_t(n) \to$ constant eventually.
(3) What happens if $c = 1$ and $A = 1.3$, with the spatially uniform random condition of $\psi_0(n) \in [-0.05, 0.05]$? Guess the behavior.
(4) In the case (3), if $c = .97$, what happens? Why don't you simulate the system?
(B) This type of discrete models have been used extensively in materials science. The latest and perhaps the state of the art example is M. Zapotocky, P. M. Goldbart, and N. Goldenfeld, Phys. Rev. **51**, 1216 (1995) on liquid crystals.

1.13 Discretization of Δ – intuitive meaning of Laplacian. As we will see later (→**1.15**, for example) discretization is a useful way to understand PDEs.

If we discretize the partial derivative (→**2B.1**) as

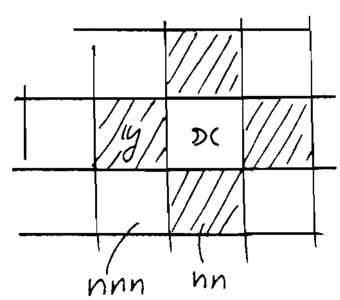
$$\frac{\partial \psi}{\partial x_i} \to \frac{\psi(\boldsymbol{x} + \delta\boldsymbol{x}_i) - \psi(\boldsymbol{x})}{\delta x_i}, \tag{1.39}$$

where $\delta\boldsymbol{x}_i$ is a small increment of the i-th coordinate component, $\boldsymbol{x}$ is understood as the position vector, and $\delta x_i = |\delta\boldsymbol{x}_i|$, then the second derivative reads

$$\frac{\partial^2 \psi}{\partial x_i^2} \to \frac{\psi(\boldsymbol{x} + \delta\boldsymbol{x}_i) + \psi(\boldsymbol{x} - \delta\boldsymbol{x}_i) - 2\psi(\boldsymbol{x})}{\delta x_i^2}. \tag{1.40}$$

Hence, the Laplacian (1.1) can be approximated as

$$\Delta\psi(\boldsymbol{x}) \to \frac{2d}{\delta^2}\left\{\frac{1}{2d}\sum_{nn}\psi(\boldsymbol{y}) - \psi(\boldsymbol{x})\right\}, \tag{1.41}$$



where nn denotes all the nearest neighbor lattice points $\boldsymbol{y}$ of $\boldsymbol{x}$, and we have assumed that all $\delta\boldsymbol{x}_i$ have the same magnitude δ.

[13] Y. Oono and S. Puri, Phys. Rev. Lett. **58**, 836 (1987).

Notice that the first term in the curly brackets is the average value of ψ on the lattice points surrounding $\boldsymbol{x}$. Hence, the intuitive (and physical) meaning of the Laplacian is the difference of a function value and its local average:

$$\Delta\psi \propto \langle\langle\psi\rangle\rangle - \psi, \tag{1.42}$$

where $\langle\langle\ \rangle\rangle$ denotes the local average.[14] Actually, (1.42) is a useful relation in the numerical study of PDE.[15] Since most PDEs cannot be solved analytically, numerical means is vitally important. We will discuss rudiments of numerical solvers of typical PDEs later (→**31**).

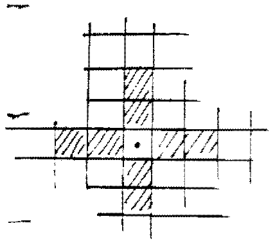
Discussion

(A) A general method to see the order of accuracy (in the small increment limit) of a discretization scheme is to use the Taylor expansion formula like

$$f(t+\delta t) = f(t) + \delta t f'(t) + \frac{1}{2}\delta t^2 f''(t) + R. \tag{1.43}$$

Here R is the remainder term. Notice that the formula can be used if f is twice differentiable (→**A3.15**).

Important Remark. However, the Taylor expansion method tells us about the accuracy of the scheme only in the small increment limit. If the increments are not infinitesimal, then the practical accuracy is a much more subtle problem. An example is the following discretization.



(B) Consider the following discretization of the Laplacian in 2-space;

$$\frac{1}{12\delta^2}\{-60\psi[0,0]+16(\psi[1,0]+\psi[-1,0]+\psi[0,1])+\psi[0,-1])-(\psi[2,0]+\psi[-2,0]+\psi[0,2]+\psi[0,-2])\}. \tag{1.44}$$

which is supposedly fourth order accurate (here $\psi[a,b] \equiv \psi(a\delta,b\delta)$). Confirm this statement. Then, discuss whether this is a good formula to adopt when we cannot afford a very small spatial increment δ.

(C) Consider the discretization of the Laplacian on the face centered cubic lattice (use only the nearest neighbor points). (A simpler version of this problem is the 2-space counterpart on the triangular lattice.) How accurate is it?

1.14 Ubiquity of Laplacian. Let us suppose that ψ is a small displacement of an isotropic and homogeneous membrane in the z-direction (perpendicular to the xy-plane). Then, according to **1.13**, $\Delta\psi$ is the difference between $\psi(x)$ and its averaged surrounding heights. If the membrane wishes to maintain a flat shape (i.e., the flat state is stable or the lowest energy state),[16] then there must be a restoring

---

[14] See the spherical mean value theorem (→**29.4**) for harmonic functions. The reader will realize how fundamental this interpretation is.

[15] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover 1972). 25.3 summarizes discretization of partial derivatives (p883-), but notice that the listed schemes are very conventional.

[16] Anthropomorphic modes of thinking are often effective in understanding physics, after all we are matter, no less, no more.

force in the z-direction which is an increasing function of $\Delta\psi$. If the displacement is small, then the restoring force must be proportional to $\Delta\psi$. This explains why the Laplacian is ubiquitous.

If the membrane can be regarded as a mechanical system, then Newton's equation must be true at each space point, so that we get the wave equation: $\partial_t^2\psi \propto \Delta\psi$.

If the displacement governs the relaxation of ψ, then we get the diffusion equation: $\partial_t\psi \propto \Delta\psi$.[17] In this case, the final stationary state should be time-independent, so that it must be governed by the Laplace equation: $\Delta\psi = 0$.

Exercise.
(1) Show that the Laplacian is the only spherically symmetric second order differential operator (cf. **16.2**).
(2) Compute $\Delta(1/r)$ in d-space ($r \neq 0$. $\to$**16A.2-3**).
(3) What happens for \D r^\a with \a = 2-d?

1.15 Utilize discretization to understand PDE. Differentiation with respect to time can also be discretized as

$$\frac{\partial\psi}{\partial t} \to \frac{\psi(t+\delta t) - \psi(t)}{\delta t}. \tag{1.45}$$

Consider the diffusion equation in 2-space. Space-time discretization of the diffusion equation gives

$$\frac{\psi(t+\delta t) - \psi(t)}{\delta t} = D\frac{\psi(\boldsymbol{x}+\delta\boldsymbol{x}_1) + \psi(\boldsymbol{x}-\delta\boldsymbol{x}_1) + \psi(\boldsymbol{x}+\delta\boldsymbol{x}_2) + \psi(\boldsymbol{x}-\delta\boldsymbol{x}_2) - 4\psi(\boldsymbol{x})}{\delta x^2}, \tag{1.46}$$

where $\delta\boldsymbol{x}_i$ is a small increment along the i-th coordinate. This discretization scheme is called the simple Euler scheme ($\to$**31.8**). Notice that on the simple square if the values of the function on a point $\boldsymbol{x}$ and on its all nearest neighbor points at time t are known, then the value of the function at $\boldsymbol{x}$ at the next time step is uniquely determined. This implies that we must have as an initial condition all the function values on the lattice points at the initial time. This suggests (correctly[18]) that to solve the diffusion equation, we must impose the initial condition: ψ all over the domain of the problem at the initial time. We will discuss the boundary conditions with the aid of simple discretization later ($\to$**1.18**(4)).

Warning. When we wish to solve a PDE numerically, the increments cannot be

[17] Relaxation processes must be irreversible, so that the equation must not be time reversal symmetric. In contrast, the equation of motion must be symmetric.

[18] For most physicists, this type of heuristic reasoning is practically very useful. In most cases the conclusions obtained by such a heuristic approach is fully justified mathematically.

infinitesimal. Hence, even if conservation laws (such as the one for probability, energy, mass, etc.) are satisfied and the symmetry (like spatial translational symmetry, isotropy) recovered in the zero increment limit for a scheme, it does not mean that the scheme is practically usable. However, for our purpose of understanding the behaviors of a PDE under various auxiliary conditions, we need not worry about these practically very important questions, so we may use the simplest discretization scheme as Euler's scheme.□

Discussion.
(1) Apply the similar reasoning to the wave equation, and discuss the initial condition.
(2) For the wave equation, can we impose initial and the final conditions?

1.16 Classification of second order linear constant coefficient PDE, principal part. Most generally, the following equation

$$F\left(x_1, \cdots, x_n, \psi, \frac{\partial \psi}{\partial x_1}, \cdots, \frac{\partial \psi}{\partial x_n}, \frac{\partial^2 \psi}{\partial x_1^2}, \cdots\right) = 0 \qquad (1.47)$$

This is only for linear cases.

is called a partial differential equation, where F is a function of independent variables, $x_1, \cdots, x_n$, the dependent variable ψ and its partial derivatives.[19]

The order of the highest order derivative in the PDE is called its *order*. If F is linear w.r.t. derivatives, we call the PDE a *linear PDE*.[20]

A linear second order PDE with constant coefficients have the following form:

$$A\frac{\partial^2 \psi}{\partial x_1^2} + B\frac{\partial^2 \psi}{\partial x_1 \partial x_2} + C\frac{\partial^2 \psi}{\partial x_2^2} + \cdots + a\frac{\partial \psi}{\partial x_1} + \cdots + b\psi + c = 0, \quad (1.48)$$

where $A, \cdots$ are constants. The highest order derivative terms (the portion consisting of them is called the *principal part*) dictate the character of the PDE, so we pay attention to the principal part (cf. **28.10**).
(1) If, by an affine transformation of the independent variables, we can transform the principal part to $\Delta\psi$, we call the PDE an *elliptic equation*.
(2) If, by an affine transformation of the independent variables, the principal part is transformed to that of the wave equation, we call it a *hyperbolic equation*.

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[19]Although we will review some topics related to partial differentiation (→**2B**), the reader should be familiar with elementary analysis of functions with many variables.
[20]Notice that this definition is different from the physicists' use of the word 'linear'.

(3) If, by an affine transformation of the independent variables, the principal part can be transformed to that of the Laplace equation on the hyperplane (a hyperplane of the d-space,[21] that is, a $(d-1)$-subspace of d-space), then the equation is called a *parabolic equation.* For 3-dimensional diffusion equation, we consider it in 4-space (space + time) and its principal part is the spatial Laplace equation. 3-space is a hyperplane of 4-space. Hence, the diffusion equation is a parabolic equation.

In 2-space, let

$$D \equiv AC - B^2, \tag{1.49}$$

which is called the *discriminant.* Depending on its sign the type of a second order linear PDE is classified as follows:[22]
(1) if $D > 0$, we call the PDE an *elliptic equation.*
(2) if $D < 0$, we call the PDE a *hyperbolic equation.*
(3) if $D = 0$, then we call the PDE a *parabolic equation.*

Exercise.
Consider

$$A\frac{\partial^2 v}{\partial x_1^2} + B\frac{\partial^2 v}{\partial x_1 \partial x_2} + C\frac{\partial^2 v}{\partial x_2^2} = 0, \tag{1.50}$$

where $A, \cdots$ are constants. For each case of (1)-(3) find an actual change of variables to convert the above equation to the 'canonical form' (i.e., the form without mixed second order derivatives).

Discussion
(A) The following equation is called Lagrange's equation for minimal surfaces:

$$(1 + u_y^2)u_{xx} + (1 + u_x^2)u_{yy} - 2u_x y_y u_{xy} = 0, \tag{1.51}$$

where $z = u(x, y)$ is the equation of a minimal surface, the surface whose area is minimum under a given shape of the boundary (like a soap film). This is an elliptic equation.
Warning. However, for nonlinear PDE, its type can change dependent on the solution.[23]
(B) If the coefficients in (1.50) are not constant, then the type depends on the space-time domain. The most famous example may be the Tricomi equation:

$$y\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{1.52}$$

Describe its type according to the spatial domains.

---

[21] d-object means d-dimensional 'object'; thus 2-sphere is a two dimensional sphere (the skin of 3-ball), etc

[22] Even if the coefficients are the functions of space and time, locally, we can classify the equation.

[23] Fluid dynamics and rheology are full of such examples. See, for example, D. D. Joseph, *Fluid Dynamics of Viscoelastic Liquids* (Springer, 1990). This book contains a readable introductory part.

1.17 Linear PDE for practitioners. For a conventional physicist, second order linear equations are the most important. **1.16** implies that. in essence, she needs to understand only three equations.

In the traditional courses, the study of general properties of these equations have been totally neglected. Their characteristic features are summarized in **28-30**.

Since most problems cannot be solved analytically, from a very pragmatic point of view,
(1) Perhaps the most important step is to set up a PDE problem so that it has a unique solution. After that, she can consult numerical analysts.
(2) To demonstrate the unique existence of the solution is a main topic of mathematical study of PDE after setting it up ($\to$**1.21**). but for physicists. who trust the correctness of PDE. this is not a major question.
(3) Practically. the easiest way to understand the uniqueness of the solution is to consider an actual physical situation relevant to the PDE and/or consider a discretization of the PDE ($\to$**1.15**) to ask whether we can solve the resultant algebraic equation uniquely or not ($\to$**31**).
In the following entries. representative equations will be discussed from these practical view points.

1.18 Parabolic equation. To understand parabolic equations. we may use the diffusion equation as their representative ($\to$**1.16**(3)). We may understand the equation as describing the relaxation ($\to$**1.14**) of the temperature of a body ($\to$**a1B.2**)

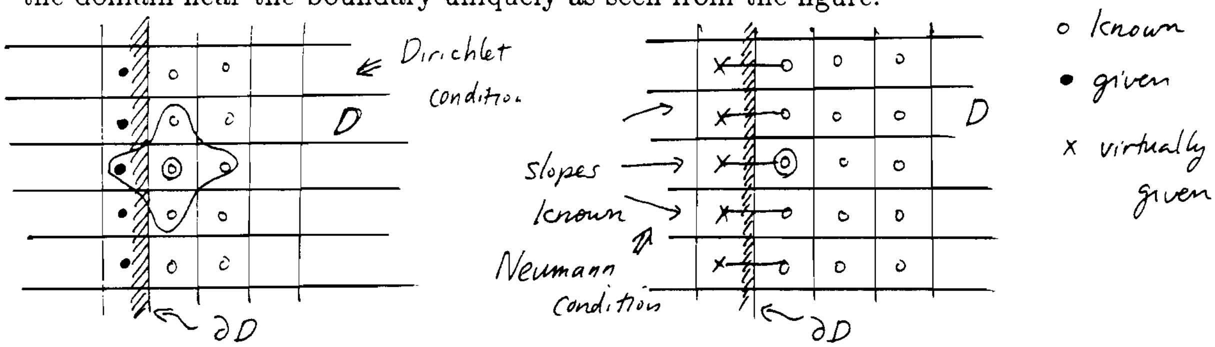
$$\frac{\partial T}{\partial t} = D\Delta T. \tag{1.53}$$

We need auxiliary conditions to single-out the solution:
(1) Initial condition. The value of temperature T everywhere on the body at $t = 0$ is certainly needed to specify the future completely. This should be enough mathematically as we have seen in **1.15**.
(2) Dirichlet condition. Our daily experience tells us that if the temperature at the boundary[24] of the body is specified, this should uniquely fix the temperature inside for all later times (here we assume **(1)**). Specifying the values at the boundary is called a *Dirichlet condition.*

Suppose T_1 and T_2 are the solutions to the same diffusion equation with the same initial and boundary conditions. Then, $u \equiv T_1 - T_2$ obeys

[24] The boundary must be 'nice'. In these notes. we assume the boundary is piecewise smooth. If the boundary has a singularity like a spine. then the solution may not exist. There is a famous counter example by Lebesgue. See the cone condition in **1.19** (2) Discussion.

the same diffusion equation with the zero boundary condition and the zero initial condition (homogeneous auxiliary conditions).[25] The temperature of a body surrounded by a zero degree ice with the same initial temperature should be zero forever. That is, $T_1 \equiv T_2$ forever, or the solution for the original initial-(Dirichlet) boundary value problem should be unique. This argument is a standard, rigorizable argument to demonstrate the uniqueness (→**28.4**). once one has established the existence of a solution.
(3) Neumann condition. It is a fundamental postulate of thermodynamics (the so-called zeroth law)[26] that if a system is thermally isolated. it eventually reaches a unique equilibrium state. Hence, no heat flux condition (= the adiabatic condition) at the boundary should also uniquely specify the future. The consideration of the difference of two systems as in **(2)** tells us that if the heat flux (that is. the (outward) normal gradient of the temperature often denoted by $\partial T/\partial n$) at the boundary is specified (this boundary condition is called a *Neumann condition*). then the future of the system should be uniquely determined.
(4) Use of discretization to understand boundary conditions. These boundary conditions can be understood with the aid of space discretization. Use the simple Euler scheme to discretize the Laplacian as in **1.15**. Consider a lattice point whose nearest neighbor point(s) are boundary points. If we can uniquely determine the function values at the next time step on the inside lattice points, we may conclude that the boundary conditon is a good condition. In this way we can understand the Dirichlet condition easily. The Neumann condition is not this explicit. but the boundary slope helps to fix the value inside the domain near the boundary uniquely as seen from the figure.



From this consideration we realize that a more general condition can be imposed as a good boundary condition called:

[25] Auxiliary conditions specifying the function to be zero are called *homogeneous conditions*.
[26] H. B. Callen. *Thermodynamics* (Wiley. 1960) is perhaps the best introduction to thermodynamics.

(5) Robin condition.[27] Spatial discretization allows us to understand the above mentioned boundary conditions more algorithmically. Use the discretization of the Laplacian in **1.13**. To determine the RHS of (1.46) in **1.15** we must be able to evaluate the Laplacian everywhere in D. The Dirichlet and Neumann conditions allow us to uniquely fix the value of the Laplacian even near the boundary. The same consideration tells us that a more general condition allows us to fix the solution uniquely:

$$g(x)\frac{\partial T}{\partial \nu} + f(x)T = h(x) \text{ on } \partial D, \tag{1.54}$$

where f and g are nonzero functions. This is called a *Robin condition.*

Warning [Unbounded domain]. The intuitive discussion in the text works generally when the domain is bounded. However, if the domain is not bounded, then bizarre things could happen. See the following example (see also **1.19** below).[28] The following function u is a solution to the initial value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{1.55}$$

on $\boldsymbol{R}$ for $t > 0$ with the initial condition $u(x, 0) \equiv 0$:

$$u(x,t) = \int_0^\infty dy\, \left[e^{xy}\cos(xy + 2ty^2) + e^{-xy}\cos(xy - 2ty^2)\right] ye^{-y^{4/3}}\cos y^{4/3}. \tag{1.56}$$

The energy conservation apparently fails. However, this is physically explainable, because the propagation speed of heat is infinite (!) according to the diffusion equation ($\rightarrow$**28.10**). We must forbid large amount of heat coming from infinity to avoid such an unphysical situation. One way is to require that the solution to be bounded everywhere.[29]

Exercise.
(1) According to Newton's radiation law, the boundary condition for the temperature of a body is given by

$$\frac{\partial T}{\partial n} = K(T_0 - T), \tag{1.57}$$

where K is a positive constant, and T_0 is the ambient temperature. Physically discuss that the temperature evolution inside the body is uniquely determined. Then, discuss the weak point of the physical argument.
(2) Use the discretization approach to demonstrate that a Robin condition can uniquely determine the solution for a diffusion equation. That is, do explicitly what

[27] About Gustave Robin, 1855-1897, see T. Abe and I. Onda, *G. Robin and his contribution to mathematics*, Internat. Symp. of History of Mathematics and Mathematical Education, Aug. 7-10, Gunma University (1987). Robin seemed to have burnt all his papers before he died at the age of 45.

[28] P. C. Rosenbloom and D. V. Widder, Am. Math. Month., **65**, 607 (1958).

[29] Actually, a much weaker condition will do.

is in **3** above.

(3) We wish to consider the following equation (the Cahn-Hilliard equation) on [0.1] for $t > 0$. What kind of auxiliary conditions do we need to ensure unique solutions?

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}\left\{-u + u^3 - \frac{\partial^2 u}{\partial x^2}\right\}. \tag{1.58}$$

Is $u \equiv 0$ stable?

Also consider a periodic boundary condition for the equation.

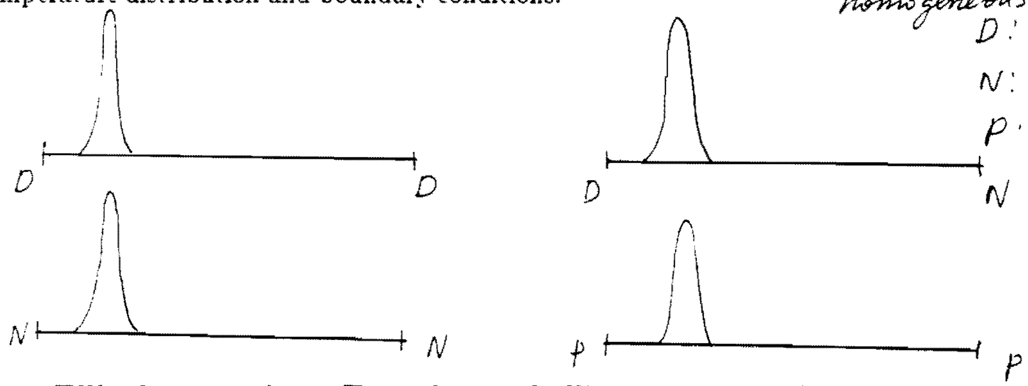
(4) The following equation is called the time-dependent Ginzburg-Landau (TDGL) equation[30]

$$\frac{\partial v}{\partial t} = av - bv^3 + D\Delta v, \tag{1.59}$$

where D and b are positive constants. and a is a numerical constant. Consider the equation on the 2D square $[0.1] \times [0.1]$. We wish to impose a periodic boundary condition. Write down the boundary condition explicitly.

(5) Sketch the time evolution of the temperature field with the following initial temperature distribution and boundary conditions.

homogeneous ∂-condition
D: Dirichlet
N: Neumann
P: periodic



1.19 Elliptic equation. To understand elliptic equations (→**1.16**). we may use the Laplace equation (→**1.2**) as the representative. We may understand the equation as describing the equilibrium temperature distribution of a body which is governed by the diffusion equation (→**a1B.3**). Hence. the boundary conditions discussed in **1.18** should allow us to determine the solution uniquely. More directly. we can repeat similar arguments as in **1.18** (→**29.9**).

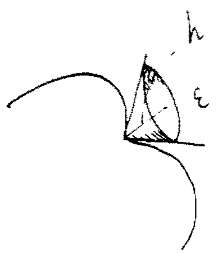
(1) Interior and exterior problems. The boundary value problem whose domain is bounded (i.e., no infinity is contained) is called the *interior problem* (cf. **26B.2**). If the domain is not bounded. the problem is called an *exterior problem* (cf. **26B.3**). Exterior problems require not only the boundary conditions but also some constraints on the growth rate of the solution toward infinity (see **2C.17**footnote. **26B.4**).

Discussion.

[30]This is a standard equation describing the phase ordering kinetics.

There are two infinite parallel walls spaced with a constant distance (say, 1m) and the walls are grounded. Discuss the electric potential in the gap, or, more concretely, discuss that it is not unique. [That is, demonstrate that the homogeneous Dirichlet condition for an unbounded region D cannot single out a unique solution to the Laplace equation on D.]

(2) Dirichlet condition. It is intuitively obvious that the Dirichlet condition which fixes the surrounding wall (i.e., this is an interior problem) temperature determines uniquely the entire steady temperature distribution. This uniqueness can be demonstrated as in **1.18**; if the wall temperature is always at 0, then the inside must reach 0 eventually.



Discussion: External cone condition.
A sufficient condition for the solvability of a Dirichlet problem is the so-called *external cone condition*. At each point x on the boundary, we must be able to place a circular cone of solid angle $\epsilon > 0$ and height $h > 0$ lying completely outside the domain except its apex at x.[31] .

(3) Neumann condition. We may consider the steady temperature of a body under a prescribed heat flux distribution on the wall, if the net heat input is zero (otherwise, the temperature keeps changing). Under a Neumann condition with this net constraint, we can argue that the final temperature is unique up to the additive constant.

(4) A Robin condition can be discussed analogously as in the diffusion equation case **1.18**(5).

Exercise:
Write down a periodic boundary condition for the Laplace equation on $[0, 1] \times [0, 1]]$. Is the solution unique?

1.20 Hyperbolic equation. To understand hyperbolic equations, we may use the wave equation as the representative. This is a second order differential equation in time, so our experience with the second order ODE (e.g., Newton's equation of motion) tells us that we need not only the initial displacement but also the initial displacement speed all over the body: $\psi_{t=0}$ and $\partial_t \psi_{t=0}$. This can be understood from the following discretization as well:

$$\begin{aligned} \psi(t + \delta t, x) &= \psi(t, x) + \delta t \partial_t \psi(t, x), \\ \partial_t \psi(t + \delta t, x) &= \partial_t \psi(t, x) + \delta t [\Delta \psi]_t. \end{aligned} \tag{1.60}$$

The boundary condition can be understood exactly the same way as the diffusion equation (→**1.18**). We can impose a Dirichlet, Neumann or Robin condition (→**30.5**).

[31] Courant-Hilbert.

Exercise.
How can we realize (physically) a Robin condition at the boundary for a wave equation?

1.21 Then, what are mathematicians doing? The reader might have obtained the impression that the unique existence of the solutions to these equations is obvious. If these models are actually very faithful models of our experience in Nature, probably the reader's impression is not an oversimplification of the situation. However, how can we be sure that these models really do model Nature accurately?

Appealing to our physical intuition to justify a PDE as a model of a given phenomenon is a circular argument, because we assume that the model behaves in accordance with our physical intuition. Therefore, the only way to justify a PDE as a model of Nature is to check that the purely mathematical outcome of the equations are consistent with our understanding of the systems being modeled. Consequently, we need a full mathematical theory of PDE (→**29.11** for example). However, practitioners can almost always trust in the extreme precision of the PDE models (at least for classical linear problems), so that often we can ignore rigorous arguments.

Discussion [Navier-Stokes equation].
The Navier-Stokes equation (→**a1E.6**) is supposedly the fundamental equation governing fluid motion. It is, however, derived (or rather, written down) intuitively by Navier. So far no one has been able to derive it in a well controlled fashion from the particle picture of fluid. Even if we cannot derive it, if it works (agrees with empirical results), the equation should be legitimate.

We cannot solve the equation analytically in most cases, so that we must rely on numerical schemes. However, it is not necessarily clear whether the numerical scheme is faithful to the original equation. Hence, the agreement of numerical results with observed results does not necessarily empirically justify the equation. For the justification, the unique existence of the solution is a prerequisite.

In the case of the 3D Navier-Stokes equation, so far no one knows whether the unique deterministic solution exists or not. One group of mathematicians (majority?) believe that the initial boundary value problem of the 3D Navier-Stokes equation is well-posed (→**28.3**). However, there are mathematicians who believe that the 3D Navier-Stokes equation is defective (mathematically meaningless for large Reynolds numbers).[32]

[32] Representative references by the champions of both views may be: P. Constantin and C. Foias, *Navier-Stokes Equations* (Chicago UP, 1988); R. Temam, *Navier-Stokes Equations: theory and numerical analysis* (North-Holland, 1977); O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow* (Gordon and Breach, 1969).